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Some Examples Concerning Linear Continuity of Solutions to Programming Problems

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In this note we construct some counterexamples concerning upper semicontinuity and linear upper and lower semicontinuity of the solution sets and ϵ-solution sets of nonlinear programming programs: max $f(x)$, subject to $g(x) \leqq b$. These examples answer some of the questions in a recent paper by Stern and Topkis.

WE CONSIDER the programming problem max $f(x)$, $g(x) \leqq b$, where f is a function $\mathcal{R}^n \to \mathcal{R}$, g a function $\mathcal{R}^n \to \mathcal{R}^m$, and b an m-vector. Here $g(x) \leqq b$ means $g_j(x) \leqq b_j$ for all $j = 1, \cdots, m$. We define $S_b = \{x \in \mathcal{R}^n | g(x) \leqq b\}$ and for all $\epsilon \geqq 0$ we define the ϵ-solution set

$$S^*_{b,\epsilon} = \{x \in \mathcal{R}^n | x \in S_b \quad \text{and} \quad f(x) \geqq -\epsilon + \max_{z \in S_b} f(z)\}.$$

We are interested in upper and lower semicontinuity and linear upper and lower semicontinuity of $S^*_{b,\epsilon}$ as b varies. For a definition of these notions see [2]. We will construct some examples that answer some of the questions asked in [2].

1. EXAMPLES

Example 1. In this example g_j is linear for all j, $-f$ is a convex differentiable function, S_b is compact, but $S^*_{b,0}$ is not lower semicontinuous.

Let f be the function defined by $f(x_1, x_2) = x_2^2/x_1$ on the open halfspace of $\mathbb{R}^2$ where $x_1<0$. The function $-f$ is convex on $\{x \in \mathbb{R}^2 | x_1 < 0\}$.

Now define $g_1(x_1, x_2) = x_1$, $g_2(x_1, x_2) = -x_1$, $g_3(x_1, x_2) = x_2$, $g_4(x_1, x_2) = -x_2$. Let $b(0)$ be the vector $(-\frac{1}{2}, 1, 0, 1)$ and $b(\delta)$ the vector $(-\frac{1}{2}, 1, -\delta, 1)$. Then $S_{b(\delta)} = \{(x_1, x_2) | -1 \leqq x_1 \leqq -\frac{1}{2}, -1 \leqq x_2 \leqq -\delta\}$ and the solution sets $S^*_{b(\delta),0}$ of the problem $\max f(x)$, $g(x) \leqq b(\delta)$ are $S^*_{b(0),0} = \{(x_1, x_2) | x_2 = 0, -1 \leqq x_1 \leqq -\frac{1}{2}\}$ and $S^*_{b(\delta),0} = \{(-1, -\delta)\}$ for all $\delta > 0$. Therefore, $S^*_{b,0}$ is not lower semicontinuous with respect to b in $b(0)$.

Example 2. In this example g_j is linear for all j, $-f$ is a convex differentiable function, S_b is compact, but $S^*_{b,0}$ is not linearly upper semicontinuous. Let $D \subset \mathbb{R}^2$ be the region $x_1^2 + x_2^2 \leqq 1$. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $f(x) = -(d(x))^2$, where $d(x)$ is the distance of $x = (x_1, x_2)$ to D. A formula for $f(x)$ is $f(x) = -(\max\{0, (x_1^2 + x_2^2)^{1/2} - 1\})^2$. The function f is continuously differentiable. Further, $-f$ is a convex function. Now define $g_1(x) = x_2$, $g_2(x) = x_1$, $g_3(x) = -x_2$, $g_4(x) = -x_1$. Let $b(0)$ be the vector $(2, 1, -1, 1)$ and $b(\delta) = (2, 1, -1+\delta, 1)$, $\delta > 0$. Then we have $S^*_{b(0),0} = \{(0, 1)\}$ and $S^*_{b(\delta),0} = \{(x_1, x_2) | x_1^2 + x_2^2 \leqq 1, x_2 \geqq 1-\delta\}$.

The point $B = ((2\delta - \delta^2)^{1/2}, 1-\delta)$ is in $S^*_{b(\delta),0}$ and has distance $\geqq (2\delta - \delta^2)^{1/2}$ to $A = (0, 1)$. It follows that $S_{b,0}$ is not linearly upper semicontinuous in b at $b(0)$.

