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THREE (COUNTER) EXAMPLES CONCERNING (LINEAR) CONTINUITY OF SOLUTIONS TO PROGRAMMING PROBLEMS

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We consider programming problems: maximize $f(x)$ subject to $g(x) \leq b$, where f is a function $\mathbb{R}^n \to \mathbb{R}$, g a function $\mathbb{R}^n \to \mathbb{R}^m$ and b an m-vector. ($a \leq b$ for m-vectors is taken to mean $a_j \leq b_j$ for all j). One defines

$$S_b = \{x \in \mathbb{R}^n | g(x) \leq b\}$$

$$S^*_{b,o} = \{x \in \mathbb{R}^n | x \in S_b \text{ and } f(x) = \max_{z \in S_b} f(z)\}, \text{ the solution set,}$$

and for given $\varepsilon \geq 0$.

$$S^*_{b,\varepsilon} = \{x \in \mathbb{R}^n | x \in S_b \text{ and } f(x) \geq -\varepsilon + \max_{z \in S_b} f(z)\}, \text{ the } \varepsilon\text{-solution set.}$$

One is interested in upper and lower semicontinuity and linear upper and lower semicontuinity of S_b, $S^*_{b,o}$ and $S^*_{b,\varepsilon}$ as b varies. (Linear semicontinuity is defined below). The setting and motivation for the examples below is provided by [1]. These examples answer (in the negative) all the questions on page 26 of [1].

1. Definition. A set valued function $s \mapsto T_s$ from $S \subset \mathbb{R}^m$ to the power set of $\mathbb{R}^n$ is said to be linearly upper semicontinuous at $s \in S$ if there exist $\delta > 0$ and $K > 0$ such that

$$\rho(z', T_s) \leq K||s-s'|| \text{ for all } z' \in T_{s'}, \quad s' \in S, \ ||s-s'|| < \delta.$$

The function $s \mapsto T_s$ is said to be linearly lower semicontinuous at $s \in S$ if there exist $\delta > 0$ and $K > 0$ such that

$$\rho(z, T_{s'}) \leq K||s-s'|| \text{ for all } z \in T_s, \ s' \in S, \ ||s'-s|| < \delta$$

These definitions are taken from [1]. The symbol $|| \ ||$ is the usual norm on $\mathbb{R}^m$ and $\rho(z,T)$ is the distance of $z \in \mathbb{R}^n$ to $T \subset \mathbb{R}^n$, i.e.

$$\rho(z,T) = \inf_{t \in T} ||z-t||.$$

2. Example. (g_j linear for all j, $-f$ convex, S_b compact but $S^*_{b,o}$ not lower semicontinuous).

In $\mathbb{R}^3$ with coordinates (x_1,x_2,x_3) let P be the parabola $x_1 = -1$, $x_3 = -x_2^2$. Let Q be the surface obtained by drawing all the lines joining the origin in $\mathbb{R}^3$ to a point of P. Let $(f(x_1,x_2)$ be the function defined by the surface Q. The function f is welldefined on the open halfspace $x_1 < 0$, and $-f$ is clearly convex. It is not difficult to write down a formula for f.

Now let $g_1(x_1,x_2) = x_1$, $g_2(x_1,x_2) = -x_1$, $g_3(x_1,x_2) = x_2$, $g_4(x_1,x_2) = -x_2$, and let b(o) be the vector $(-\frac{1}{2},1,0,1)$ and $b(\varepsilon)$ the vector $(-\frac{1}{2},1,-\varepsilon,1)$. Then $S_{b(\varepsilon)} = \{x|g(x) \leq b(\varepsilon)\}$ is the box $\{(x_1,x_2)|\ -1 \leq x_1 \leq -\frac{1}{2},\ -1 \leq x_2 \leq -\varepsilon\}$ and (the) solution sets $S^*_{b(\varepsilon),0}$ of max f(x), subject to $g(x) \leq b(\varepsilon)$, are equal to

$$S^*_{b(0),0} = \{(x_1,x_2)|\ x_2 = 0,\ -1 \leq x_1 \leq -\tfrac{1}{2}\}$$

$$S^*_{b(\varepsilon),0} = \{(-1,-\varepsilon)\} \text{ for all } \varepsilon > 0 .$$

Therefore $S^*_{b,o}$ is not lower semicontinuous (with respect to b) in b(0).

3. Example. (g_j linear for all j, $-f$ (strictly) convex, S_b compact, but $S^*_{b,o}$ not linearly upper semicontinuous).

