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ON STABILITY IN MATHEMATICAL PROGRAMMING
(first order theory)

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Preliminary and Confidential

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1. INTRODUCTION

Usually a mathematical programming problem (MP) is given in the form:

$$\begin{aligned} & \max f(x), x \in \mathbb{R}^n \\ & \text{subject to } g_i(x) \leq 0 ; i = 1, \dots, m \end{aligned}$$

where f, g_i are functions $\mathbb{R}^n \rightarrow \mathbb{R}$. We shall most of the time assume that the functions f and g_i are C^1 , i.e. continuously differentiable. Now, of course, in practice f and g_i are usually only imperfectly known (due to inaccuracies in the measuring of various constraints involved e.g.). Hence, instead of dealing with the "true" problem (MP) one will usually have a slightly perturbed problem.

$$\begin{aligned} (\text{MP}') & \quad \max f'(x), x \in \mathbb{R}^n \\ & \quad \text{subject to } g'_i(x) \leq 0 ; i = 1, \dots, m \end{aligned}$$

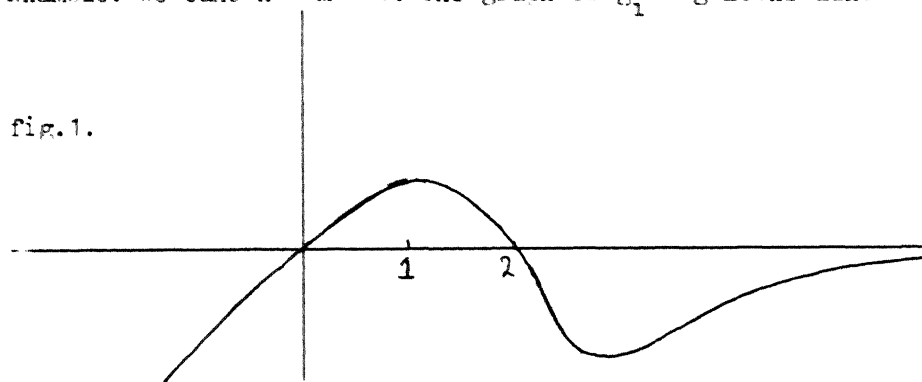
There are now three natural questions. Denoting with $\text{Sol}(\text{MP})$, resp. $\text{Sol}(\text{MP}')$ the solutions of (MP), resp. (MP') we can ask:

- (i) Given $\hat{x} \in \text{Sol}(\text{MP})$ is there always an $\hat{x}' \in \text{Sol}(\text{MP}')$ close to \hat{x} if (MP') is close enough to (MP) (lower semicontinuity of the function $\text{Sol} : (\text{MP}) \rightarrow \text{Sol}(\text{MP})$).
- (ii) Given an $\hat{x}' \in \text{Sol}(\text{MP}')$ for (MP') close enough to (MP) is there an $\hat{x} \in \text{Sol}(\text{MP})$ close to \hat{x}' (upper semicontinuity of Sol)
- (iii) Is $f(\text{Sol}(\text{MP}))$ a continuous function of (MP) (continuity of returns).

Unfortunately (or fortunately, depending on one's point of view) the answer to all these questions is not always yes; i.e. there are programming problems (MP) with arbitrary small perturbations (MP') for which the statements (i) - (iii) are all false.

One reason for this is that an arbitrary small change in the g_i can result in a very large change in the feasible region $D = \{x \mid x \in \mathbb{R}^n; g_i(x) \leq 0; i = 1, \dots, m\}$, as the following elementary example shows:

Example. We take $n = m = 1$. The graph of $g_i = g$ looks like



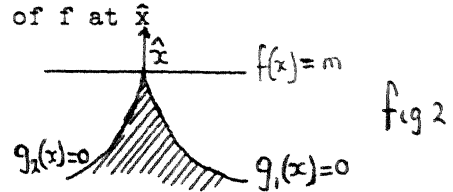
Then $D = \{x \in \mathbb{R} \mid g(x) \geq 0\} = [0, 2]$. g_ϵ , defined by $g_\epsilon(x) = g(x) + \epsilon$ is an arbitrary small perturbation of g (in all reasonable meanings of the word), but $D_\epsilon = \{x \in \mathbb{R} \mid g_\epsilon(x) \geq 0\}$ contains an interval $[c, \infty)$ for some c . If one now takes $f(x) = x$, it is trivial to see that the answers to all the questions (i) - (iii) is no. This example can be found in Evans & Gould [2]. Results on this facet of the stability problem can be found in [2] and also in [3].

If this phenomenon does not occur, i.e. if the feasible region D' of (MP') is close to D (in the Hausdorff-distance sense e.g.) and D is compact, the answer to questions (ii) and (iii) is yes, as is well-known (cf. §3). In this case therefore one is in a fairly good position if one has calculated the solutions \hat{x}' of (MP'), provided one does not mind overlooking (possibly very nice) solutions \hat{x} of (MP), that may be far away from $\text{Sol}(\text{MP}')$ (This phenomenon occurs already in linear programming problems).

