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ON THE NONEXISTENCE OF CONTINUOUS CANONICAL FORMS  
FOR LINEAR DYNAMICAL SYSTEMS

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1. INTRODUCTION AND STATEMENT OF RESULTS

A real linear, constant, finite dimensional dynamical system is thought of as being represented by a triple of real matrices  $(F,G,H)$  where  $F$  is an  $n \times n$  matrix,  $G$  an  $n \times m$  matrix and  $H$  an  $p \times n$  matrix; i.e. there are  $m$  inputs,  $p$  outputs and the state space dimension is  $n$ . The dynamical system itself is then

$$(1.1) \quad \dot{x} = Fx + Gu, \quad y = Hx$$

in the continuous case, or

$$(1.2) \quad x(t+1) = Fx(t) + Gu(t), \quad y(t) = Hx(t)$$

in the discrete case. A change of coordinates in state space changes the triple of matrices  $(F,G,H)$  into the triple  $(SFS^{-1}, SG, HS^{-1})$ . We are interested in continuous canonical forms for this action of  $GL_n(\mathbb{R})$ , the group of real invertible  $n \times n$  matrices. Cf. 3.3 below for a precise definition of what a canonical form is.

The triple  $(F,G,H)$  is completely reachable if the matrix

$$(1.3) \quad R(F,G) = (G \quad FG \quad \dots \quad F^{n-1}G)$$

consisting of all the columns of the matrices  $F^i G$ ,  $i = 0, \dots, n-1$  has rank  $n$ . The triple  $(F,G,H)$  is completely observable if the matrix

$$Q(F,H) = (H^T \quad F^T H^T \quad \dots \quad (F^{n-1})^T H^T)$$

where the upper T denotes transposes has rank  $n$ . Cf. [6] for these notions.

Let  $\mathcal{F}(F,G,H) \subset \mathbb{R}^{(n,m,p)}$  denote the space of all triples of matrices  $(F,G,H)$ ,  $\mathcal{F}(F,G,H)_{cr}$  the subspace of all completely reachable triples and  $\mathcal{F}(F,G,H)_{cr,co}$  the subspace of all triples which are completely observable and completely reachable.

In [ 2 ], [ 3 ] we studied pairs of completely reachable matrices  $(F,G) \in \mathbb{R}G_{cr}$  (over arbitrary fields) by algebraic geometric methods and proved that there are no algebraic continuous canonical forms on  $\mathbb{R}G_{cr}$  if  $m \geq 2$ . We can embed  $\mathbb{R}G_{cr}$  into  $\mathbb{R}GH_{cr}$  by means of the  $GL_n$  invariant map

$$(1.4) \quad (F,G) \mapsto (F,G,0)$$

So this result implies the nonexistence of algebraic continuous canonical forms on  $\mathbb{R}GH_{cr}$  (and  $\mathbb{R}GH(\mathbb{C})_{cr}$ ) but gives at first sight no information on the existence of canonical forms on  $\mathbb{R}GH(\mathbb{R})_{cr,co}$ . Firstly, because the results of [2], [3] as stated there do not rule out the existence of nonalgebraic continuous canonical forms on  $\mathbb{R}G(\mathbb{R})_{cr}$  and  $\mathbb{R}G(\mathbb{C})_{cr}$ , and secondly because there seems to be no  $GL_n(\mathbb{R})$  invariant embedding  $\mathbb{R}G(\mathbb{R})_{cr} \hookrightarrow \mathbb{R}GH(\mathbb{R})_{cr,co}$ .

So that the results of [2], [3] leave it open whether  $\mathbb{R}GH(\mathbb{R})_{cr,co}$  admits a canonical form or not. In fact, this had better be the case because there does exist an (algebraic) continuous canonical form on  $\mathbb{R}GH(\mathbb{R})_{cr,co}$  if  $p = 1$ .

We have:

#### 1.5. Theorem.

There does not exist a continuous canonical form on  $\mathbb{R}GH(\mathbb{R})_{cr,co}$  if and only if  $m \geq 2$  and  $p \geq 2$ . A fortiori there are no continuous canonical forms on  $\mathbb{R}G(\mathbb{R})$ ,  $\mathbb{R}G(\mathbb{R})_{cr}$ ,  $\mathbb{R}G(\mathbb{R})_{co}$  if  $m \geq 2$  and  $p \geq 2$ .

In this paper we show how one can use results on the nonexistence of canonical forms on  $\mathbb{R}G(\mathbb{R})_{cr}$  to deduce results on the nonexistence of canonical forms on  $\mathbb{R}GH(\mathbb{R})_{cr,co}$  for suitable  $p$ . We are thus able to prove theorem 1.5 for the case  $m \geq 2$ ,  $p \geq 2n$  and we indicate a similar proof for the cases  $p \geq 2$ ,  $m \geq 2n$  and  $p, m > n$ . For the general case cf. [4]. The basic idea is very simple. The Gramm-Schmidt orthonormalization process shows that there exists a continuous  $GL_n(\mathbb{R})$  canonical form on  $\mathbb{R}G(\mathbb{R})_{cr}$  (resp.  $\mathbb{R}GH(\mathbb{R})_{cr,co}$ ) if and only if there exists an  $O_n(\mathbb{R})$  canonical form on  $\mathbb{R}G(\mathbb{R})_{cr}^{ortho}$  (resp.  $\mathbb{R}GH(\mathbb{R})_{cr,co}^{ortho}$ ) where the superscript "ortho" means that we consider only those pairs (resp. triples) such that  $R(F,G)$  has orthonormal row vectors, and where  $O_n(\mathbb{R})$  is the group of orthogonal  $n \times n$  matrices.