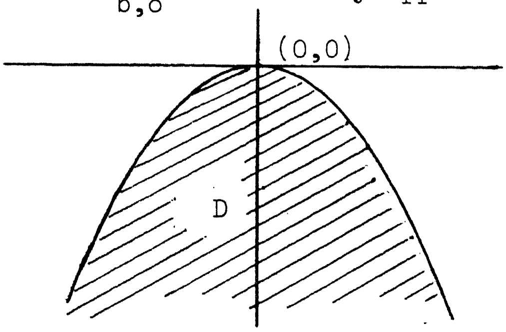

fig. 1

Let D in $\mathbb{R}^2$ be the region $x_2 + x_1^2 \leq 0$
Let $f: \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $f(x) = -d(x)$, where d(x) is the distance of (x_1,x_2) to D. Then $-f$ is a convex function because D is a convex region. (cf. the lemma below).

Let $g_1(x) = x_1$, $g_2(x) = -x_1$, $g_3(x) = x_2$, $g_4(x) = -x_2$ and let b(0) and $b(\varepsilon)$ be the vectors (1,1,1,0) and $(1,1,1,\varepsilon)$. The feasible regions $S_{b(\varepsilon)}$ are the boxes

$$S_{b(\varepsilon)} = \{x|g(x) \leq b(\varepsilon)\} = \{(x_1,x_2)|\ -1 \leq x_1 \leq 1,\ -\varepsilon \leq x_2 \leq 1\}$$

and the solution sets of max f(x) subject to $x \in S_{b(\varepsilon)}$ are

$$S^*_{b(o),o} = \{(0,0)\}$$

$$S^*_{b(\varepsilon),0} = \{(x_1,x_2) \mid x_2 + x_1^2 \leq 0, -\varepsilon \leq x_2 \leq 0\}$$

Therefore

$$\max_{z \in S^*_{b(\varepsilon),o}} \rho(z,S^*_{b(o),o}) = (\varepsilon+\varepsilon^2)^{\frac{1}{2}} ,$$

where $\rho(z,S^*_{b(o),o})$ is the distance of z to $S^*_{b(o),o}$ and as $\|b(o) - b(\varepsilon)\| = \varepsilon$ it follows that $S^*_{b.o}$ is not linearly upper continuous in b(o).

4. Lemma. Let C be a closed convex subject of $\mathbb{R}^n$ and let d(x) denote the distance of $x \in \mathbb{R}^n$ to C. Then d: $\mathbb{R}^n \to \mathbb{R}$ is a convex function.

Proof. Let x, y $\in \mathbb{R}^n$ and let a and b in C be points such that $\|x-a\| = d(n)$, $\|y-b\| = d(y)$. Let z be a point $z = \lambda x + (1-\lambda)y$, $0 \leq \lambda \leq 1$. Let w be the point on the line through a and y obtained by intersecting this line with the line through z parrallel to the line through x and a. Let c be the point on the line through a and b obtained by intersecting this line with the line through w parrallel to the line through y and b. Then

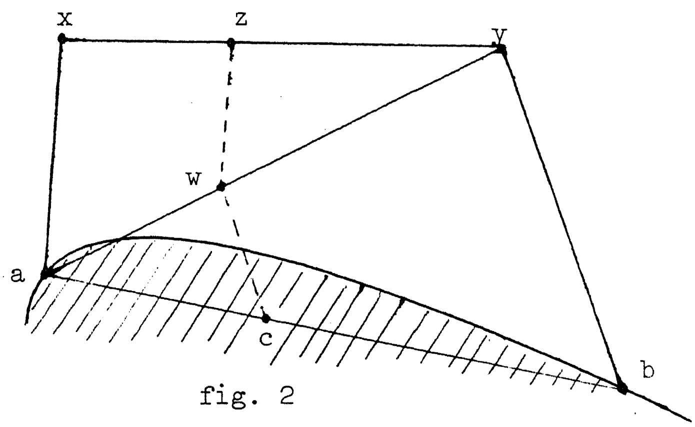

fig. 2

$$\|z-w\| = \lambda d(x) \text{ and } \|w-c\| = (1-\lambda)d(y)$$

and as c is a point of C (C being convex) we have that

$$d(z) \leq \|z-c\| \leq \|z-w\| + \|w-c\| = \lambda d(x) + (1-\lambda)d(y)$$

5. Example. (-f (strictly) convex, S_b compact and uniformly linearly continuous, but for all $0 \leq \varepsilon \leq 1$, $S^*_{b,\varepsilon}$ not linearly upper semicontinuous).