Example 3. In this example $-f$ is a convex differentiable function, S_b is compact and uniformly linearly continuous. For every ϵ with $0 \leqq \epsilon \leqq \frac{1}{16}$ there is a $b(\epsilon, 0)$ such that $S^*_{b,\epsilon}$ is not linearly upper semicontinuous in b at $b(\epsilon, 0)$.

The function f in this example is the same as the function f in example 2 above. There are five restriction functions:

$$g_1(x) = \begin{cases} -(\frac{1}{2}x_1 + \frac{1}{2}x_2 + \frac{1}{8}) + \frac{1}{4}(\frac{1}{4} - 4(x_1 - x_2)(x_1 - x_2 + \frac{1}{2}))^{1/2} \\ \qquad \text{if } x_1 - x_2 \leqq 0 \text{ and } x_2 - x_1 \leqq \frac{1}{2} \\ -x_1 \quad \text{if } x_1 - x_2 \geqq 0 \\ -x_2 + \frac{1}{4} \quad \text{if } x_2 - x_1 \geqq \frac{1}{2}, \end{cases}$$

$g_2(x) = x_1$, $g_3(x) = -x_2$, $g_4(x) = x_2$, and $g_5(x) = -x_1$.

The functions g_j are all continuously differentiable. The level curves of g_1 are straight lines joined by a quarter circle of radius $\frac{1}{4}$ (see Figure 1). Choose $0 \leqq \epsilon \leqq \frac{1}{16}$. Let $b(\epsilon, 0)$ be the vector $(-\frac{3}{4} - \epsilon^{1/2}, \frac{9}{4}, \frac{3}{2}, \frac{9}{4}, \frac{3}{2})$ and $b(\epsilon, \delta) = (-\frac{3}{4} - \epsilon^{1/2} + \delta, \frac{9}{4}, \frac{3}{2}, \frac{9}{4}, \frac{3}{2})$. We then have for $\delta \geqq 0$ sufficiently small $S^*_{b(\epsilon,\delta),\epsilon} = \{(x_1, x_2) | x_1^2 + x_2^2 \leqq 1+\epsilon, x_1 \geqq \frac{3}{4} + \epsilon^{1/2} - \delta\} \cup \{(x_1, x_2) | x_1^2 + x_2^2 \leqq 1+\epsilon, x_2 \geqq 1 + \epsilon^{1/2} - \delta\}$. In Figure 1 the case $\epsilon = \frac{1}{16}$, $\delta = \frac{1}{10}$ is shown. $S^*_{b(\epsilon,0),\epsilon}$ consists of the cross-hatched area and the point A and $S^*_{b(\epsilon,\delta),\epsilon}$ is the union of the cross-hatched and shaded areas.

It now follows as in example 2 that $S^*_{b,\epsilon}$ is not linearly upper semicontinuous in b at $b(\epsilon, 0)$.

2. REMARKS

Example 4.1. If one changes the function f of example 3 to $\bar{f}(x) = -(e(x))^2$, where $e(x)$ is the distance of x to the region $\{(x_1, x_2) | x_2 \leqq -x_1^2 + 1\}$, one can construct an example similar to example 3 such that for *every* $\epsilon \geqq 0$ there is a $b(\epsilon)$ such that $S^*_{b,\epsilon}$ is not linearly upper semicontinuous in b at $b(\epsilon, 0)$.

