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ON $Ext_k^i(G_a, G_a)$ FOR NOT NECESSARILY PERFECT BASE FIELDS k

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1. INTRODUCTION

Let k be a not necessarily perfect base field; κ an extension of k In this note we continue our study of the natural map

$$\phi^i: \kappa \otimes_k \mathrm{Ext}^i_k(G, G_a) \quad \mathrm{Ext}^i_\kappa(G_\kappa, G_{a,\kappa})$$

where G_a is the additive group over k, $G \in \underline{G}_k$, the category of commutative algebraic group schemes over k. For definition of the left-k-vector space structure on $\mathrm{Ext}^i_k(G, G_a)$, cf. [5] of Section 2 below.

As was remarked in [5] it only remained to show that ϕ^i is an isomurphism for all i in the case $G = G_a$, to prove that ϕ^i is an isomurphism for all algebraic extensions κ/k, with κ perfect and all $G \in \underline{G}_k$.

This is the subject matter of Sections 2-4 below.

In Section 5 we study the cohomological dimension of $\underline{G}_k$ in case k is not necessarily perfect. Let $\underline{\mathrm{Un}}_k$ respectively $\underline{\mathrm{Mult}}_k$ be the full subcategories of $\underline{G}_k$ consisting of the unipotent groups respectively the groups of multiplicative type. If k is perfect it turns out that

$$\mathrm{cohdim}\ (\underline{G}_k) = \max\ (\mathrm{cohdim}\ \underline{\mathrm{Un}}_k,\ \mathrm{cohdim}\ \underline{G}_k)$$

cf. Section 5 and [6].

If k is not perfect this is no longer necessarily true (cf. 5.11). This is due to the existence of nonsplitting exact sequences of type $0 \to M \to L \to U \to 0$, $M \in \underline{\mathrm{Mult}}_k$, $U \in \underline{\mathrm{Un}}_k$. The best we are able to prove is that

$$\mathrm{cohdim}\ (\underline{G}_k) \leq \mathrm{cohdim}\ \underline{\mathrm{Un}}_k + \mathrm{cohdim}\ \underline{\mathrm{Mult}}_k + 1$$

A diagram is called exact if it is commutative and all its rows and columns are exact.

$Ext_k^i(-, -)$ denotes the i-th Yoneda extension group in the category $\underline{G}_k$. The characteristic p of k is assumed to be positive. We use the same notations and conventions as in [5]; $\bar{k}$ denotes the algebraic closure of k. The results concerning the properties ESC and ESD of [5] and [5] (3.11) and (4.7) will be used occasionally without explicit reference.

2. $Ext_k^1(G_a, G_a)$ AND $Ext_k^1(W_\infty, G_a)$

G_a denotes the additive group. For each $n \in N$ let W_n be the ring scheme over k of the Witt vectors of length n. There are various natural group scheme homomorphisms between the W_n, given on points by (S a k-algebra)

$$F: W_n(S) \to W_n(S), \quad (x_0, \dots, x_{n-1}) \to (x_0^p, \dots, x_{n-1}^p)$$
$$V: W_n(S) \to W_n(S), \quad (x_0, \dots, x_{n-1}) \to (0, x_0, \dots, x_{n-2})$$
$$T: W_n(S) \to W_{n+1}(S), \quad (x_0, \dots, x_{n-1}) \to (0, x_0, \dots, x_{n-1})$$
$$R: W_{n+1}(S) \to W_n(S), \quad (x_0, \dots, x_n) \to (x_0, \dots, x_{n-1})$$

W_∞ denotes the progroup scheme $(W_n; R^{n-m}: W_n \to W_m, m < n)$.

Let $\xi \in Ext_k^1(U, G_a)$ be represented by (*): $0 \to G_a \to E \to U \to 0$ for each $c \in k$, let $\phi_c: G_a \to G_a$ be the natural map "multiplication with c". Let $c\xi$ be the element represented by the pushout of (*) along ϕ_c and let $F\xi$ be the element represented by the pushout of (*) along $F: G_a \to G_a$. This turns $Ext_k^1(U, G_a)$ into a left k[F] module, where

$$k[F] = \{ \sum^{<\infty} a_i F^i \mid a_i \in k\}; \text{ multiplication rule: } Fa = a^p F$$

(cf. also [5] and [9], section 7.