The function f of this example is the same as the function f in Example 4. There are five restriction functions

$g_1(x) = \min(x_2-x_1+3,-x_2)$
$g_2(x) = x_2$
$g_3(x) = -x_1$
$g_4(x) = x_1$

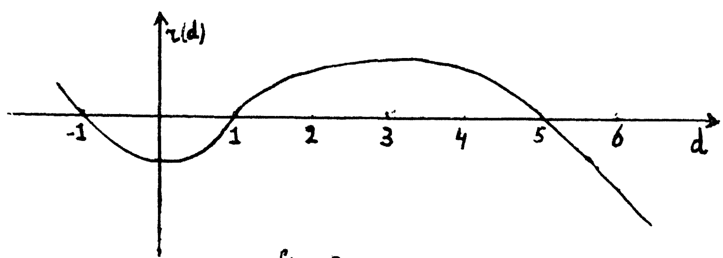

fig. 3

$g_5(x) = r(||x-(0,6)||)$, where r is a function of one variable like the one sketched in fig. 3.

Let $b(\varepsilon,0)$ be the vector $(-\varepsilon,2,1,1,0)$. In fig. 4 below the region of $x \in \mathbb{R}^2$ satisfying $g_i(x) \leq b_i(\varepsilon,0)$ for i = 1,2,3,4 is shaded and the region of $x \in \mathbb{R}^2$ satisfying $g(x) \leq b(\varepsilon,0)$ is crosshatched. The dotted line are the points where $f(x) = -\varepsilon$.

