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SOME OF THE GENERATORS OF THE COMPLEX COBORDISM RING.

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In [1] we wrote down a recursion formula for a set of generators of the complex cobordismring. In this little note we write down some of these formulae in full by way of a trivial addendum (and corrigendum) to [1].

1. The general formula.

Let P_i be the cobordism class of $\mathbf{CP}^n$. Write $q_i = P_{i-1}$, $q_1 = 1$. The complex cobordismring is then isomorphic to $\mathbf{Z}[t_2,t_3,\ldots]$ (note that we start with index 2 !), where t_i is given by the recursion formula:

$$(1.1)\quad t_s = \mu(s)q_s - \overset{(1)}{\Sigma}\, \nu(s,d_1)\frac{q_d}{d}\, t_{d_1}^{d} + \ldots + (-1)^i \overset{(i)}{\Sigma}\, \nu(s,d_1)\frac{q_d}{d}\, t_{d_i}^{d}\, t_{d_{i-1}}^{dd_i} \ldots \ldots t_{d_1}^{dd_i\ldots d_2} + \ldots$$

Here $\overset{(1)}{\Sigma}$ is the sum over all pairs (d,d_1) such that $dd_1 = s$, $d_1 \neq 1,s; d,d_1 \in \mathbb{N}$ and $\overset{(i)}{\Sigma}$ is the sum over all sequences $(d,d_i,d_{i-1},\ldots,d_1)$ such that $d, d_i, \ldots, d_1 \in \mathbb{N}$, $dd_i\ldots d_1 = s$; $d_1 \neq 1, s$; d_j a composite number for all $j = 2, \ldots, i$. (i.e. there are at least two different prime numbers dividing d_j). (Note that there may be contributions with $d = 1$ in $\overset{(i)}{\Sigma}$ if $i \geq 2$).

The integers $\mu(s)$ and $\nu(s,d)$ which occur in (1.1) can be obtained as follows. For every pair of prime numbers p,p' let $c(p,p')$ be an integer such that

$$(1.2)\quad c(p,p) = 1 \quad \text{and} \quad c(p,p') \equiv 1 \bmod p,\ c(p,p') \equiv 0 \bmod p' \text{ if } p \neq p'$$

We now define for d dividing s, $d \neq s$:

$$m(s,d) = 1 \text{ if } d \text{ is composite or } d = 1$$

$$(1.3)\quad m(s,p^r) = \prod_{p' \in J_s} c(p',p), \text{ if } p \text{ prime and } r > 0, \text{ where}$$

$$J_s = \{\ell \mid \ell \text{ is prime number and } \ell \mid s\}$$

$$(1.4)\quad \begin{aligned} &\mu(d) = 1 \text{ if } d \text{ composite or } d = 1 \\ &\mu(p^r) = p \text{ if } p \text{ is prime (and } r > 0) \end{aligned}$$

$$(1.5) \qquad \nu(s,d) = \frac{\mu(s)}{\mu(d)}\, m(s,d)$$

2. TWO TABLES

A possible choice for the numbers $c(p,p')$ for small p is given by the following table.

	2	3	5	7	11	13	17	19	23	29	31	37	41	43	47
2	1	3	5	7	11	13	17	19	23	29	31	37	41	43	47
3	4	1	10	7	22	13	34	19	46	58	31				
5	6	6	1	21	11	26	51	76							
7	8	15	15	1	22	78									
11	12	12	45	56	1										
13	14	27	40	14											
17	18	18	35												
19	20	39	20												
23	24	24													
29	30	30													
31	32														
37	38														
41	42														
43	44														
47	48														

(2.1) Table for $c(p,p')$, p vertical, p' horizontal

This table together with $c(p,2) = p+1$ and $c(2,p) = p$ if $p \neq 2$ gives possible values for $c(p,p')$ for all p,p' for which $pp' \leq 100$.