2.1. Lemma

Let $\bar{e}_n$ be the element of $\mathrm{Ext}_k^1(W_n, G_a)$ represented by the exact sequence

$$0 \to G_a \overset{T^n}{\to} W_{n+1} \overset{R}{\to} W \to 0$$

(i) $\mathrm{Ext}_k^1(W_n, G_a)$ is the free left k[F]-module generated by $\bar{e}_n$.
(ii) $R^*: \mathrm{Ext}_k^1(W_n, G_a) \to \mathrm{Ext}_k^1(W_{n+1}, G_a)$ is the zero map.

Proof.

Statement (i) for n = 1 is part (i) of [9] Theorem 7.3. Suppose that (i) holds for n. The exact sequence $\bar{e}_n$ gives an exact sequence (of k[F] modules)

$$\mathrm{Ext}_k^1(W_n, G_a) \overset{R^*}{\to} \mathrm{Ext}_k^1(W_{n+1}, G_a) \overset{T^{n*}}{\to} \mathrm{Ext}_k^1(G_a, G_a)$$

Pulling $\bar{e}_n$ back along $W_{n+1} \to W_n$ yields an exact diagram

$$\begin{array}{ccccccccc} 0 & \to & G_a & \to & W_{n+1} & \to & W_n & \to & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \to & G_a & \to & E & \to & W_{n+1} & \to & 0 \end{array}$$

and it is obvious that the lower exact sequence splits. This shows that $R^*\bar{e}_n = 0$, which proves (ii) for n, and shows that T^{n*} is injective.

There is also an exact diagram

$$\begin{array}{ccccccccc} 0 & \to & G_a & \overset{T}{\to} & W_2 & \overset{R}{\to} & G_a & \to & 0 \\ & & \| & & \downarrow T^n & & \downarrow T^n & & \\ 0 & \to & G_a & \to & W_{n+2} & \overset{R}{\to} & W_{n+1} & \to & 0 \end{array}$$

which proves that $T^{n*}(\bar{e}_{n+1}) = \bar{e}_1$, proving that T^{n*} is also surjective and hence bijective. As $\bar{e}_{n+1}$ corresponds to $\bar{e}_1$ this proves (i) for n + 1.
q.e.d.

2.2. Corollary

The natural map $\kappa \otimes_k \mathrm{Ext}^1_k (G_a, G_a) \xrightarrow{m} \mathrm{Ext}^1_\kappa (G_{a\kappa}, G_{a\kappa})$ is an isomorphism for field extensions κ/k.

2.3. Corollary

$$\mathrm{Ext}^1_k (W_\infty, G_a) = 0$$

3. THE PROGROUP SCHEMES $\bar{W}$ AND $\bar{\bar{W}}$

Let $L_{\infty,\infty}$ be the profinite group scheme consisting of the $L_{n,m} := \mathrm{Ker}\,(W_n \xrightarrow{F^n} W_n)$; $L_{\infty,\infty}$ is defined over every field k. Over every algebraically closed field we have an exact sequence

$$0 \to \pi_1(W_\infty) \to \bar{W} \to W_\infty \to 0$$

(cf. [GP], [CGS]; $\pi_1(W_\infty)$ is profinite, the local component of $\pi_1(W\ )$ is $L_{\infty,\infty}$, its etale component is a projective limit of groups $\mathrm{Hom}\,(W_n, Z/p^nZ)$.

This sequence is in fact already defined over the prime field and hence over every field (cf.[7], [4]).

If k is perfect, $\bar{W}$ and $\pi_1(W_\infty)$ are projective. If k is not perfect this is not the case (cf. [5]).

Let A be the ring of $\bar{W}$.

3.1. Lemma

$$\mathrm{Ext}\,(\bar{W}, N) = 0 \text{ if } N \in \underline{\mathrm{FUn}_k} \text{ is etale}$$

Proof.

It suffices to prove this for N a twisted version of $(Z/pZ)^r$. Let

(3.1) $$0 \to N \to E \to \bar{W} \to 0$$

be an exact sequence. Let B be the ring of E and C the ring of N. We know, because $\bar{W}$ is projective over perfect fields, that (3.1) splits after a finite purely inseparable extension ℓ/k. I.e. there exists a morphism of rings

$$C_\ell \overset{r}{\to} B_\ell$$

which composed with the morphism $B \to C$ defining $N \to E$ yields the identity on C. The morphism F^n: $N \to N^{(p^n)}$ is an isomorphism because N is etale (cf. [1], Chapter IV, 5, (5.3). On the ring level we have $N^{(p^n)}$ is spectrum of $C \otimes_k k^{(p^n)}$, and F^n: $N \to N^{(p^n)}$ is given by

$$\phi^{(n)}: C \otimes_k k^{(p^n)} \to C, \ c \otimes 1 \to c^{p^n}$$

it follows, because ℓ/k is purely separable that $r o \phi^{(n)}$ is defined over k for n large enough. Which means that

$$F^n_*: \text{Ext}\ (\bar{W}, N) \to \text{Ext}\ (\bar{W}, N^{(p^n)})$$

kills the element represented by (3.1), and because F^n is an isomorphism this implies that (3.1) represents zero.