This trick is useful because there does exist an  $O_n(\mathbb{R})$  invariant

embedding  $(\mathbb{C}\mathbb{G}(\mathbb{R})_{\text{cr}})^{\text{ortho}} \rightarrow (\mathbb{C}\mathbb{H}(\mathbb{R})_{\text{cr,co}})^{\text{ortho}}$  for suitable  $p$ , viz.

$$(1.6) \quad (F, G) \rightarrow (F, G, \bar{R}(F, G)^T)$$

where  $\bar{R}(F, G)$  is the matrix

$$(1.7) \quad \bar{R}(F, G) = (G \quad FG \quad \dots \quad F^{n-1}G)$$

However, Gram-Schmidt orthonormalization is essentially nonalgebraic which is one more reason why we cannot use the results of [2], [3] as they stand, but have to extend them to prove nonexistence of continuous (possibly nonalgebraic) canonical forms on  $(\mathbb{C}\mathbb{G}(\mathbb{R})_{\text{cr}})$ . This is done in section 2 below. The methods are the same as those of [2], [3]: the quotient  $(\mathbb{C}\mathbb{G}(\mathbb{R})_{\text{cr}})/GL_n(\mathbb{R})$  is shown to exist and to admit a universal family of completely reachable pairs over it. Then we need a new proof that the underlying bundle of the universal family is nontrivial if  $m \geq 2$ , because a priori there is no reason why the bundle of  $\mathbb{R}$ -points  $E(\mathbb{R}) \rightarrow B(\mathbb{R})$  of a nontrivial algebraic bundle  $E \rightarrow B$  should be nontrivial. The rest of the nonexistence proof is then as in [3], 6.1. Section 3 contains the orthonormalization trick alluded to above and in section 4 theorem 1.5 is proved for suitable  $m, n, p$ .

## 2. A FINE MODULI SPACE FOR CONTINUOUS FAMILIES OF REAL LINEAR DYNAMICAL SYSTEMS.

In this section we consider completely reachable pairs of real matrices  $(F, G)$  of size  $n \times n$  and  $m \times n$  respectively. As usual  $R(F, G)$  is the matrix  $(G \quad FG \quad F^2G \quad \dots \quad F^{n-1}G)$  with columns  $F^i g_j$ ,  $j = 1, \dots, m$ ;

$i = 0, \dots, n$  where  $g_j$  is the  $j$ -th column of  $G$ . We number the columns of  $R(F, G)$  by means of the pairs  $(i, j)$  ordered lexicographically. Let  $J$  be this set of indices.

### 2.1. Nice Selections and Successor Indices.

A nice selection is a subset  $\alpha$  of  $J$  with the property that  $(i, j) \in \alpha \Rightarrow (i', j) \in \alpha$  for all  $i' \leq i$ . A successor index  $k = (i, j)$  of a nice selection  $\alpha$

is an element  $(i,j) \in J$  such that  $(i',j) \in \alpha$  for all  $i' \leq i$ . Note that there is precisely one successor index of the form  $(i,j)$  for  $\alpha$  for every  $j = 1, \dots, m$ . This successor index is denoted  $s(\alpha,j)$ .

## 2.2. Construction of the Differentiable Manifold $\widehat{M}_{m,n}(\mathbb{R})$ .

For each nice selection  $\alpha$  let  $U_\alpha = \mathbb{R}^{mn}$ . For  $x \in U_\alpha$  with components  $x_k$ ,  $k = 1, \dots, mn$  let  $x(i)$ ,  $i = 1, \dots, m$  denote the columnvector with entries  $x(i)_j = x_{(i-1)n+j}$ ,  $j = 1, \dots, m$ . (I.e. we write  $x$  as an  $n \times m$  array). For each  $x \in U_\alpha = \mathbb{R}^{mn}$  there is a unique pair of real matrices  $(F,G) \in FG(\mathbb{R})_{cr}$  such that

$$(2.2.1) \quad R(F,G)_\alpha = I_n$$

where  $R(F,G)_\alpha$  is the matrix consisting of the columns of  $R(F,G)$  with indices in  $\alpha$  (in their original order), and such that

$$(2.2.2) \quad R(F,G)_{s(\alpha,j)} = x(j), \quad j = 1, \dots, m$$

where  $R(F,G)_{s(\alpha,j)}$  is the column of  $R(F,G)$  with index  $s(\alpha,j)$ , the  $j$ -th successor index of  $\alpha$ . For a proof cf. [3] sections 3.4, 3.5. This pair of matrices is denoted  $\psi_\alpha(x)$ .

For each ordered pair of nice selections  $\alpha$  and  $\beta$  we define

$$(2.2.3) \quad U_{\alpha\beta} = \{x \in U_\alpha \mid (R\psi_\alpha(x))_\beta \text{ is nonsingular}\}$$

and we identify  $U_{\alpha\beta}$  and  $U_{\beta\alpha}$  by means of the correspondence

$$(2.2.4) \quad x \leftrightarrow y \iff (R\psi_\alpha(x))_\beta^{-1} (R\psi_\alpha(x)) = R\psi_\beta(y)$$

These identifications define a differentiable manifold denoted  $\widehat{M}_{m,n}(\mathbb{R})$  which is covered by the coordinate patches  $U_\alpha = \mathbb{R}^{mn}$ ,  $\alpha$  a nice selection.

There is a natural map

$$(2.2.5) \quad \pi : FG(\mathbb{R})_{cr} \rightarrow \widehat{M}_{m,n}(\mathbb{R})$$

which is defined as follows. For each  $(F,G) \in FG(\mathbb{R})_{cr}$  there is a nice

selection  $\alpha$  such that  $R(F,G)_\alpha$  is nonsingular ([2] lemma 2.4.1).