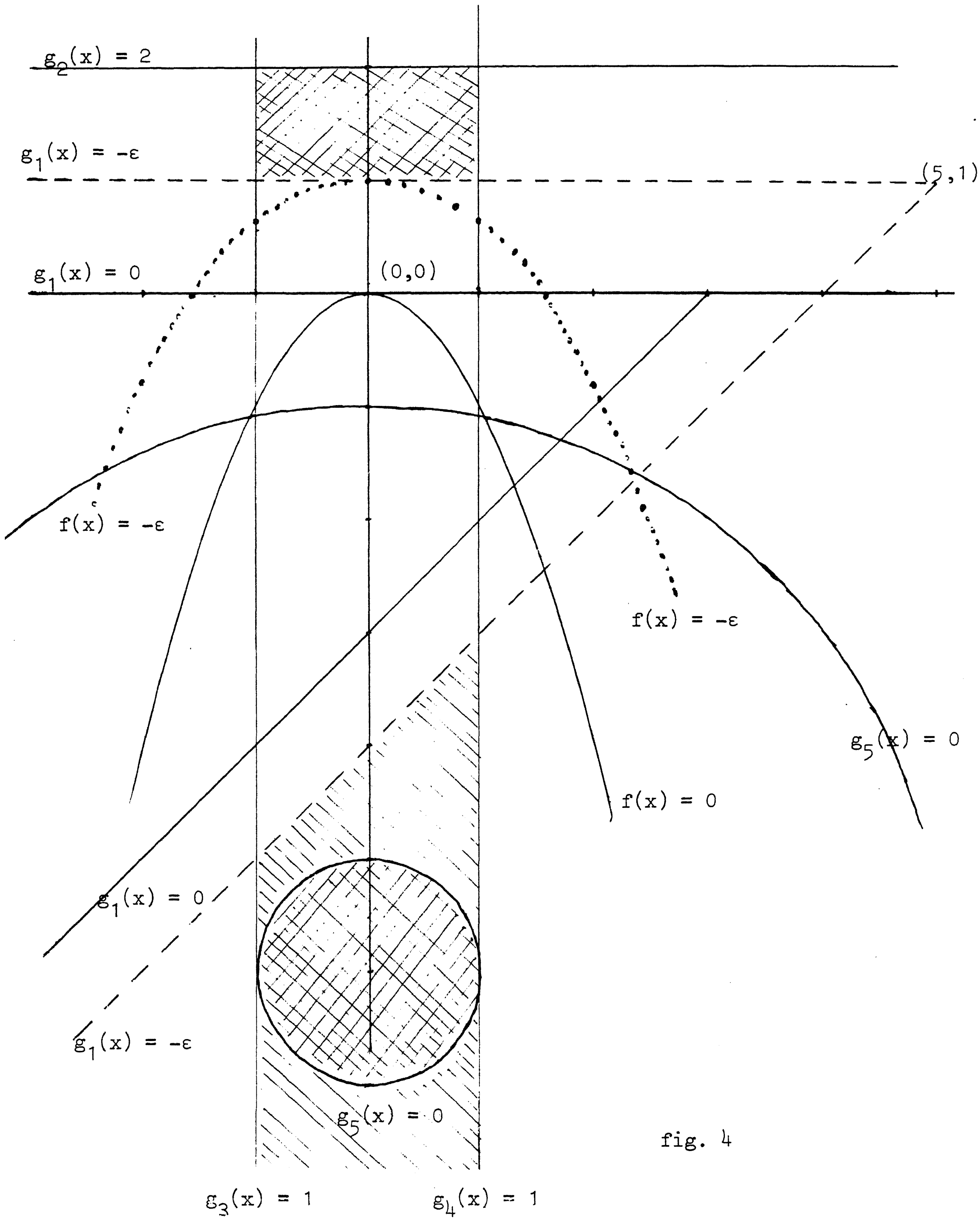

fig. 4

Now let

$$b(\varepsilon,c) = (\varepsilon(4c+1)^{-\frac{1}{2}} - c, 2, 1, 1, 0)$$

The sets $S^*_{b(\varepsilon,c),\varepsilon}$ are now equal to

$$S^*_{b(\varepsilon,0),\varepsilon} = A \cup \{(0,\varepsilon)\}$$

$$S^*_{b(\varepsilon,c),\varepsilon} = A \cup T$$

where T is the shaded region in fig. 5 below, and $A = \{x \in \mathbb{R}^2 \mid \|x-(0,b)\| \leq 1\}$.

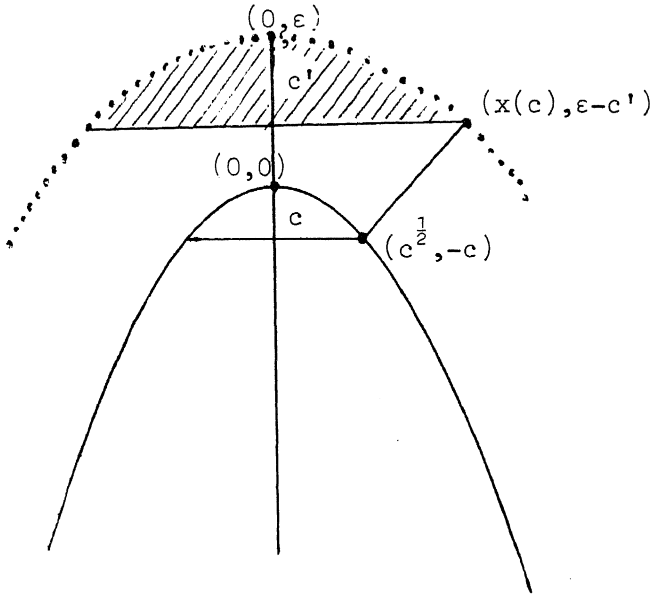

fig. 5

The point of $S^*_{b(\varepsilon,c),\varepsilon}$ furthest away from $S^*_{b(\varepsilon,0),\varepsilon}$ is the point $(x_1(c),\ \varepsilon-c')$ where

$$x_1(c) = c^{\frac{1}{2}} + 2\varepsilon c^{\frac{1}{2}}(1+4c)^{-\frac{1}{2}} \text{ and } c' = c + \varepsilon(1-(4c+1)^{-\frac{1}{2}})$$

(The point of the parabola $x_2 = -x_1^2$ closest to $(x_1(c),\ \varepsilon-c')$ is the point $(c^{\frac{1}{2}}, -c)$).

The distance of $(x_1(c),\ \varepsilon-c')$ (to $S^*_{b(\varepsilon,0),\varepsilon}$) is therefore $\geq c^{\frac{1}{2}}$. The distance between $b(\varepsilon,c)$ and $b(\varepsilon,0)$ is

$$c + \varepsilon(1-(4c+1)^{-\frac{1}{2}})$$

It follows that

$$\lim_{c \to o} \frac{\rho((x(c), \varepsilon - c'), S^{*}_{b(\varepsilon,o),\varepsilon})}{||b(\varepsilon,c) - b(\varepsilon,o)||} = \infty$$

and therefore $S_{b,\varepsilon}$ is not linearly upper semicontinuous in $b(\varepsilon,o)$.

REFERENCES

1. M.H. Stern, D.M. Topkis — Rates of stability in nonlinear programming.