3.2. Group like Elements and Witt-like Sequences of Elements of A

Let B be a bialgebra over k. An element $x \in B$ is called grouplike if $m(x) = 1 \otimes x + x \otimes 1$ where $m: B \to B \otimes_k B$ is the comultiplication on B. This is equivalent to saying that the k-algebra homomorphism $k[X] \to B$, defined by $X \mapsto x$ is a morphism of bialgebras, where $k[X]$ has the coalgebra structure $X \mapsto 1 \otimes X + X \otimes 1$.

A sequence of elements $(x_0, \ldots, x_{n-1})$, $x_i \to B$ is called n-Witt-like if the algebra homomorphism $k[X_0, \ldots, X_{n-1}] \to B$, $X_i \mapsto x_i$ is a morphism of bialgebras where $k[X_0, \ldots, X_{n-1}]$ is given the additive Witt coalgebra structure; i.e. $X_i \to \sigma_i(X_0 \otimes 1, \ldots, X_i \otimes 1; 1 \otimes X_0, \ldots, 1 \otimes X_i)$ where the $\sigma_0, \ldots, \sigma_{n-1}$, are the polynomials defining the Witt addition. Note that 1-Witt-like is the same as grouplike. A sequence $(x_0, x_1, \ldots)$ is called Witt-like if $(x_0, x_1, \ldots, x_{n-1})$ is n-Witt-like for all n.

Lemma.

Let A be the algebra of $\bar{W}_k$. For every grouplike element $a \in A$ there exists a Witt like sequence $(a_0, a_1, \ldots)$, $a_i \in A$, $a = a_0$.

Proof.

Let $(a_0, \ldots, a_{n-1})$, $a = a_0$ be n-Witt-like. There is an exact sequence

$$0 \to G_a \to W_{n+1} \to W_n \to 0$$

The sequence $(a_{01}, \ldots, a_{n-1})$ defines a homomorphism $f\colon \bar{W} \to W_n$. We have an exact sequence

$$\text{Ham}(\bar{W}, W_{n+1}) \to \text{Ham}(\bar{W}, W_n) \to \text{Ext}(\bar{W}, G_a)$$

Because $\text{Ext}(\bar{W}, G_a) = 0$ there exists a lift of f, i.e. an (n + 1)-Witt-like sequence $(a_0, \ldots, a_{n-1}, a_n)$ prolonging $(a_0, a_1, \ldots, a_{n-1})$

3.3. The Progroup Scheme $\bar{\bar{W}}$

As above let A be the algebra of $\bar{W}_k$. Let V be the k-vector space of grouplike elements of A. Let $V^{(p)}$ be the k-vector space obtained from V by the change of rings $k \to k$, $\alpha \to \alpha^p$. The homomorphism $A \to A$, $a \to a^p$ maps V into itself and the image W is a sub-k-vector space of $V^{(p)}$. Choose a basis $\{w_\alpha\}$ of $V^{(p)}/W$ and for each w_α let $v_\alpha \in V$ be a lift of w_α. For each v_α let $(v_{\alpha 0}, v_{\alpha 1}, \ldots)$ be a Witt-like sequence prolonging v_α. Define A' as

$$A' = \varinjlim_{n} A[\ldots, X_{\alpha i}, \ldots]/(\ldots, X_{\alpha i}^{p^n} - v_{\alpha i}, \ldots)$$

with the co-algebra structure given by

$$a \to m(a),\ X_{\alpha i} \to \sigma_i(X_{\alpha 0} \otimes 1, \ldots, X_{\alpha i} \otimes 1; 1 \otimes X_{\alpha 0}, \ldots, 1 \otimes X_{\alpha i})$$

where σ_i is the i-th Witt-addition polynomial. This is well defined because the $(v_{\alpha 0}, v_{\alpha 1}, \ldots)$ are Witt-like.