The corresponding values for $\nu(s,d)$ are given by the following table (2.2) if one takes into account that $\nu(s,p^r) = \nu(s,p)$ and $\nu(s,d) = 1$ if d is composite or 1. All the unlisted $\nu(s,p)$, p prime, $s \leq 50$, $s = 60, 72, 96, 216$ are equal to 1.

$\nu(6,2) = 2$	$\nu(24,2) = 2$	$\nu(35,7) = 3$	$\nu(45,5) = 2$
$\nu(10,2) = 3$	$\nu(26,2) = 7$	$\nu(36,2) = 2$	$\nu(46,2) = 12$
$\nu(12,2) = 2$	$\nu(28,2) = 4$	$\nu(38,2) = 10$	$\nu(48,2) = 2$
$\nu(14,2) = 4$	$\nu(30,2) = 12$	$\nu(39,3) = 9$	$\nu(50,2) = 3$
$\nu(15,3) = 2$	$\nu(30,3) = 6$	$\nu(40,2) = 3$	$\nu(60,2) = 12$
$\nu(15,5) = 2$	$\nu(30,5) = 10$	$\nu(42,2) = 16$	$\nu(60,3) = 6$
$\nu(19,2) = 2$	$\nu(33,3) = 1$	$\nu(42,3) = 15$	$\nu(60,5) = 10$
$\nu(20,2) = 3$	$\nu(33,11) = 2$	$\nu(42,7) = 7$	$\nu(72,2) = 2$
$\nu(21,3) = 5$	$\nu(34,2) = 9$	$\nu(44,2) = 6$	$\nu(96,2) = 2$
$\nu(22,2) = 6$	$\nu(35,5) = 3$	$\nu(45,3) = 2$	$\nu(216,2) = 2$

(2.2) Table for $\nu(s,d)$

3. SOME OF THE RECURSION FORMULAE EXPLICITLY

If $s = p^r$, p prime, $r \geq 1$ then

$$(3.1)\qquad t_{p^r} = \frac{q_{p^r}}{p^{r-1}} - \frac{q_{p^{r-1}}}{p^{r-1}} \cdot t_p^{p^{r-1}} - \frac{q_{p^{r-2}}}{p^{r-2}} t_{p^2}^{p^{r-2}} - \dots - \frac{q_p}{p} t_{p^{r-1}}^{p}$$

E.g.

$$(3.2)\qquad t_{32} = \frac{q_{32}}{16} - \frac{q_{16}}{16} \cdot t_2^{16} - \frac{q_8}{8} \cdot t_4^{8} - \frac{q_4}{4} t_8^{4} - \frac{q_2}{2} t_{16}^{2}$$

If $s = pp'$, p and p' prime and $p \neq p'$, then

$$(3.3)\qquad t_{pp'} = \frac{q_{pp'}}{pp'} - \nu(pp',p) \frac{q_{p'}}{p'} t_p^{p'} - \nu(pp',p') \frac{q_p}{p} t_{p'}^{p}$$

E.g.

$$(3.4)\qquad t_{14} = \frac{q_{14}}{14} - \frac{4q_7}{7} t_2^7 - \frac{q_2}{2} t_7^2, \quad t_{35} = \frac{q_{35}}{35} - \frac{3q_7}{7} t_5^7 - \frac{3q_5}{5} t_7^5$$

All the other t_s also involve terms from $\overset{(i)}{\Sigma}$ with $i \geq 2$. The first s for which there is a contribution in t_s from $\overset{(2)}{\Sigma}$ is $s = 12$; the first s for which there is a contribution in t_s from $\overset{(3)}{\Sigma}$ is $s = 72$; and the first s for which there is a contribution in t_s from $\overset{(4)}{\Sigma}$ is $s = 436$.