Because of this, it seemed to us reasonable to separate the stability problem in two parts: a) When does D depend continuously on the functions g_1, \dots, g_m ? b) when does $\text{Sol}(\text{MP})$ depend continuously on (D, f) ? ((D, f) is the programming problem with feasible region D and goal-function f . In this note we concentrate on b).

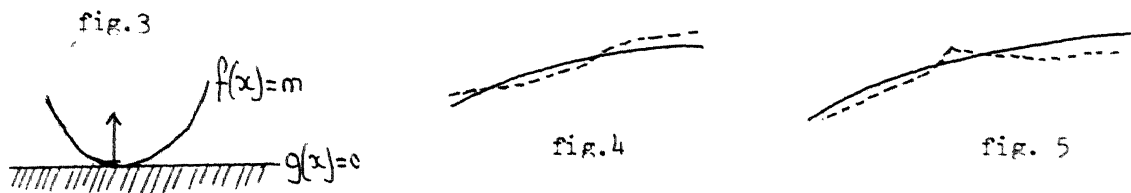
Then, of course, one has to define feasible regions D and perturbations of them without using the functions g_i . This has led to the entirely obvious notions of a differentiable set - roughly a curvilinear polyhedron- and perturbations of a differentiable set. Cf §2.

In this paper we deal so to speak with first order phenomena only. For example, if the feasible region looks as in the drawing on the right (fig.2) and $\nabla f(\hat{x})$, the gradient of f at \hat{x} points in the direction of the arrow, then a slight perturbation in the C^1 -sense results in much the same situation.



But in the case of a solution to a programming problem, as shown in fig.3 there are always C^1 -perturbations such that the perturbed problem has many solutions. In this case C^2 -perturbations give much the same situation. Here we are in a mixed first order-second order situation, (because the second order properties are only important in the tangent direction in \hat{x} to $g(x) = 0$).

Given a differentiable set, there are two natural notions of perturbation. They are exemplified by fig. 4 and fig.5, where the drawn line indicates the boundary of the original set and the one the boundary of the perturbed set.



In the case of fig. 4 we speak of C^1 -perturbations and in the case of fig.5 of (C^1) -generalized perturbations. The corner creating trick of fig. 5 permits us to change a situation like the one of fig. 3 into one which is stable under C^1 -perturbations.

We give the set \mathcal{MP} of programming problems with compact feasible regions the topology corresponding to the notion of a C^1 -generalized perturbation. Let $\mathcal{MP}^{(h)}$ be the subspace of the problems (D, f) that have a solution in ∂D . It should now be intuitively clear that the subset of $\mathcal{MP}^{(h)}$ of programming problems with precisely one solution contains an open, dense subset of $\mathcal{MP}^{(h)}$. Which implies that $\text{Sol} : \mathcal{MP}^{(h)} \rightarrow \text{Pow}(\mathbb{R}^n)$ is continuous on an open, dense subset of

$m\phi^{(h)}$). It is the aim of this note to prove this. For a similar theorem concerning C^2 -perturbations, cf [4]. Unless it is stated otherwise, a programming problem (MP) is always assumed to be such that its feasible region is compact. All programming problems considered are in \mathbb{R}^n . If $x, y \in \mathbb{R}^n$, $\langle x, y \rangle$ denotes the innerproduct of x and y .

2. DIFFERENTIABLE SETS

2.1. Definition.

A differentiable set $D \subset \mathbb{R}^n$ is a set $D \subset \mathbb{R}^n$ such that there exist an open covering $\{U_i\}$ of D and halfspaces $H_{i,j}^n$, differentially imbedded in U_i , such that

- (i) $D \cap U_i = \bigcap_j H_{i,j}^n$
- (ii) for each i, i' let $s(i, i') = \{t | H_{i,t} \cap U_i \cap U_{i'} \neq \emptyset\}$.

Then there is a bijection $\sigma_{ii'} : s(i, i') \rightarrow s(i', i)$ such that

$$H_{i,t} \cap U_i \cap U_{i'} = H_{i', \sigma_{ii'}(t)} \cap U_{i'} \cap U_i \text{ for all } t \in s(i, i')$$

2.2. Remark.

As the $H_{i,j}$ are differentially embedded in the U_i there exist functions $g_{i,j} : U_i \rightarrow \mathbb{R}$ such that $H_{i,j} = \{x \in U_i | g_{i,j} \leq 0\}$ and $\nabla g_{i,j}(x) \neq 0$ if $g_{i,j}(x) = 0$ (cf. e.g. [5], §5). Of course the $g_{i,j}$ are not uniquely determined by the $H_{i,j}$.

2.3. Definitions.

Let $x \in U_i \cap D$. A constraint H_{i,j_0} is called effective in x if $x \in \partial H_{i,j_0}$; it is called essential in x if for every open neighbourhood $x \in U \subset U_i$ we have that

$$\bigcap_j H_{i,j}^n \cap U \neq \bigcap_{j \neq j_0} H_{i,j}^n \cap U$$

In other words: H_{i,j_0} is really necessary to define D in a neighbourhood of x .