Let $\bar{W}'$ be the pro-algebraic group scheme corresponding to A'. There is an exact sequence

$$0 \to L' \to \bar{W}' \to \bar{W} \to 0$$

where L' is a product of copies of $L_{\infty,\infty}$. Hence Ext $(\bar{W}', G_a) = 0$ because Ext $(\bar{W}, G_a) = 0$ and Ext $(L_{\infty,\infty}, G_a) = 0$ (use [5] (4.7)). This means that also every group like element of A' can be prolonged as a Witt'like sequence (cf. proof of lemma 3.2). Now construct A" from A', in the same way as A' from A. Continuing in this way we find a unipotent pro algebraic group scheme $\bar{\bar{W}}$ with algebra $\bar{A}$ such that the map $V(\bar{A}) \to V(\bar{A})$, $a \to a^p$ of the vector space of grouplike elements of $\bar{A}$ into itself is surjective.

Remark.

In 3.2 and 3.3 we have permitted ourselves to confuse inductive limits of finitely generated bialgebras and inductive systems of finitely generated bialgebras.

This is harmless because the category of commutative unipotent group schemes $\underline{BUn}_k$ is proartinian; its full subcategory of artinian objects is $\underline{Un}_k$, the category of algebraic commutative group schemes, which is artinian, so that

$$\text{Pro}\ (\underline{Un}_k) \simeq \text{Sex}\ (\underline{Un}_k, \text{Ab})^0 \simeq \underline{BUn}_k$$

(cf. [1], Ch. V, §2, [7] I.4, [9] §3).

3.4. Proposition

$\bar{\bar{W}}$ is projective in Pro($\underline{Un}_k$)

Proof.

$\bar{\bar{W}}$ is an extension of $\bar{W}$ by a projective limit of copies of $L_{\infty,\infty}$ and hence Ext $(\bar{\bar{W}}, G_a) = 0$ because Ext $(\bar{W}, G_a) = 0$ and Ext $(L_{\infty,\infty}, G_a) = 0$ Further Ext $(\bar{\bar{W}}, E) = 0$ if $E \in \underline{FUn}_k$ is etale because Ext $(\bar{W}, E) = 0$ (3.1) and Ext $(L_{\infty,\infty}, E) = 0$, because every exact sequence $0 \to E \to E' \to E'' \to 0$, E etale and E" local splits. It remains to show that Ext $(\bar{\bar{W}}, \alpha_p) = 0$. The exact sequence $0 \to \alpha_p \to G_a \xrightarrow{F} G_a \to 0$ gives an exact sequence

$$\mathrm{Hom}(\bar{\bar{W}}, G_a) \xrightarrow{F*} \mathrm{Hom}(\bar{\bar{W}}, G_a) \to \mathrm{Ext}(\bar{\bar{W}}, \alpha_p) \to \mathrm{Ext}(\bar{\bar{W}}, G_a)$$

The map $F*$ is surjective because $V(\bar{A}) \to V(\bar{A})$, $a \to a^p$ is surjective; and $\mathrm{Ext}(\bar{\bar{W}}, G_a) = 0$. This shows that $\mathrm{Ext}(\bar{\bar{W}}, \alpha_p) = 0$.

3.5. Remark

Let Π be the kernel of $\bar{\bar{W}} \to W_\infty$. The pro-algebraic group schemes $\Pi_{\bar{k}}$ and $\bar{\bar{W}}_{\bar{k}}$ are then projective in Pro $(G_{\bar{k}})$, because $\bar{W}_{\bar{k}}$, $\pi_1(W_\infty)_{\bar{k}}$ and $L_{\infty,\infty,\bar{k}}$ are projective in Pro $(G_{\bar{k}})$.

4. THE GROUPS $\mathrm{Ext}^i_k(G_a, G_a)$ for $i \geq 2$

Because the embedding $\underline{Un}_k \to \underline{Lin}_k$ is ESC and $\underline{Lin}_k \to \underline{G}_k$ is ESD we have $\mathrm{Ext}^i_k(G_a, G_a) \simeq \mathrm{Ext}^i_{\underline{Un}_k}(G_a, G_a)$. Both extension functions shall be denoted Ext^i.

4.1. Lemma

$$\mathrm{Ext}^i(W_\infty, G_a) = 0 \quad \text{for all} \quad i = 1, 2, \ldots$$

Proof.