The simplest s for which t_s has a contribution from $\overset{(2)}{\Sigma}$ are of the form $s = p^2p'$ where p, p' are two different prime numbers. For these t_s one has the formula

$$t_{p^2p'} = \frac{q_{p^2p'}}{p^2p'} - \nu(p^2p',p')\,\frac{q_{p^2}}{p^2}\,t^{p^2}_{p'} - \nu(p^2p',p)\frac{q_{pp'}}{pp'}\,t^{pp'}_{p}$$

$$- \nu(p^2p'.p^2)\frac{q_{p'}}{p'}\,t^{p'}_{p^2}$$

(3.5)
$$-\nu(p^2p',pp')\,\frac{q_p}{p}\,t^{p}_{pp'} + \nu(p^2p',p)t_{pp'}t^{pp'}_{p}$$

$$\begin{bmatrix} d \\ d_1 \end{bmatrix} = \begin{bmatrix} p^2 & pp' & p' & p \\ p' & p & p^2 & pp' \end{bmatrix}, \qquad \begin{bmatrix} d \\ d_2 \\ d_1 \end{bmatrix} = \begin{bmatrix} 1 \\ pp' \\ p \end{bmatrix}$$

E.g.

$$t_{12} = \frac{q_{12}}{12} - \frac{2q_6}{6}\,t^6_2 - \frac{q_4}{4}\,t^4_3 - \frac{2q_3}{3}\,t^3_4 - \frac{q_2}{2}\,t^2_6 + 2t_6t^6_2$$

$$t_{18} = \frac{q_{18}}{18} - \frac{2q_9}{9}\,t^9_2 - \frac{q_6}{6}\,t^6_3 - \frac{q_3}{3}\,t^3_6 - \frac{q_2}{2}\,t^2_9 + t_6t^6_3$$

In the table below are the recursion formulae for the t_s with $s \leq 39$ and not of the type covered by (3.1), (3.3) and (3.5) and the recursion formulae for t_{48}, t_{60}, t_{72}, t_{216}. Below each formula are listed the "vectors" $(d, d_i, \ldots, d_2, d_1)$ which "contribute to t_s".

The $s \leqslant 100$ which are not of the type covered by (3.1), (3.3), (3.5) are listed below in groups of the same type.

24, 40, 54, 56, 88	type: p^3p'
30, 42, 66, 70, 78	type: $p''p'p$
36, 100	type: $p^2p'^2$
48,80	type: p^4p'
60, 84, 90	type: $p^2p'p''$
72	type: $p^3p'^2$
96	type: p^5p'

Below are the recursions formulae for t_s for s = 24, 30, 36, 48, 60, 72, 96

$$t_{24} = \frac{q_{24}}{24} - \frac{2q_{12}}{12} t_2^{12} - \frac{q_8}{8} t_3^8 - \frac{2q_6}{6} t_4^6 - \frac{q_4}{4} t_6^4 - \frac{2q_3}{3} t_8^3 - \frac{q_2}{2} t_{12}^2$$

$$+ \frac{2q_2}{2} t_6^2 t_2^{12} + 2t_{12}t_2^{12} + 2t_6 t_4^6$$

d	12	8	6	4	3	2
d_1	2	3	4	6	8	12

d	2	1	1
d_2	6	12	6
d_1	2	2	4

$$t_{30} = \frac{q_{30}}{30} - \frac{12q_{15}}{15} t_2^{15} - \frac{6q_{10}}{10} t_3^{10} - \frac{10q_6}{10} t_5^6 - \frac{q_5}{5} t_6^5 - \frac{q_3}{3} t_{10}^3 - \frac{q_2}{2} t_{15}^2$$

$$+ 12t_{15}t_2^{15} + 6t_{10}t_3^{10} + 10t_6 t_5^6$$

d	15	10	6	5	3	2
d_1	2	3	5	6	10	15

d	1	1	1
d_2	15	10	6
d_1	2	3	5

$$t_{36} = \frac{q_{36}}{36} - \frac{2q_{18}}{18} t_2^{18} - \frac{q_{12}}{12} t_3^{12} - \frac{2q_9}{9} t_4^9 - \frac{q_6}{6} t_6^6 - \frac{q_4}{4} t_9^4 - \frac{q_3}{3} t_{12}^3 - \frac{q_2}{2} t_{18}^2$$