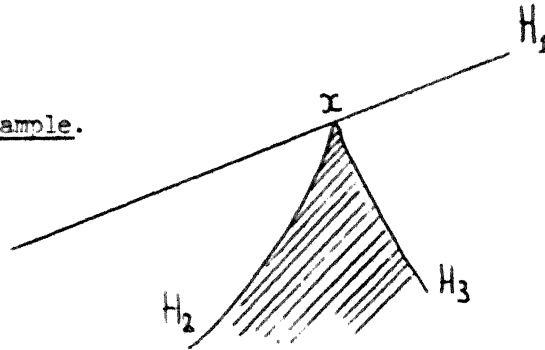
2.4. Example.

fig.6

H_1 is effective in x but not essential. H_2 and H_3 are essential in x

2.5. Lemma.

If H is essential in x , then H is also effective.

Proof: If H is not effective in x , then there is an open U such that

$$x \in U \subset H,$$

q.e.d.

2.6. Normals: Let $x \in D \cap U_i$ and let $H_{i,j}$ be essential in x . As $H_{i,j}$ is differentially embedded in U_i and $x \in \partial H_{i,j}$ there is an outward pointing normal vector in x . Using the metric of \mathbb{R}^n we define $n_{i,j}(x)$ as the normal vector in x to $H_{i,j}$ with $\|x\| = 1$. If we use functions as in Remark (2.2) then $n_{i,j}(x) = c \cdot \nabla g_{i,j}(x)$ for some non-zero constant c .

For each $x \in D$, let $N_x(D)$ be the set of endpoints of the vectors $n_{i,j}(x)$ (where $x \in U_i$ and $H_{i,j}$ essential in x). If U is any set, we define $N_U(D) = \bigcup_{x \in U} N_x(D)$.

2.7. Hausdorff-distance.

For two sets $A, B \in \mathbb{R}^n$ one defines the Hausdorff-distance $Hd(A, B)$ as $Hd(A, B) = \inf\{d \mid A \subset S_d(B), B \subset S_d(A)\}$ where $S_d(A) = \{x \in \mathbb{R}^n \mid$

$$\inf_{y \in A} \|x - y\| \leq d\}.$$

2.8. Definition. (perturbation of a differentiable set)

Let D, D' be two differentiable sets in \mathbb{R}^n and let $\epsilon > 0$. We say that D' is an ϵ -generalized-perturbation of D or that D and D' are ϵ -close to each other if

(i) $Hd(D, D') < \epsilon$

(ii) there is a covering $\{V_i\}$ of $D \cup D'$ such that

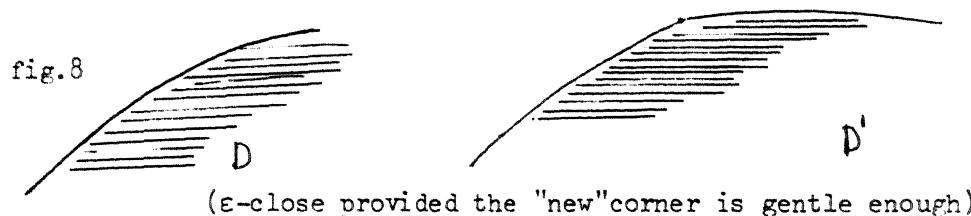
$$Hd(N_{V_i}(D), N_{V_i}(D')) < \epsilon \text{ for all } i$$

We say that D' is an ϵ - C^1 -perturbation of D (not generalized) if everywhere (locally) there is a 1-1-correspondence between the halfplanes defining D and the halfplanes defining D' (I.e. D and D' are (locally) given by the same number of restrictions)

2.9. Examples.



(not ϵ -close)



(ϵ -close provided the "new" corner is gentle enough)

2.10. Polyhedral cones.

We recall that a (closed) polyhedral cone in \mathbb{R}^n (with vertex in 0) is defined by a system of inequalities $C = \{x \in \mathbb{R}^n \mid \langle x, a_i \rangle \geq 0, i = 1, \dots, m\}$ where the a_i are a fixed finite set of vectors. The cone C is called pointed if $x \in C$ and $-x \in C \Rightarrow x = 0$.

2.11. Linear differentiable sets

The differentiable set D is linear if the embedded halfplanes $H_{i,j} \subset U_i$ are linearly embedded. One can then choose affine functions $g_i(x) = \langle x, a_i \rangle + b_i$ such that $D = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0\}$. Small changes in the a_i and the b_i give rise to (a special kind of) small C^1 -perturbations of D .