We now map  $(F,G)$  to the point  $x \in U_\alpha \subset \mathbb{M}_{m,n}(\mathbb{R})$  determined by

$$(2.2.6) \quad \pi(F,G) = x \in U_\alpha \subset \mathbb{M}_{m,n}(\mathbb{R}) \iff \psi_\alpha(x) = R(F,G)_\alpha^{-1}R(F,G)$$

This is independent of the choice of  $\alpha$  because of the identifications

$$(2.2.4). \quad \text{The map } \pi \text{ is surjective because } \pi\psi_\alpha(x) = x \text{ for } x \in U_\alpha,$$

and we have for  $x \in U_\alpha$

$$(2.2.7) \quad \pi^{-1}(x) = \{(SFS^{-1}, SG) \mid S \in GL_n(\mathbb{R})\} \text{ if } (F,G) = \psi_\alpha(x).$$

In other words  $\mathbb{M}_{m,n}(\mathbb{R})$  is the quotient of  $\mathbb{FG}(\mathbb{R})_{cr}$  under the action of  $GL_n(\mathbb{R})$ . Cf. [2], 3.3 and [3], 3.5 - 3.7 for proofs.

### 2.3. Continuous Families of Completely Reachable Pairs.

Let  $X$  be a topological space. A continuous family of pairs over  $X$  is an  $n$ -dimensional real vector bundle  $E$  over  $X$  together with a vectorbundle endomorphism  $F: E \rightarrow E$  and  $m$  sections  $g_1, \dots, g_m: X \rightarrow E$ . For each  $x \in X$  we have an endomorphism  $F(x): E(x) = \mathbb{R}^n \rightarrow E(x)$  and  $m$  vectors  $g_1(x), \dots, g_m(x) \in E(x) = \mathbb{R}^n$ . After a choice of basis in  $E(x)$  these vectors and this endomorphism define a pair of matrices, i.e. an element of  $\mathbb{FG}(\mathbb{R})$ . Note that the element so defined is welldefined up to the action of  $GL_n(\mathbb{R})$  (= change of basis). The family  $(E, F, g_1, \dots, g_m)$  is said to be completely reachable if all these elements of  $\mathbb{FG}(\mathbb{R})$  are in fact in  $\mathbb{FG}(\mathbb{R})_{cr}$ .

Two continuous families over  $X$ ,  $(E, F, g_1, \dots, g_m)$ ,  $(E', F', g'_1, \dots, g'_m)$  are said to be isomorphic if there is a vectorbundle isomorphism  $\phi: E \rightarrow E'$  such that

$$(2.3.1) \quad \phi F = F' \phi$$

$$(2.3.2) \quad \phi g_i = g'_i \quad i = 1, \dots, m$$

For every space  $X$  let  $A(X)$  be the set of isomorphism classes of continuous families of completely reachable pairs over  $X$ . By means of the pullback construction which associates to a continuous map  $f: Y \rightarrow X$  and a family  $(E, F, g_1, \dots, g_m)$  over  $X$ , the family  $(f^!E, f^!g_1, \dots, f^!g_m)$  over  $Y$ , we can turn  $A$  into a contravariant functor

from the category of topological spaces to the category of sets.  
Cf. [5] for background material on vectorbundles and pullback.

#### 2.4. The Canonical Map Associated to a Completely Reachable Family.

Let  $\Sigma = (E, F, g_1, \dots, g_m)$  be a family of completely reachable pairs over  $X$ . For each  $x \in X$  we then have a completely reachable pair  $F(x), G(x)$  over  $x$  (cf. 2.3 above) which is determined up to a choice of basis in  $E(x)$ . This means that  $\pi(F(x), g(x))$  is welldefined. (Cf (2.2.5), (2.2.6) above for the definition of  $\pi$ ). Associated to  $\Sigma$  we have thus defined a continuous map  $f(\Sigma) : X \rightarrow \mathbb{M}_{m,n}(\mathbb{R})$ . Note that isomorphic families give rise to the same maps  $X \rightarrow \mathbb{M}_{m,n}(\mathbb{R})$ .

#### 2.5. Definition of the Universal Family.

For each nice selection  $\alpha$  let  $E_\alpha = U_\alpha \times \mathbb{R}^n$  be the trivial vectorbundle over  $U_\alpha$ . We define the bundle endomorphism  $F_\alpha : E_\alpha \rightarrow E_\alpha$  and the sections  $g_{1\alpha}, \dots, g_{m\alpha} : U_\alpha \rightarrow E_\alpha$  as follows. For  $x \in U_\alpha$  write

$$(2.5.1) \quad \psi_\alpha(x) = (F_\alpha(x), G_\alpha(x))$$

We then define

$$(2.5.2) \quad F_\alpha(x, v) = (x, F_\alpha(x)v)$$

$$(2.5.3) \quad g_{i\alpha}(x) = (x, G_\alpha(x)_i) \quad i = 1, \dots, m$$

where  $G_\alpha(x)_i$  is the  $i$ -th column of  $G_\alpha(x)$ .

We now construct a family  $\Sigma^u = (E^u, F^u, g_1^u, \dots, g_m^u)$  over  $\mathbb{M}_{m,n}(\mathbb{R})$  by patching together the partial families  $(E_\alpha, F_\alpha, g_{1\alpha}, \dots, g_{m\alpha})$ . This is done as follows. Let  $E_{\alpha\beta} = \{(x, v) \in E_\alpha \mid x \in U_{\alpha\beta}\}$  and let

$\phi_{\alpha\beta} : U_{\alpha\beta} \rightarrow U_{\beta\alpha}$  be the diffeomorphism defined in (2.2.4) above. We now define the isomorphism

$$(2.5.4) \quad \tilde{\phi}_{\alpha\beta} : E_{\alpha\beta} \rightarrow E_{\beta\alpha}$$

by the formula

$$(2.5.5) \quad \tilde{\phi}_{\alpha\beta}(x, v) = (\phi_{\alpha\beta}(x), (R\psi_\alpha(x))_\beta^{-1}v)$$

It is easy to check that these isomorphisms are compatible with the endomorphisms  $F_\alpha, F_\beta$  and the sections  $g_{i\alpha}, g_{i\beta}, i = 1, \dots, m$ , so that these identifications yield a family  $\Sigma^u$  such that the restriction of  $\Sigma^u$  to  $U_\alpha$  is isomorphic to the family  $(E_\alpha, F_\alpha, g_{1\alpha}, \dots, g_{m\alpha})$  for all nice selections  $\alpha$ .