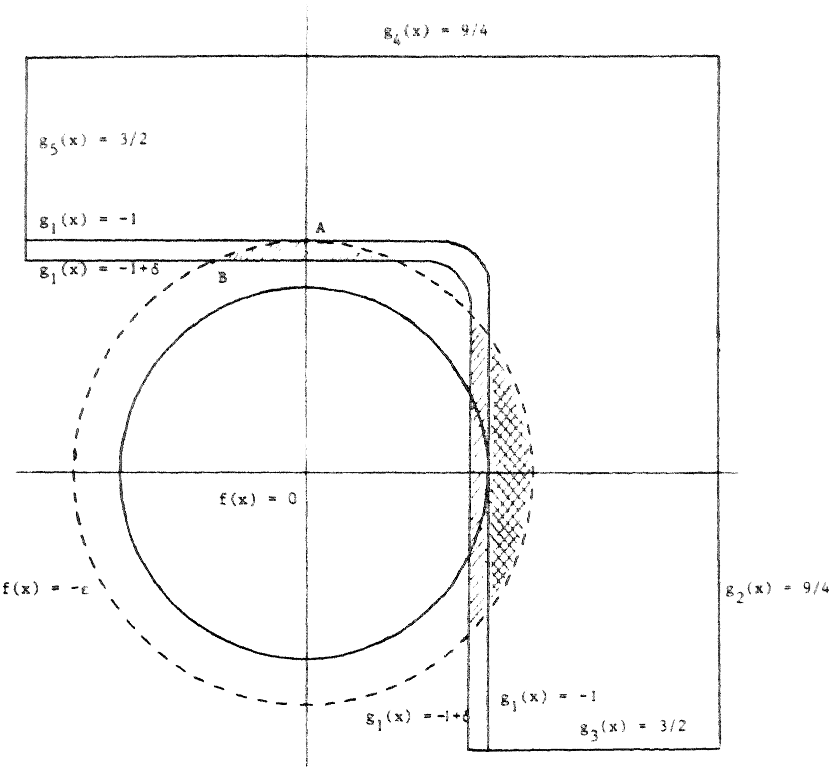

Figure 1

Example 4.2. By repeating infinitely often the trick by which the function g_1 of example 3 was constructed, one can construct an example with the following properties: 1) $-f$ is a convex differentiable function; 2) g_1 is a continuous differentiable function; 3) $g_2, \cdots, g_5$ are linear functions; 4) S_b is uniformly linearly continuous for b in a suitable hypercube B; and 5) for every $b(0)$ in the 4-dimensional subcube of B where $b_1 = -1$ there is an infinite sequence $\{\epsilon(n)\}_n$, $\epsilon(n) > 0$ such that $S^*_{b,\epsilon(n)}$ is not linearly upper semicontinuous in b at $b(0)$ for all $n \in \mathfrak{N}$. Moreover, $\lim_{n\to\infty} \epsilon(n) = 0$ and $S^*_{b,0}$ is not linearly upper semicontinuous in b at $b(0)$.

In this example $f, g_2, \cdots, g_5$ are as in example 3. The function g_1 is constructed as follows. We first define the level curve $g_1(x) = -1$. Draw rays l_n from the origin in $\mathfrak{R}^2$ at an angle of $2^{-n-1}\pi$ with the positive X_1-axis. Let A_n be the point on l_n at distance $1+a_n$ from the origin. In the points A_n