2.12. Perturbations given by functions.

Let D be a differentiable set, given locally in U by the halfplanes $H_{i,j}$. Then there are C^1 -functions $g_i (i = 1, \dots, p)$ such that $D \cap U = \{x \in U \mid g_i(x) \leq 0\}$ and $\nabla g_i(x) \neq 0$ if $g_i(x) = 0$. If $\|\nabla g_i(x)\| \geq \frac{1}{2}$ for all $x \in U$ (by shrinking U if necessary and changing the g_i this can be arranged),

and h_1, \dots, h_m are a set of functions such that for every i there is a $i(i)$ such that $|h_i(x) - g_{i(i)}(x)| < \delta$

$$||\nabla h_i(x) - \nabla g_{i(i)}(x)|| < \delta$$

and for every j there is an $i(j)$ such that

$$|g_j(x) - h_{i(j)}(x)| < \delta$$

$$||\nabla g_j(x) - \nabla h_{i(j)}(x)|| < \delta$$

then the set $D' \cap U$ is an ε - C^1 -perturbation of $D \cap U$ if δ is small enough. Quite often this is the most natural way to construct perturbations.

3. CONTINUITY PROPERTIES OF MATHEMATICAL PROGRAMMING PROBLEMS.

In this section we consider mathematical programming problems (D, f) where D is a compact (differentiable) set in \mathbb{R}^n and f a function on \mathbb{R}^n . The material of this section is quite well-known.

Let $\mathcal{M}^{\mathcal{P}}$ be the set of all programming problems (D, f) , D compact, $D \subset \mathbb{R}^n$. Two programming problems (D, f) and (D', f') are ε -close in the C^0 sense if $\text{Hd}(D, D') < \varepsilon$ and $|f(x) - f'(x)| < \varepsilon$ for all $x \in D \cup D'$. Taking as open nbd's of $(D, f) \in \mathcal{M}^{\mathcal{P}}$ the sets of all (D', f') that are ε -close in the C^0 -sense for all $\varepsilon > 0$, defines a topology on $\mathcal{M}^{\mathcal{P}}$. The set $\mathcal{M}^{\mathcal{P}}$ with this topology will be denoted $\mathcal{M}^{\mathcal{P}}_0$.

3.1. Proposition.

The function $\text{Sol} : \mathcal{M}^{\mathcal{P}}_0 \rightarrow \text{Pow}(\mathbb{R}^n)$, $(D, f) \mapsto \text{Sol}(D, f)$ (solutions of (D, f)) is upper semi-continuous (Here, $\text{Pow}(\mathbb{R}^n)$ is the set of subsets of \mathbb{R}^n).

Proof: Let U be an open set containing $\text{Sol}(D, f)$. $\text{Sol}(D, f)$ is compact,

so there exists an open V such that $\text{Sol}(D, f) \subset V \subset \bar{V} \subset U$, \bar{V}

compact. Let $M = \sup_{x \in D} f(x)$ and $M' = \sup_{x \in D \setminus V} f(x)$. Then $M = f(\bar{x})$ if

$\bar{x} \in \text{Sol}(D, f)$ and $M' < M$. Because \bar{V} is compact there exists an ε_1

such that $x \in \bar{V}$, $||v-x|| < \varepsilon_1 \Rightarrow v \in U$

f is uniformly continuous on $D_1 = \{x \in \mathbb{R}^n \mid \inf_{y \in D} ||x-y|| < 1\}$. So there

exists an ε_2 such that : $x, x' \in D_1, \|x-x'\| < \varepsilon_2 \Rightarrow |f(x)-f(x')| < \frac{1}{2}(M-M')$

Let $\varepsilon = \min\{1, \frac{1}{2}(M-M'), \varepsilon_1, \varepsilon_2\}$ and let (D', f') be ε -close in the C^0 -sense

to (D, f) . There is a point $\hat{y} \in D'$ and a point $\hat{x} \in \text{Sol}(D, f)$ such that

$\|\hat{y}-\hat{x}\| < \varepsilon$, therefore $f'(\hat{y}) > M' + \frac{1}{2}(M-M')$ for a certain $\hat{y} \in D'$.

Now let $v \in D' \setminus U$. There is an $x \in D$ such that $\|x-v\| < \varepsilon$. Then $x \notin \bar{V}$

because we would otherwise have that $v \in U$. Therefore $f(x) \leq M'$ and

hence $f'(v) \leq M' + \frac{1}{2}(M-M')$. This proves that $\text{Sol}(D', f') \subset U$,

q.e.d.

The proof above also shows that

3.2. Proposition.

The function $\text{Return} : \mathcal{M}_0 \rightarrow \mathbb{R}, \text{Return}(D, f) = \sup_{x \in D} f(x)$

is continuous.

And, as there is always at least one solution of (D, f) (because D is compact), we also have:

3.3. Corollary. If (D, f) has exactly one solution, then $\text{Sol} : \mathcal{M}_0 \rightarrow \text{Pow}(\mathbb{R}^n)$ is continuous in (D, f) .

4. STABILITY (LINEAR CASE)

As we are interested in first-order phenomena in this note, the case of linear programming problems should give a good indication of what to expect to general.