It follows that

$$(2.5.6) \quad f(\Sigma^u) = \text{identity on } \mathbb{M}_{m,n}(\mathbb{R})$$

(Cf. 2.4 and (2.2.7)).

## 2.6. Theorem.

$\mathbb{M}_{m,n}(\mathbb{R})$  is a fine moduli space for the functor  $A$ .

This means the following. Let  $\text{Top}(X, Y)$  be the set of continuous maps from the topological space  $X$  to the topological space  $Y$ . Then theorem 2.6 says that the map  $\Sigma \mapsto f(\Sigma)$  of section 2.4 above induces a bijection from  $A(X)$  to  $\text{Top}(X, \mathbb{M}_{m,n}(\mathbb{R}))$  for all topological spaces  $X$ . More precisely theorem 2.6 says that: (i) For every  $f \in \text{Top}(X, \mathbb{M}_{m,n}(\mathbb{R}))$  there is a family  $\Sigma^f$  such that  $f(\Sigma^f) = f$ . (N.B. The family  $f^! \Sigma^u$  is such a family), and (ii) for every family of completely reachable pairs  $\Sigma$  over a space  $X$  there is a unique map  $f: X \rightarrow \mathbb{M}_{m,n}(\mathbb{R})$  such that  $f^! \Sigma^u$  is isomorphic to  $\Sigma$ . This map is of course  $f(\Sigma): X \rightarrow \mathbb{M}_{m,n}(\mathbb{R})$  and what is left to prove is that  $f(\Sigma)^! \Sigma^u$  and  $\Sigma$  are isomorphic families. This is done exactly as in [2], 3.6.

## 2.7. An Embedding $S^1 \rightarrow \mathbb{M}_{m,n}(\mathbb{R})$

The next thing we want to do is to show that the bundle  $E^u$  underlying the universal family  $\Sigma^u$  over  $\mathbb{M}_{m,n}(\mathbb{R})$  is not the trivial bundle if  $m \geq 2$ . (If  $m = 1$  there is only one nice selection and it follows that the bundle is trivial in that case). To this end we first construct an explicit embedding  $\phi: S^1 = \mathbb{P}^1(\mathbb{R}) \rightarrow \mathbb{M}_{m,n}(\mathbb{R})$  for  $m, n \geq 2$ . This is done as follows. Define a continuous map

$$(2.7.1) \quad \phi_1: \mathbb{R} \rightarrow \mathbb{E}\mathbb{C}(\mathbb{R})_{\text{cr}}, \quad t \mapsto (F(t,1), G(t,1))$$

where  $F(t,1)$  is equal to the matrix consisting of the columnvectors



$$(2.7.2) \quad F(t,1)_1 = e_1, F(t,1)_2 = e_1 + e_2, F(t,1)_i = e_{i+1} \text{ for } i = 3, \dots, n-1$$

$$F(t,1)_n = 2e_3 \text{ if } n \geq 3,$$

where  $e_j$  is the  $j$ -th unit columnvector. The matrix  $G(t,1)$  consists of the columnvectors

$$(2.7.3) \quad G(t,1)_1 = te_1, G(t,1)_2 = e_1 + e_2, G(t,1)_i = 0 \text{ if } i \geq 3$$

$$\text{if } n = 2 \text{ and } \text{if } n \geq 3$$

$$(2.7.4) \quad G(t,1)_1 = te_1 + e_3, G(t,1)_2 = e_1 + e_2, G(t,1)_i = 0 \text{ if } i \geq 3$$

Note that  $R(\phi_1(t))_\alpha$  is nonsingular for all  $t$  for the nice selection

$$(2.7.5) \quad \alpha = \{(0,1), \dots, (n-3,1), (0,2), (1,2)\}$$

We also define a continuous map

$$(2.7.6) \quad \phi_2 : \mathbb{R} \rightarrow \mathbb{C}(\mathbb{R})_{\text{cr}}, s \mapsto (F(s,2), G(s,2))$$

with

$$(2.7.7) \quad F(s,2)_1 = e_1, F(s,2)_2 = se_1 + e_2, F(s,2)_i = e_{i+1} \text{ for } i = 3, \dots, n-1$$

$$F(s,2)_n = 2e_3 \text{ if } n \geq 3,$$

and with

$$(2.7.8) \quad G(s,2)_1 = e_1, G(s,2)_2 = se_1 + e_2, G(s,2)_i = 0 \text{ for } i \geq 3$$

$$\text{if } n = 2 \text{ and } \text{if } n \geq 3$$

$$(2.7.9) \quad G(s,2)_1 = e_1 + e_3, G(s,2) = se_1 + e_2, G(s,2)_i = 0 \text{ for } i \geq 3$$

Note that  $R(\phi_2(s))_\beta$  is nonsingular for all  $s$  for the nice selection

$$(2.7.10) \quad \beta = \{(0,1), \dots, (n-2,1), (0,2)\}$$

The pairs of matrices  $\phi_1(t)$  and  $\phi_2(s)$  are equivalent pairs if  $t \neq 0$ ,  $s \neq 0$  and  $ts = 1$ . The matrix transforming the pair  $\phi_1(t)$  into  $\phi_2(s)$  is then equal to

$$\begin{pmatrix} t^{-1} & 0 & & 0 \\ 0 & 1 & & \\ \hline & & & \\ 0 & & & I_{n-2} \end{pmatrix} = \begin{pmatrix} s & 0 & & 0 \\ 0 & 1 & & \\ \hline & & & \\ 0 & & & I_{n-2} \end{pmatrix}$$