$$+ \frac{2q_3}{3} t_6^3 t_2^{18} + \frac{q_2}{2} t_6^2 t_3^{12} + 2t_{18}t_2^{18} + t_{12}t_3^{12} + t_6 t_6^6$$

d	18	12	9	6	4	3	2
d_1	2	3	4	6	9	12	18

d	3	2	1	1	1
d_2	6	6	18	12	6
d_1	2	3	2	3	6

$$t_{48} = \frac{q_{48}}{48} - \frac{2q_{24}}{24} t_2^{24} - \frac{q_{16}}{16} t_3^{18} - \frac{2q_{12}}{12} t_4^{12} - \frac{q_8}{8} t_6^8 - \frac{2q_6}{6} t_8^6 - \frac{q_4}{4} t_{12}^4$$

$$- \frac{2q_3}{3} t_{16}^3 - \frac{q_2}{2} t_{24}^2 + \frac{2q_4}{4} t_6^4 t_2^{24} + \frac{2q_2}{2} t_{12}^2 t_2^{24}$$

$$+ \frac{2q_2}{2} t_6^2 t_4^{12} + 2t_{24}t_2^{24} + 2t_{12}t_4^{12} + 2t_6 t_8^6$$

d	24	16	12	8	6	4	3	2
d_1	2	3	4	6	8	12	16	24

d	4	2	2	1	1	1
d_2	6	12	6	24	12	6
d_1	2	2	4	2	4	8

$$t_{60} = \frac{q_{60}}{60} - \frac{12q_{30}}{30} t_2^{30} - \frac{6q_{20}}{20} t_3^{20} - \frac{12q_{15}}{15} t_4^{15} - \frac{10q_{12}}{12} t_5^{12} - \frac{q_{10}}{10} t_6^{10} - \frac{q_6}{6} t_{10}^6$$

$$- \frac{q_5}{5} t_{12}^5 - \frac{q_4}{4} t_{15}^4 - \frac{q_3}{3} t_{20}^3 - \frac{q_2}{2} t_{30}^2 + \frac{12q_5}{5} t_6^5 t_2^{30} + \frac{6q_2}{2} t_{10}^2 t_3^{20}$$

$$+ \frac{10q_2}{2} t_6^2 t_5^{12} + 12t_{30}t_2^{30} + 6t_{20}t_3^{20} + 12t_{15}t_4^{15} + 10t_{12}t_5^{12} + t_{10}t_6^{10} + t_6t_{10}^6$$

d	30	20	15	12	10	6	5	4	3	2
d_1	2	3	4	5	6	10	12	15	20	30

d	5	2	2	1	1	1	1	1	1
d_2	6	10	6	30	20	15	12	10	6
d_1	2	3	5	2	3	4	5	6	10

$$t_{72} = \frac{q_{72}}{72} - \frac{2q_{36}}{36} t_2^{36} - \frac{q_{24}}{24} t_3^{24} - \frac{2q_{18}}{18} t_4^{18} - \frac{q_{12}}{12} t_6^{12} - \frac{2q_9}{9} t_8^9 - \frac{q_8}{8} t_9^8$$

$$- \frac{q_6}{6} t_{12}^6 - \frac{q_4}{4} t_{18}^4 - \frac{q_3}{3} t_{24}^3 - \frac{q_2}{2} t_{36}^2 + \frac{2q_6}{6} t_6^6 t_2^{36} + \frac{q_4}{4} t_6^4 t_3^{24}$$