4.1. Definition. A linear programming problem (D, f) , $D \subset \mathbb{R}^n$ compact, is called nice if

- (i) (D, f) has precisely one solution \hat{x}
- (ii) There are precisely n restraints f_{i_1}, \dots, f_{i_n} effective in \hat{x} ; they are all essential and $\nabla f_{i_1}(\hat{x}), \dots, \nabla f_{i_n}(\hat{x})$ are linearly independent
- (iii) $\nabla f(\hat{x}) = \lambda_1 \nabla f_{i_1}(\hat{x}) + \dots + \lambda_n \nabla f_{i_n}(\hat{x})$ for certain $\lambda_1, \dots, \lambda_n, \lambda_i > 0$

4.2. Proposition. Let (D, f) be a nice linear programming problem. Then there is an $\varepsilon > 0$ such that

- (i) If (D', f') is a LP-problem which is a generalized $\epsilon-C^1$ -perturbation of (D, f) then (D', f') has precisely one solution.
- (ii) If (D', f') is a LP-problem which is a $\epsilon-C^1$ -perturbation of (D, f) then (D', f') is nice.
 (The definition of a (generalized) $\epsilon-C^1$ -perturbation of (D, f) is obvious).

Proof:

(i) For a sufficiently small neighbourhood U of \hat{x} , $D \cap U$ is defined by n linear constraints $g_{i_1}(x) \leq 0; g_{i_2}(x) \leq 0; \dots; g_{i_n}(x) \leq 0$. If ϵ is small enough and (D', f') a generalized $\epsilon-C^1$ -perturbation of (D, f) then all solutions of (D', f') are in U . Suppose (D', f') is given in U by the linear functions $h_1, \dots, h_m; m \geq n$. There is a solution of (D', f') in U , say \hat{x}' . Let $h_1 \dots h_n$ be the constraints essential in \hat{x}' .

For every h_i there is an $i(j)$ such that $\nabla h_i(\hat{x}')$ is within ϵ of $\nabla g_{i(j)}(\hat{x})$ (and inversely). Hence we have that $p \geq n$ and that $\nabla f'(\hat{x}') = \lambda_1 \nabla h_{i_1}(\hat{x}') + \dots + \lambda_n \nabla h_{i_n}(\hat{x}')$ if ϵ is small enough. (Consider

(ii) and (iii) of definition 4.1 and the fact that $\nabla f'(\hat{x}')$ is within ϵ of $\nabla f(\hat{x}) \neq 0$ - therefore $\nabla f'(\hat{x}') \neq 0$ if ϵ is small enough). Furthermore $\nabla h_{i_1}(\hat{x}'), \dots, \nabla h_{i_n}(\hat{x}')$ are linearly independent.

It follows that (D, f) looks as in the drawing on the right (fig.9) in a nbd of \hat{x} . If a second point \hat{x}'' was also a solution of (D', f') then- because $\hat{x}'' \in U$,

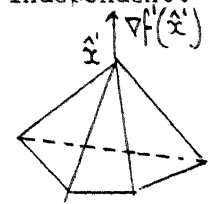


fig.9

in a neighbourhood of \hat{x}'' D would look as in a neighbourhood of \hat{x}' . We would get a situation like in fig.10 which is impossible.

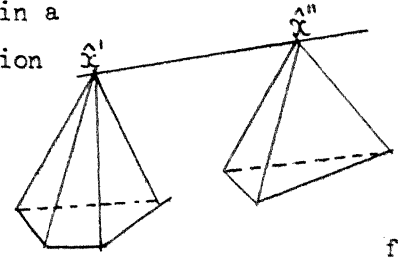


fig.10.

Then, \hat{x} and \hat{x}'' would be in the hyperplane $(\nabla f')^T y = c$. It follows from the convexity that this hyperplane would be a bounding hyperplane of D' which is contradictory).

The proof of (ii) is similar but easier because there are exactly n functions h .

4.3. Proposition.

Let (D, f) be an LP-problem. Then for every $\varepsilon > 0$ there is a ε - C^1 -perturbation (D', f') of f which is nice.

Proof. Let D be defined by the linear constraints $g_1(x) \leq 0, \dots, g_m(x) \leq 0$

As D is compact (we only consider compact problems) there is a solution \hat{x} of (D, f) and because (D, f) is linear we can choose \hat{x} such that at least n constraints are essential in \hat{x} .

Let $g_1(x) \leq 0, \dots, g_n(x) \leq 0$ be the essential constraints in \hat{x} .

Because \hat{x} is a solution of (D, f) we have : $\nabla f(\hat{x}) = \lambda_1 \nabla g_1(\hat{x}) + \dots + \lambda_n \nabla g_n(\hat{x})$,

$\lambda_i \geq 0$. It follows (Caratheodory's theorem, cf [1], Th.18) that there are $\nabla g_{i_1}(\hat{x}), \dots, \nabla g_{i_n}(\hat{x})$ such that

$$(4.3.1) \quad \nabla f(\hat{x}) = \mu_1 \nabla g_{i_1}(\hat{x}) + \dots + \mu_n \nabla g_{i_n}(\hat{x}); \mu_i \geq 0$$