This means that the composed maps

$$\pi\phi_1 : \mathbb{R} \rightarrow \widehat{\mathbb{C}}(\mathbb{R})_{\text{cr}} \rightarrow \widehat{\mathbb{M}}_{m,n}(\mathbb{R})$$

and

$$\pi\phi_2 : \mathbb{R} \rightarrow \widehat{\mathbb{C}}(\mathbb{R})_{\text{cr}} \rightarrow \widehat{\mathbb{M}}_{m,n}(\mathbb{R})$$

combine to define a continuous map

$$(2.7.11) \quad \phi : S^1 = \mathbb{P}^1(\mathbb{R}) \rightarrow \widehat{\mathbb{M}}_{m,n}(\mathbb{R})$$

Let  $(t:s)$  be homogeneous coordinates for  $\mathbb{P}^1(\mathbb{R})$ . Then

$$(2.7.12) \quad \begin{aligned} \phi(t:s) &\in U_\alpha \quad \text{if } s \neq 0 \\ \phi(t:s) &\in U_\beta \quad \text{if } t \neq 0 \end{aligned}$$

where  $\alpha$  and  $\beta$  are the nice selections given by (2.7.5) and (2.7.10) above. It remains to construct an embedding  $\mathbb{P}^1(\mathbb{R}) \rightarrow \widehat{\mathbb{M}}_{m,n}(\mathbb{R})$  in the case  $n = 1$ . This is done as follows. We define

$$\phi_1 : \mathbb{R} \rightarrow \widehat{\mathbb{C}}(\mathbb{R})_{\text{cr}}, \quad t \rightarrow (F(t,1), G(t,1))$$

where  $G(t,1)_1 = t$ ,  $G(t,1)_2 = 1$ ,  $G(t,1)_i = 0$   $i \geq 3$  and  $F(t,1) = 0$ , and

$$\phi_2 : \mathbb{R} \rightarrow \widehat{\mathbb{C}}(\mathbb{R})_{\text{cr}}, \quad s \rightarrow (F(s,2), G(s,2))$$

where  $G(s,2)_1 = 1$ ,  $G(s,2)_2 = s$ ,  $G(s,2)_i = 0$ ,  $i \geq 3$  and  $F(s,2) = 0$

As above these two applications combine to define a continuous map

$$\phi : \mathbb{P}^1(\mathbb{R}) \rightarrow \mathbb{M}_{m,1}(\mathbb{R})$$

### 2.8. Proposition.

The underlying vector bundle of the universal family  $\Sigma^u$  over  $\mathbb{M}_{m,n}(\mathbb{R})$  is nontrivial iff  $m \geq 2$ .

Proof. The only if part is trivial as there is only one nice selection if  $m = 1$ . There are several ways to prove the if part. One is by algebraic geometry as follows:  $\mathbb{M}_{m,n}(\mathbb{C})$  embeds naturally into the Grassmann variety of complex  $n$ -planes in complex  $(n+1)m$  space which in turn is a closed subvariety of projective space of (complex) dimension  $N$  with  $N + 1$  equal to the binomial coefficient  $\binom{(n+1)m}{n}$ . Cf. [2] for details. The underlying bundle  $E^u$  of  $\Sigma^u$  is the restriction to  $\mathbb{M}_{m,n}(\mathbb{C})$  of the canonical bundle over the Grassmann variety. The  $n$ -th exterior product of this bundle is the restriction of the canonical line bundle  $\xi_1$  over  $\mathbb{P}^N(\mathbb{C})$  which is very ample. Now the map  $\phi$  defined above is defined by polynomials and defines an algebraic geometric embedding  $\mathbb{P}^1(\mathbb{C}) \xrightarrow{\phi} \mathbb{M}_{m,n}(\mathbb{C}) \xrightarrow{i} \text{Grassmann} \xrightarrow{j} \mathbb{P}^N(\mathbb{C})$ . It follows that  $(j \circ i \circ \phi)^* \xi_1$  is very ample and its real restriction to  $\mathbb{P}^1(\mathbb{R})$  is then also nontrivial. I.e. the  $n$ -th exterior product of  $\phi^* E^u$  is nontrivial which proves that  $E^u$  is nontrivial.

Alternatively one simply calculates the bundle  $\wedge^n \phi^* E^u$  explicitly. This line bundle over  $\mathbb{P}^1(\mathbb{R})$  is trivial over the pieces  $\{(t:s) | s \neq 0\} \subset \mathbb{P}^1(\mathbb{R})$  and  $\{(t,s) | t \neq 0\} \subset \mathbb{P}^1(\mathbb{R})$  by (2.7.12). And if  $n \geq 2$  these trivial pieces are identified on the intersection  $\{(t,s) | t \neq 0, s \neq 0\}$  by means of multiplication with the number

$$\det \begin{pmatrix} t^{-1}s & 0 & & 0 \\ & 0 & 1 & \\ \hline & 0 & & I_{n-2} \end{pmatrix} = t^{-1}s$$

Similarly if  $n = 1$  these pieces are also identified by multiplication with the number  $t^{-1}s$ .

This defines a nontrivial bundle over  $\mathbb{P}^1(\mathbb{R})$ , which proves that the bundle  $E^u$  was also nontrivial.

### 3. THE GRAMM-SCHMIDT ORTHONORMALIZATION PROCESS AND CANONICAL FORMS

In this section we discuss the equivalence given by the Gramm-Schmidt orthonormalization process between the existence of  $GL_n(\mathbb{R})$  canonical forms for all pairs and triples of matrices and the existence of  $O_n(\mathbb{R})$  canonical forms for orthonormal pairs and triples of matrices.