$$+ \frac{2q_3}{3} t_{12}^3 t_2^{36} + \frac{2q_3}{3} t_6^3 t_4^{18} + \frac{2q_2}{2} t_{18}^2 t_2^{36} + \frac{q_2}{2} t_{12}^2 t_3^{24} + \frac{q_2}{2} t_6^2 t_6^{12} + 2t_{36}t_2^{36}$$

$$+ t_{24}t_3^{24} + 2t_{18}t_4^{18} + t_{12}t_6^{12} + t_6t_{12}^6 - 2t_6t_6^6t_2^{36}$$

d	36	24	18	12	9	8	6	4	3	2
d_1	2	3	4	6	8	9	12	18	24	36

d	6	4	3	3	2	2	2	1	1	1	1	1
d_2	6	6	12	6	18	12	6	36	24	18	12	6
d_1	2	3	2	4	2	3	6	2	3	4	6	12

d	1
d_3	6
d_2	6
d_1	2

$$t_{96} = \frac{q_{96}}{96} - \frac{2q_{48}}{48} t_2^{48} - \frac{q_{32}}{32} t_3^{32} - \frac{2q_{24}}{24} t_4^{24} - \frac{q_{16}}{16} t_6^{16} - \frac{2q_{12}}{12} t_8^{12} - \frac{q_8}{8} t_{12}^{8}$$

$$- \frac{2q_6}{6} t_{16}^{6} - \frac{q_4}{4} t_{24}^{4} - \frac{2q_3}{3} t_{32}^{3} - \frac{q_2}{2} t_{48}^{2} + \frac{2q_8}{8} t_6^8 t_2^{48} + \frac{2q_4}{4} t_{12}^4 t_2^{48}$$

$$+ \frac{2q_4}{4} t_6^4 t_4^{24} + \frac{2q_2}{2} t_{24}^2 t_2^{48} + \frac{2q_2}{2} t_{12}^2 t_4^{24} + \frac{2q_2}{2} t_6^2 t_8^{12} + 2t_{48} t_2^{48}$$

$$+ 2t_{24} t_4^{24} + 2t_{12} t_8^{12} + 2t_6 t_{16}^{6}$$

d	48	32	24	16	12	8	6	4	3	2
d_1	2	3	4	6	8	12	16	24	32	48

d	8	4	4	2	2	2	1	1	1	1
d_2	6	12	6	24	12	6	48	24	12	6
d_1	2	2	4	2	4	8	2	4	8	16

For s = 216 de "d-vectors" contributing to t_{216} are

d	108	72	54	36	27	24	18	12	9	8	6	4	3	2
d_1	2	3	4	6	8	9	12	18	24	27	36	54	72	108

d	18	12	9	9	6	6	6	4	4	3	3	3	3	3	2	2	2	2
d_2	6	6	12	6	18	12	6	18	6	36	24	18	12	6	54	36	18	12
d_1	2	3	2	4	2	3	6	3	9	2	3	4	6	12	2	3	6	9

2	1	1	1	1	1	1	1	1
6	108	72	54	36	24	18	12	6
18	2	3	4	6	9	12	18	36

d	3	2	1	1	1	1	1
d_3	6	6	18	12	6	6	6
d_2	6	6	6	6	18	12	6
d_1	2	3	2	3	2	3	6

So that we get terms like $-\frac{2q_3}{3} t_6^3 t_6^{18} t_2^{108}$ and $-\frac{q_2}{2} t_6^2 t_6^{12} t_3^{72}$

4. CORRIGENDUM.

The formula for t_{18} on page 13 of [1] is not correct as stated.

A term

$$-\frac{P_1}{2}\left(\frac{P_8}{3}-\frac{P_2^4}{3}\right)^2$$

should be added. (Cf. the formula for t_{18} above).

REFERENCES

[1]. M. Hazewinkel (1972), Constructing Formal Groups II: over Z-Algebras. Report 7201, Econometric Institute, Erasmus University Rotterdam, The Netherlands.