Now choose n linear functions h_1, \dots, h_n such that $h_1(\hat{x}) = \dots = h_n(\hat{x}) = 0$ and $\nabla h_j(\hat{x})$ is within δ_1 of $\nabla g_{i_j}(\hat{x})$, and $\nabla h_j(\hat{x}); j = 1, \dots, n$ are

linearly independent. For $k \in \{1, \dots, p\} \setminus \{i_1, \dots, i_n\}$ let b_k be a

real number; $|b_k| < \delta_2$ such that $g_k(\hat{x}) + b_k < 0$. Let D' be the linear set defined by

$$h_1(x) \leq 0, \dots, h_n(x) \leq 0, g_k(x) + b_k \leq 0 \text{ for } k \in \{1, \dots, p\} \setminus \{i_1, \dots, i_n\},$$

$$g_{n+1}(x) \leq 0, \dots, g_m(x) \leq 0.$$

Because of (4.3.1) there is a small vector v , $\|v\| < \varepsilon$ such that (if δ_1 is small enough).

$$\nabla f(\hat{x}) + v = \mu'_1 \nabla h_1(\hat{x}) + \dots + \mu'_n \nabla h_n(\hat{x}); \mu'_i > 0$$

Let $f'(x) = f(x) + \langle v, x \rangle$. Then (D', f') is an ε - C^1 -perturbation of (D, f) , $\hat{x} \in D'$, and (D', f') is nice.

Let \mathcal{LP}_m be the set of LP problems, defined by m restrictions and let \mathcal{LP} be the set of all programming problems (in \mathbb{R}^n). A topology on

\mathcal{LP}_m is defined by taking as a basis for the neighbourhood of (D, f) the set of all ε - C^1 -perturbations of (D, f) for all $\varepsilon > 0$. A topology

on \mathcal{LP} is defined by taking as a basis for the open nbd's of (D, f) the set of all generalized ϵ - C^1 -perturbations of (D, f) for all $\epsilon > 0$. Then we have the following corollaries of (4.3) and (4.2).

- 4.4. Corollary. The set of nice \mathcal{LP} problems in \mathcal{LP}_m is open and dense.
- 4.5. Corollary. There is an open and dense set in \mathcal{LP} of problems with exactly one solution.
- 4.6. Corollary. The set of points where $\text{Sol} : \mathcal{LP} \rightarrow \text{Pow}(\mathbb{R}^n)$ is continuous contains an open and dense set.
- 4.7. Remark. The same arguments as in (4.2), (4.3) show also that the measure of the LP-problems in \mathcal{LP}_m which don't have an unique solution is zero (natural measure on \mathcal{LP}_m).

5. STABILITY (first order theory)

In this section we consider programming problems (D, f) consisting of a compact differentiable set $D \subset \mathbb{R}^n$ and C^1 -function $f : \mathbb{R}^n \rightarrow \mathbb{R}^1$; such that there is a solution \hat{x} of (D, f) with $\hat{x} \in \partial D$. We topologize the set of such programming problems by taking as a basis for the open neighbourhood of (D, f) the sets of all generalized ϵ - C^1 -perturbations of (D, f) for all $\epsilon > 0$. Let \mathcal{MP}_1^b be the resulting topological space

5.1. Definition. A programming problem (D, f) is nice if

- (i) (D, f) has precisely one solution \hat{x}
- (ii) There are exactly n effective constraints in \hat{x} ; they are all essential and the set of their normals in \hat{x} is linearly independent.
- (iii) $\nabla f(\hat{x}) = \lambda_1 v_1 + \dots + \lambda_n v_n$; $\lambda_i > 0$, where v_i are the normal vectors in \hat{x} .

5.2. Proposition. The set of nice programming problems is dense in \mathcal{MP}_1^b .

Proof. For the proof of this and also further on we need the existence of certain functions. There exists a C^1 -function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\phi(0) = 1$; $0 \leq \phi(x) \leq 1$ for all $x \neq 0$ and $\phi(x) = 0$ if $\|x\| \geq 1$. Then $\|\nabla \phi(x)\|$ is bounded; let c_1 be such that $\|\nabla \phi(x)\| \leq c_1$ for all $x \in \mathbb{R}^n$.

There also exists a function $\psi: \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\psi(x) = 1$ if

$$\|x\| \leq 1, 0 \leq \psi(x) \leq 1 \text{ for all } x \in \mathbb{R}^n, \psi(x) = 0 \text{ if } \|x\| \geq 2.$$

Then also $\|\nabla\psi(x)\|$ is bounded. Let c_2 be such that $\|\nabla\psi(x)\| \leq c_2$ for all $x \in \mathbb{R}^n$.

Now, let $(D, f) \in \mathcal{M}_1^b$ be a programming problem and let \hat{x} be a solution

Let f' be defined by $f'(x) = f(x) - \delta\phi(x-\hat{x})$. For δ small enough (D, f') is an ε - C^1 -perturbation of (D, f) and (D, f') has exactly one solution \hat{x} . Let U be a neighbourhood of \hat{x} such that $D \cap U$ is defined by functions g_1, \dots, g_m (cf. 2.2 and 2.12). By shrinking (if necessary) the other sets of an open covering which serves to define D we can assume that there is an open set V such that $\hat{x} \in V$ and $V \cap U' = \emptyset$ from all U' different from U in the open cover used in the definition of D .