#### 3.1. The Spaces $(FG(\mathbb{R}))_{cr}^{ortho}$ , $(FGH(\mathbb{R}))_{cr,co}$ and $(FGH(\mathbb{R}))_{cr,co}^{ortho}$

We define  $(FG(\mathbb{R}))_{cr}^{ortho}$  as the space of all pairs of matrices  $(F,G)$  such that the rows of  $R(F,G)$  are a set of orthonormal vectors (in  $\mathbb{R}^{(n+1)m}$ ).

Note that  $(FG(\mathbb{R}))_{cr}^{ortho} \subset (FG(\mathbb{R}))_{cr}$ .

We define  $(FGH(\mathbb{R}))_{cr,co}$  as the space of all triples of real matrices  $F,G,H$  of sizes  $n \times n$ ,  $n \times m$ ,  $p \times n$  and  $(FGH(\mathbb{R}))_{cr,co}^{ortho}$  is the subspace of all completely observable and completely reachable triples. I.e.

$(F,G,H) \in (FGH(\mathbb{R}))_{cr,co}^{ortho}$  iff the matrices  $R(F,G) = (G \ FG \ \dots \ F^n G)$  and  $Q(F,H) = (H^T \ F^T H^T \ \dots \ (F^T)^n H^T)$  are both of rank  $n$ . Here  $H^T, F^T$  are the transposes of  $H, F$ . Cf. [6] for more details about these notions.

Finally we define  $(FGH(\mathbb{R}))_{cr,co}^{ortho}$  as the subspace of  $(FGH(\mathbb{R}))_{cr,co}$  consisting of the triples of matrices in  $(FGH(\mathbb{R}))_{cr,co}$  such that moreover the rows of  $R(F,G)$  are orthonormal.

#### 3.2. Lemma.

Let  $A, B$  be two  $n \times r$  matrices of rank  $n$ , where  $r \geq n$ . Then we have

- (i) If the rows of  $A$  are orthonormal and  $U \in O_n(\mathbb{R})$  is an orthogonal  $n \times n$  matrix, then the rows of  $UA$  are orthonormal
- (ii) If the rows of  $A$  and the rows of  $B$  are both orthonormal and if the rows of  $A$  and the rows of  $B$  span the same subspace of  $\mathbb{R}^r$ , then there is an orthonormal  $n \times n$  matrix  $U \in O_n(\mathbb{R})$  such that  $B = UA$ .

Proof. Easy.

#### 3.3. Canonical Forms.

The group  $GL_n(\mathbb{R})$  acts on  $(FG(\mathbb{R}))_{cr}$  and  $(FGH(\mathbb{R}))_{cr,co}$  respectively as follows

$$(3.3.1) \quad (F,G)^S = (SFS^{-1}, SG), \quad (F,G,H)^S = (SFS^{-1}, SG, HS^{-1}), \quad S \in GL_n(\mathbb{R})$$

Two pairs (resp. triples) of matrices  $(F,G)$  and  $(F',G')$  (resp.  $(F,G,H)$

and  $(F', G', H')$  are equivalent under  $GL_n(\mathbb{R})$  if there is an  $S \in GL_n(\mathbb{R})$  such that  $(F, G)^S = (F', G')$  (resp.  $(F^S, G, H)^S = (F', G', H')$ ). We now define a canonical form for the action of  $GL_n(\mathbb{R})$  on  $FG(\mathbb{R})_{cr}$  as a continuous map

$$(3.3.2) \quad \gamma : \mathbb{C}\mathbb{G}(\mathbb{R})_{cr} \rightarrow \mathbb{C}\mathbb{G}(\mathbb{R})$$

such that for every two pairs  $(F, G), (F', G') \in \mathbb{C}\mathbb{G}(\mathbb{R})_{cr}$  we have

$$(3.3.3) \quad (F, G) \text{ is equivalent under } GL_n(\mathbb{R}) \text{ to } \gamma(F, G)$$

and

$$(3.3.4) \quad (F, G), (F', G') \text{ are equivalent under } GL_n(\mathbb{R}) \text{ iff} \\ \gamma(F, G) = \gamma(F', G')$$

A canonical form on  $\mathbb{C}\mathbb{G}\mathbb{H}(\mathbb{R})_{cr, co}$  under  $GL_n(\mathbb{R})$  is defined similarly. The group  $O_n(\mathbb{R})$  of real orthogonal  $n \times n$  matrices acts on

$\mathbb{C}\mathbb{G}(\mathbb{R})_{cr}^{ortho}$  and  $\mathbb{C}\mathbb{G}\mathbb{H}(\mathbb{R})_{cr, co}^{ortho}$  as follows

$$(3.3.5) \quad (F, G)^U = (UFU^{-1}, UG), (F, G, H)^U = (UFU^{-1}, UG, HU^{-1})$$

This follows from lemma 3.2 (i). We now define a canonical form on  $\mathbb{C}\mathbb{G}\mathbb{H}(\mathbb{R})_{cr, co}^{ortho}$  under  $O_n(\mathbb{R})$  as a continuous map

$$(3.3.6) \quad \gamma : \mathbb{C}\mathbb{G}\mathbb{H}(\mathbb{R})_{cr, co}^{ortho} \rightarrow \mathbb{C}\mathbb{G}\mathbb{H}(\mathbb{R})$$

such that for every two triples  $(F, G, H), (F', G', H')$  we have

$$(3.3.7) \quad \gamma(F, G, H) \text{ is equivalent under } O_n(\mathbb{R}) \text{ to } (F, G, H)$$

and

$$(3.3.8) \quad (F, G, H) \text{ and } (F', G', H') \text{ are equivalent under } O_n(\mathbb{R}) \text{ iff} \\ \gamma(F, G, H) = \gamma(F', G', H')$$

A canonical form on  $\mathbb{C}\mathbb{G}(\mathbb{R})_{cr}^{ortho}$  under  $O_n(\mathbb{R})$  is defined similarly.