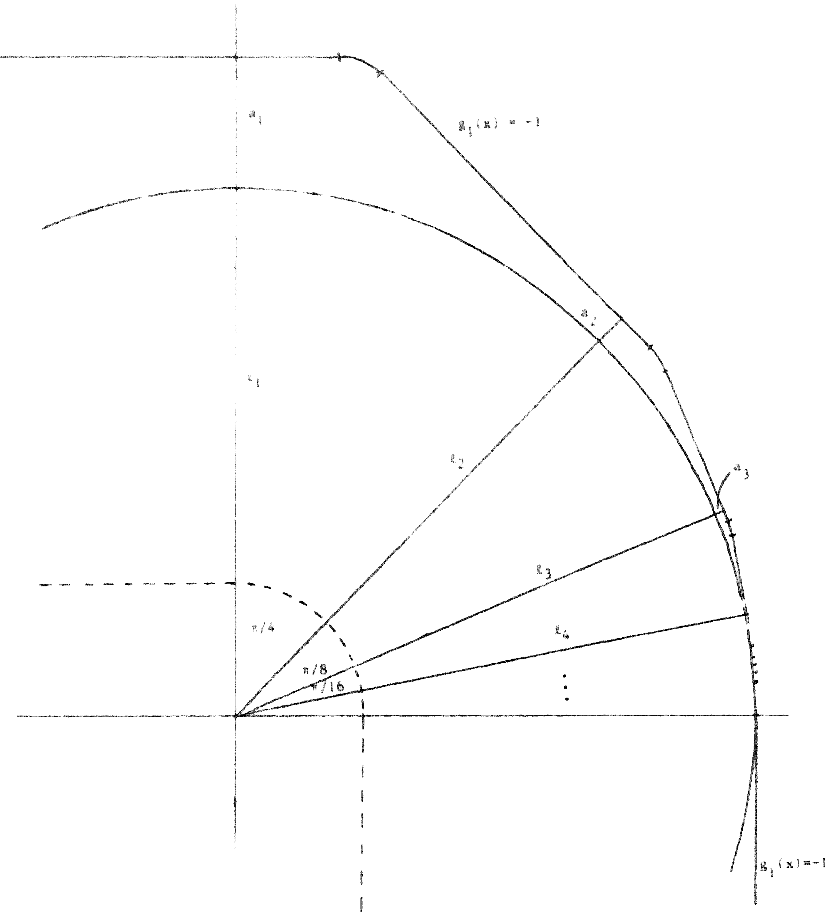

Figure 2

draw lines m_n perpendicular to l_n. If one takes, e.g., $a_n = 4^{-n-1}\pi^2/2$, then the intersection point of m_n and m_{n+1} is in the angle formed by l_n and l_{n+1}. Now join suitable segments of the m_n by "smoothing the corners in the intersection points." This can be done by means of circle arcs of radius $\frac{1}{8}$. The resulting curve C is differentiable and has the property that it is a straight line near every intersection point $C \cap l_n = A_n$, $n \in \mathfrak{N}$.

Let $E \subset \mathbb{R}^2$ be the region

$$E = \{x \in \mathbb{R}^2 | x_1 \geqq 1/4\} \cup \{x \in \mathbb{R}^2 | x_2 \geqq 1/4\} \cup \{x \in \mathbb{R}^2 | x_1 \geqq 0, \quad x_2 \geqq 0, \quad \|x\| \geqq 1/4\}.$$

The boundary of E is the dashed line in Figure 2. For each $x \in E$ let l_x be the ray from the origin passing through x, B_x be the intersection point $l_x \cap C$, and r_x be the distance of B_x to the origin. We now define $g_1(x) = -\|x\| r_x^{-1}$. This defines a differentiable function $g_1 : E \to \mathbb{R}$, which by the familiar tools of differential topology can be extended to a suitable, continuously differentiable function $g_1 : \mathbb{R}^2 \to \mathbb{R}$, such that $g_1(x) \geqq -1/2$ for all $x \in \mathbb{R}^2 \backslash E$ (see [1]).

A suitable hypercube such that property 4 holds is, e.g., defined by the inequalities $-3/4 \geqq b_1 \geqq -2$, $3 \leqq b_2 \leqq 4$, $3 \leqq b_3 \leqq 4$, $4 \leqq b_4 \leqq 5$, $2 \leqq b_5 \leqq 3$. In the example as described (see Figure 2) property 5 holds for all $b(0)$ in this cube for which $b_1 = -1$. For example, $S^*_{b,0}$ grows nonlinearly at $b = (-1, b_2, b_3, b_4, b_5)$ as b_1 becomes greater than -1. Nonlinear growth in the S^*_{b,ϵ_n} for $\epsilon_n = a_n^2$ at $b = (-1, b_2, b_3, b_4, b_5)$ as b_1 becomes greater than -1 is caused by the straight-line sections in the curve $g_1(x) = -1$ at the points A_n.

The functions f and g_1 of examples 2, 3, and 4.2 are continuously differentiable but not twice continuously differentiable. There are similar examples with f and g_1 of class C^∞, i.e., n times continuously differentiable for all n. This is done by smoothing the functions g_1 suitably, using techniques as in [1]. To obtain suitable functions f one needs only a C^∞ function h of one variable r that is convex, $\geqq 0$ everywhere and such that $h(r) = 0 \Leftrightarrow 0 \leqq r \leqq 1$. Such functions h exist, e.g., $h(r) = (r-1)^6 \exp(-(r-1)^{-2})$ if $r \geqq 1$, $h(r) = 0$ if $0 \leqq r \leqq 1$.

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