Because \hat{x} is a solution of (D, f') we have that

$$\nabla f'(\hat{x}) = \lambda_1 \nabla g_1(\hat{x}) + \dots + \lambda_m \nabla g_m(\hat{x}), \lambda_i \geq 0$$

By Caratheodory's theorem it follows that there are indices i_1, \dots, i_n

$$\text{such that } \nabla f'(\hat{x}) = \mu_1 \nabla g_{i_1}(\hat{x}) + \dots + \mu_n \nabla g_{i_n}(\hat{x}); \mu_i \geq 0$$

Let w_1, \dots, w_s be a maximal linearly independent subset of $\nabla g_{i_1}(\hat{x}), \dots, \nabla g_{i_n}(\hat{x})$.

There exist vectors w_{s+1}, \dots, w_n, w such that

- (i) $w_1, \dots, w_s, w_{s+1}, \dots, w_n$ is linearly independent
- (ii) w_j is within δ of $\nabla g_{i_j}(\hat{x})$; $j = s+1, \dots, n$
- (iii) $\|w\| < \delta$
- (iv) $\nabla f'(\hat{x}) + w = \mu'_1 w_1 + \dots + \mu'_n w_n$ for certain $\mu'_i > 0$

Now choose a real number $a > 0$ such that $\{x \in \mathbb{R}^n \mid \|x-\hat{x}\| \leq 2a\} \subset V$

and let $V' = \{x \in \mathbb{R}^n \mid \|x-\hat{x}\| < a\}$

We now define functions $h_1, \dots, h_n; U \rightarrow \mathbb{R}$ by means of the formulae

$$h_i(x) = \psi_1(x) \cdot \langle x-\hat{x}, w_i \rangle + (1-\psi_1(x)) \cdot g_{i_j}(x). \psi_1(x) \text{ is the function}$$

$\psi(\frac{1}{a}(x-\hat{x}))$ where ψ is the function defined in the beginning of the proof.

For $t \in \{1, \dots, m\} \setminus \{i_1, \dots, i_n\}$ we define $k_t(x) = g_t(x) + b_t$, where

b_t a real number such that $|b_t| < \delta_2$ and $g_t(\hat{x}) + b_t < 0$.

Finally we define f'' as $f''(x) = (f'(x) + \langle x - \hat{x}, w \rangle) \cdot \psi_1(x) + (1 - \psi_1(x))f'(x)$

Note that $h_j(x) = g_{i_j}(x)$ for $\|x - \hat{x}\| \geq 2a$. Using the fact that $g_{i_j}(x)$

is continuously differentiable and hence that

$g_{i_j}(x) = g_{i_j}(\hat{x}) + \langle \nabla g_{i_j}(\bar{x}), x - \hat{x} \rangle + o(\|x - \hat{x}\|)$, it is not difficult

to prove, that $h_{i_j}(x)$ is an $\varepsilon - C^1$ -perturbation of g_{i_j} if we choose a

and δ small enough. Similarly $f''(x)$ is an $\varepsilon - C^1$ -perturbation of f .

Choosing also the b_t small enough and defining $D' \cap U$ by the

inequalities $h_1(x) \leq 0, \dots, h_n(x) \leq 0, k_t(x) \leq 0; t = \{1, \dots, m\} \setminus \{i_1, \dots, i_n\}$

and taking $D' \cap (\mathbb{R}^n \setminus U) = D \cap (\mathbb{R}^n \setminus U)$ we find a generalized $\varepsilon - C^1$ -perturbation (D', f'') of (D, f) such that $\hat{x} \in D'$ and

$$\nabla f''(x) = \mu_1' \nabla h_1(\hat{x}) + \dots + \mu_n' \nabla h_n(\hat{x}); \mu_i > 0 \text{ and } k_t(\hat{x}) < 0$$

This means that \hat{x} is in any case a local solution of (D', f'') , and as h_1, \dots, h_n are linear in V' we know that $f''(\hat{x}) > f'(x)$ if $\|\hat{x} - x\| < a$.

Furthermore $|f''(x) - f'(x)| \leq \|w\| \cdot \|x - \hat{x}\|$. Thus by multiplying w if necessary by a very small positive constant we can assume that

$$|f''(x) - f'(x)| < a\varepsilon. \text{ Now define}$$

$$f'''(x) = f''(x) \cdot \psi_2(x) - (1 - \psi_2(x))a\varepsilon \text{ where}$$

$$\psi_2(x) = \psi\left(\frac{1}{2a}(x - \hat{x})\right). \text{ Then } f'''(x) \text{ is an } \varepsilon - C^1\text{-perturbation of } f'' \text{ and}$$

$$f'''(\hat{x}) = f''(\hat{x}), f'''(x) \leq f''(x) \text{ for all } x, \text{ and } f'''(x) \leq f''(x) \text{ for}$$

$x \notin V'$. It follows that (D', f''') has exactly one solution. Taking everything together we have found a generalized $3\varepsilon - C^1$ -perturbation (D', f''') of (D, f) which is nice.

q.e.d.