### 3.4. Gramm-Schmidt Orthonormalization.

Let  $(F,G) \in \mathbb{F}\mathbb{C}(\mathbb{R})_{cr}$ . The matrix  $R(F,G)$  is then of rank  $n$ . Applying the Gramm-Schmidt Orthonormalization to the rows of  $R(F,G)$  we find an  $n \times (n+1)m$  matrix  $R'$  with orthonormal rows, whose rows span the same subspace of  $\mathbb{R}^{(n+1)m}$  as the rows of  $R(F,G)$ . It follows that  $R' = SR(F,G)$  for a certain unique  $S \in GL_n(\mathbb{R})$ . It follows that  $R' = R(SFS^{-1}, SG)$ . Orthonormalization is also continuous. It follows that orthonormalization defines continuous (well defined) maps

$$(3.4.1) \quad \mu : \mathbb{F}\mathbb{C}(\mathbb{R})_{cr} \rightarrow \mathbb{F}\mathbb{C}(\mathbb{R})_{cr}^{ortho}$$

$$(3.4.2) \quad \nu : \mathbb{F}\mathbb{C}\mathbb{H}(\mathbb{R})_{cr,co} \rightarrow \mathbb{F}\mathbb{C}\mathbb{H}(\mathbb{R})_{cr,co}^{ortho}$$

Note that  $\mu(F,G)$  and  $(F,G)$ , and  $\nu(F,G,H)$  and  $(F,G,H)$ , are equivalent under  $GL_n(\mathbb{R})$ . The maps  $\mu, \nu$  take  $GL_n(\mathbb{R})$  equivalent elements into  $O_n(\mathbb{R})$  equivalent elements. More precisely we have

### 3.5. Lemma.

Two pairs  $(F,G), (F',G')$  in  $\mathbb{F}\mathbb{C}(\mathbb{R})_{cr}$  (resp. two triples  $(F,G,H), (F',G',H')$ ) in  $\mathbb{F}\mathbb{C}\mathbb{H}(\mathbb{R})_{cr,co}$  are equivalent under  $GL_n(\mathbb{R})$  iff the pairs  $\mu(F,G), \mu(F',G')$  (resp. the triples  $\nu(F,G,H), \nu(F',G',H')$ ) are equivalent under  $O_n(\mathbb{R})$ .

Proof. This follows from lemma 3.2

### 3.6. Proposition.

(i) There exists a canonical form on  $\mathbb{F}\mathbb{C}(\mathbb{R})_{cr}$  under  $GL_n(\mathbb{R})$  iff there exists a canonical form on  $\mathbb{F}\mathbb{C}(\mathbb{R})_{cr}^{ortho}$  under  $O_n(\mathbb{R})$

(ii) There exists a canonical form on  $\mathbb{F}\mathbb{C}\mathbb{H}(\mathbb{R})_{cr,co}$  under  $GL_n(\mathbb{R})$  iff there exists a canonical form on  $\mathbb{F}\mathbb{C}\mathbb{H}(\mathbb{R})_{cr,co}^{ortho}$  under  $O_n(\mathbb{R})$ .

Proof. Let  $\gamma : \mathbb{F}\mathbb{C}(\mathbb{R})_{cr} \rightarrow \mathbb{F}\mathbb{C}(\mathbb{R})_{cr}$  be a  $GL_n(\mathbb{R})$  canonical form.

Then

$$\mathbb{F}\mathbb{C}(\mathbb{R})_{cr}^{ortho} \xrightarrow{i} \mathbb{F}\mathbb{C}(\mathbb{R})_{cr} \xrightarrow{\gamma} \mathbb{F}\mathbb{C}(\mathbb{R})_{cr} \xrightarrow{\mu} \mathbb{F}\mathbb{C}(\mathbb{R})_{cr}^{ortho}$$

where  $i$  is the inclusion, is an  $O_n(\mathbb{R})$  canonical form on  $\mathbb{F}\mathbb{C}(\mathbb{R})_{cr}^{ortho}$ .

(Note that two elements of  $\mathbb{F}\mathbb{C}(\mathbb{R})_{cr}^{ortho}$  are  $O_n(\mathbb{R})$  equivalent iff they are  $GL_n(\mathbb{R})$  equivalent; this follows from lemma 3.2). Inversely if  $\gamma : \mathbb{F}\mathbb{C}(\mathbb{R})_{cr}^{ortho} \rightarrow \mathbb{F}\mathbb{C}(\mathbb{R})$  is an  $O_n(\mathbb{R})$  canonical form then  $\gamma \circ \mu$  is a  $GL_n(\mathbb{R})$  canonical form on  $\mathbb{F}\mathbb{C}(\mathbb{R})_{cr}$ . Part (ii) of the lemma is proved in the same way.

#### 4. ON THE NONEXISTENCE OF CANONICAL FORMS

We have now enough material to prove the nonexistence of  $GL_n(\mathbb{R})$  canonical forms on  $\mathbb{F}\mathbb{C}(\mathbb{R})_{cr,co}$  for those dimensions  $(m,n,p)$  for which  $p \geq 2n$ ,  $m \geq 2$ . The first step is the following theorem.

##### 4.1. Theorem.

There exists a  $GL_n(\mathbb{R})$  canonical form on  $\mathbb{F}\mathbb{C}(\mathbb{R})_{cr}$  iff the underlying bundle  $E^u$  of the universal family  $\Sigma^u$  is trivial

Proof. This is proved exactly as the algebraic geometric case in [3], 6.1.

##### 4.2. Corollary.

There does not exist a  $GL_n(\mathbb{R})$  canonical form on  $\mathbb{F}\mathbb{C}(\mathbb{R})_{cr}$  if  $m \geq 2$ .

his follows from 4.1 together with 2.8.