5.3. Proposition.

Let (D, f) be a nice programming problem in \mathcal{MP}_1^b . Then there

is an $\varepsilon > 0$ such that every generalized ε - C^1 -perturbation of (D, f) is in \mathcal{M}_1^h and has exactly one solution.

Proof. Because (D, f) is nice it has exactly one solution \hat{x} and $\nabla f(\hat{x}) \neq 0$. Let V be a small neighbourhood of \hat{x} such that $\nabla f(x) \neq 0$ for all $x \in V$. For sufficiently small ε all ε - C^1 -perturbations (D', f') have all their solutions in V and $\nabla f'(x) \neq 0$ for all $x \in V \cap D'$. (D', f') therefore has no solutions in the interior of D^0 . This proves the first statement.

Now let V be a neighbourhood of \hat{x} such that $D \cap V$ is described by n functions g_1, \dots, g_n . Let $\nabla f(\hat{x}) = \lambda_1 \nabla g_1(\hat{x}) + \dots + \lambda_n \nabla g_n(\hat{x})$; $\lambda_i > 0$

Let δ be a small positive number such that

(i) for all w, v_1, \dots, v_n such that $\|w - \nabla f(\hat{x})\| < \delta$, $\|v_i - \nabla g_i(\hat{x})\| < \delta$ there are $\lambda'_1, \dots, \lambda'_n$ such that $w = \lambda'_1 v_1 + \dots + \lambda'_n v_n$, $\lambda'_i > 0$

(ii) if $\|w - \nabla f(\hat{x})\| < \delta$ and v_1, \dots, v_n are such that for all i there is a j such that $\|v_i - \nabla g_j(\hat{x})\| < \delta$ and there is an i_0 such that $\|v_{i_0} - \nabla g_{i_0}(\hat{x})\| > \delta$ for all i then there are no μ_1, \dots, μ_n ,

$\mu_i \geq 0$ such that $w = \mu_1 v_1 + \dots + \mu_n v_n$.

(It is not difficult to see that such a δ exists). Shrinking V , if necessary, we can assume that $\|\nabla g_i(x) - \nabla g_i(\hat{x})\| < \frac{1}{2} \delta$ for all i and $x \in V$. Choose ε_1 such that all solutions of an ε_1 - C^1 -perturbation of (D, f) are necessarily in V . Now let $\varepsilon = \min\{\frac{1}{3}\delta, \varepsilon_1\}$ and (D', f') be a generalized ε - C^1 -perturbation of (D, f) . Let \hat{x}' be a solution of (D', f') and v_1, \dots, v_m be the normals in \hat{x}' . Because \hat{x}' is a solution we have that $\nabla f'(\hat{x}') = \mu_1 v_1 + \dots + \mu_m v_m$, $\mu_i \geq 0$

and by Caratheodory's theorem, there are v_{i_1}, \dots, v_{i_n} such that

$$(5.3.1) \quad \nabla f'(\hat{x}') = \mu'_1 v_{i_1} + \dots + \mu'_n v_{i_n}; \quad \mu'_i \geq 0.$$

Because of property (ii) this means that for each i_j there is

precisely one $\varepsilon_{k,j}$ such that $\|v_{i,j} - \nabla \varepsilon_{k,j}(\hat{x})\| < \delta$

$$(5.3.2) \quad \|v_{i,j} - \nabla \varepsilon_{k,j}(\hat{x})\| < \delta$$

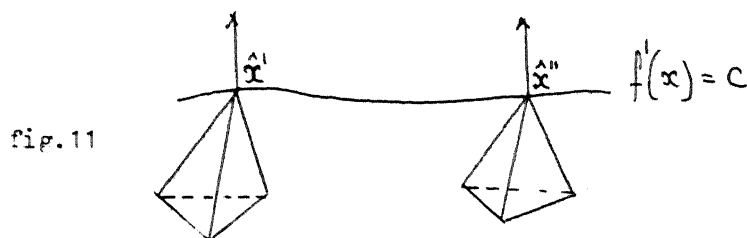
Now suppose that there is a second solution \hat{x}'' of (D', f') then we would similarly have:

$$(5.3.3) \quad \nabla f'(\hat{x}'') = \mu_1'' v_{i_1}'' + \dots + \mu_n'' v_{i_n}'' \text{ and for each } i_j \text{ a } l_j \text{ such}$$

that

$$(5.3.4) \quad \|v'_{i,j} - \nabla \varepsilon_{l_j}(\hat{x}'')\| < \delta$$

Because also $\|\nabla f'(\hat{x}) - \nabla f'(\hat{x}'')\| < \delta$ and because of property (i) we have that (D', f') must look like the drawing below



But this is not a generalized ε - C^1 -perturbation of (D, f) . Which proves the proposition.

5.4. Corollary. The set of programming problems (D, f) in \mathcal{M}_1^b with

exactly one solution contains an open dense set

5.5. Corollary. There is an open dense set in \mathcal{M}_1^b on which the function S_{01} is continuous.

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