##### 4.3. An $O_n(\mathbb{R})$ - invariant embedding.

For each  $(F,G) \in \mathbb{F}\mathbb{C}(\mathbb{R})_{cr}$  let  $\bar{R}(F,G)$  be the matrix

$$\bar{R}(F,G) = (G \ FG \ \dots \ F^{n-1}G)$$

For all  $m,n$  we can now define an  $O_n(\mathbb{R})$  invariant embedding

$$(4.3.1) \quad \rho : \mathbb{F}\mathbb{C}(\mathbb{R})_{cr}^{ortho} \rightarrow \mathbb{F}\mathbb{C}(\mathbb{R})_{cr,co}^{ortho}$$

as follows

$$(4.3.2) \quad (F,G) \rightarrow (F,G,\bar{R}(F,G)^T)$$

This is  $O_n(\mathbb{R})$  invariant because  $U^T = U^{-1}$  for  $U \in O_n(\mathbb{R})$  and  $\bar{R}((F,G)^U) = U\bar{R}(F,G)$ . The triple  $(F,G,\bar{R}(F,G)^T)$  is completely observable because  $\bar{R}(F,G)^T$  has rank  $n$ .

4.4. Theorem.

There does not exist a continuous canonical form under  $GL_n(\mathbb{R})$  for completely reachable and completely observable linear dynamical systems of dimension  $n$  with  $m$  inputs and  $p$  outputs in the cases

- (i)  $m \geq 2, p \geq 2n$
- (ii)  $p \geq 2, m \geq 2n$
- (iii)  $p, m > n$

Proof (i). We have  $O_n(\mathbb{R})$  invariant embeddings

$$(4.4.1) \quad \mathbb{C}G_{2,n}(\mathbb{R})_{\text{cr}}^{\text{ortho}} \rightarrow \mathbb{C}G_{2,n,2n}(\mathbb{R})_{\text{cr,co}}^{\text{ortho}} \rightarrow \mathbb{C}G_{m,n,p}(\mathbb{R})_{\text{cr,co}}^{\text{ortho}}$$

where the first embedding is the one defined in 4.3 above and the second one consists of adding some zero columns to  $G$  (if  $m > 2$ ) and some zero rows to  $H$  (if  $p > 2n$ ). Now suppose there existed a  $GL_n(\mathbb{R})$  canonical form for  $\mathbb{C}G_{m,n,p}(\mathbb{R})_{\text{cr,co}}^{\text{ortho}}$  then there would be an  $O_n(\mathbb{R})$

canonical form on  $\mathbb{C}G_{m,n,p}(\mathbb{R})_{\text{cr,co}}^{\text{ortho}}$  by 3.6 and by the  $O_n(\mathbb{R})$  invariant inclusions (4.4.1) above an  $O_n(\mathbb{R})$  canonical form on  $\mathbb{C}G_{2,n}(\mathbb{R})_{\text{cr}}^{\text{ortho}}$

which in turn would imply the existence of an  $GL_n(\mathbb{R})$  canonical form on  $\mathbb{C}G_{2,n}(\mathbb{R})_{\text{cr}}^{\text{ortho}}$  (again by 3.6), which contradicts 4.2.

Part (ii) of the theorem is proved by dualizing this whole paper. I.e. instead of completely reachable pairs  $(F,G)$  one studies completely observable pairs  $(F,H)$  etc. etc.

Part (iii) of the theorem uses: 1<sup>o</sup>) the nonexistence of a  $GL_n(\mathbb{R})$  canonical form on  $\mathbb{A}_{n,m}$ , the space of all  $n \times m$  matrices of rank  $n$  under the action  $A^S = SA$ , if  $m > n$ , and 2<sup>o</sup>) the  $O_n(\mathbb{R})$ -invariant embedding

$$\mathbb{A}_{n,m}^{\text{ortho}} \rightarrow \mathbb{C}G(\mathbb{R})_{\text{cr,co}}^{\text{ortho}}$$

$$A \mapsto (0, A, A^T)$$

4.5. As was already stated in the introduction theorem 4,4 holds in greater generality: there exists a continuous  $GL_n(\mathbb{R})$  canonical form for completely observable and completely reachable linear dynamical systems of dimension  $n$  with  $m$  inputs and  $p$  outputs if and only if  $m = 1$  or  $p = 1$ . Cf. [4].



Of course the nonexistence of a  $GL_n(\mathbb{R})$  canonical form on  $\mathbb{R}^{m,n,p}_{cr,co}$  implies a fortiori the nonexistence of such a form on the larger spaces  $\mathbb{R}^{n,m,p}_{(R)}$ ,  $\mathbb{R}^{(R)}_{cr}$  and  $\mathbb{R}^{(R)}_{co}$ .

## REFERENCES.

1. A. Grothendieck, J. Dieudonné. *Eléments de la Géométrie Algébrique* Ch. II Publ. Math. J.H.E.S. 8, 1961.
2. M. Hazewinkel, R.E. Kalman. *Moduli and Canonical Forms for Linear Dynamical Systems*. Report 7504, Econometric Inst., Erasmus Univ. Rotterdam, 1975.
3. M. Hazewinkel, R.E. Kalman. *On Invariants, Canonical Forms and Moduli for Linear, Constant, Finite Dimensional Dynamical Systems*. Proc. CNR-CISM Conference on Algebraic System Theory, Udine, 1975. To appear in *Lecture Notes in Math.*, Springer.
4. M. Hazewinkel. In preparation.
5. D. Husemoller. *Theory of Fibre Bundles*, McGraw Hill, 1966.
6. R.E. Kalman. *Controllability and Observability*. In; CIME Symposium 1968 on Controllability and Observability, Edizione Cremonese, Roma, 1969.