For $i = 1$ this is 2.3. For $i > 1$ we have an exact sequence

$$\mathrm{Ext}^{i+1}(\Pi, G_a) \to \mathrm{Ext}^i(W_\infty, G_a) \to \mathrm{Ext}^i(\bar{\bar{W}}, G_a)$$

deduced from $0 \to \Pi \to \bar{\bar{W}} \to W_\infty \to 0$. Now $\mathrm{Ext}^i(\bar{\bar{W}}, G_a) = 0$ because of (3.4) and $\mathrm{Ext}^{i-1}(\Pi, G_a) = 0$ because $\Pi_{\bar{k}}$ is projective in $\underline{G}_k$ and [5] (4.7) q.e.d.

4.2. Proposition

$$\mathrm{Ext}^i_k(G_a, G_a) = 0 \quad \text{for} \quad i = 2, 3, \ldots$$

Proof.

This follows from 4.1 by means of the exact sequence

$$0 \to W_\infty \to W_\infty \to G_a \to 0$$

4.3. Corollary

$$\mathrm{Ext}^i\,(U_1, U_2) = 0 \quad \text{of} \quad i \geq 3,\ U_1, U_2 \in \underline{\mathrm{Un}}_k$$

Proof.

It suffices to prove this for the four cases

(i) $\mathrm{Ext}^i\,(G_a, G_a)$

(ii) $\mathrm{Ext}^i\,(G_a, N)$

(iii) $\mathrm{Ext}^i\,(N, G_a)$

(iv) $\mathrm{Ext}^i\,(N, N')$

where N, N' are finite subgroup schemes of G_a. Case (i) follows from 4.2; case (iii) follows from [5] (4.7)

Case (ii). There is an exact sequence $0 \to N \to G_a \to G_a \to 0$ (cf. [1], Ch. IV, 2.1) which gives us an exact sequence

$$\mathrm{Ext}^{i-1}\,(G_a, G_a) \to \mathrm{Ext}^i\,(G_a, N) \to \mathrm{Ext}^i\,(G_a, G_a)$$

which proves case (ii) because of 4.2.

Case (iv). We have an exact sequence $0 \to N' \to G_a \to G_a \to 0$, yielding an exact sequence

$$\mathrm{Ext}^{i-1}\,(N, G_a) \to \mathrm{Ext}^i\,(N, N') \to \mathrm{Ext}^i\,(N, G_a)$$

It follows that $\mathrm{Ext}^i\,(N', N) = 0$ for $i \geq 3$ because $\mathrm{Ext}^i\,(N, G_a) = 0$ for $i \geq 2$ (cf. [5] (4.7)).

We recall that the cohomological dimension of an abelian category $\mathcal{C}$, denoted cohdim ($\mathcal{C}$) is the smallest number of such that $\mathrm{Ext}_{\mathcal{C}}^{d+1}\,(C_1, C_2) = 0$ for all $C_1, C_2 \in \mathcal{C}$.

4.4. Corollary

For all fields k of positive characteristic cohdim $(\underline{\mathrm{Un}}_k) = 2$.

4.5. Theorem

The natural map

$$\kappa \otimes_k \mathrm{Ext}_k^i (G, G_a) \simeq \mathrm{Ext}^i (G_\kappa, G_{a,\kappa})$$

is an isomorphism for all algebraic extensions κ/k such that κ is perfect, $G \in \underline{G}_k$.

Proof. This follows from [5], (4.7), and 4.2. and 2.2. above, cf. also [5] (5.3).

5. ON THE COHOMOLOGICAL DIMENSION OF $\underline{G}_k$ FOR NOT NECESSARILY PERFECT BASE FIELDS k

We first recall some usefull facts about Yoneda extensions. The putting together of two exact sequences

$$0 \to A \to G_1 \to G_2 \to \dots \to G_i \to C \to 0 \quad \text{and} \quad 0 \to C \to G_{i+1} \to \dots G_{i+j} \to B \to 0$$

to get a longer exact sequence

$$0 \to A \to G_1 \to \dots \to G_i \to G_{i+1} \to \dots \to G_{i+j} \to B \to 0$$

induces a multiplication (denoted with .)

$$\mathrm{Ext}^i (C, A) \times \mathrm{Ext}^j (B, C) \to \mathrm{Ext}^{i+j} (B, A)$$

With respect to this multiplication one has (immediately from the definitions)

$$g^* \xi . \eta = \xi . g_* \eta$$

if $g: G \to C$ is a homomorphism, $\xi \in \mathrm{Ext}^i (C, A)$, $\eta \in \mathrm{Ext}^j (B, G)$.

5.1. Lemma

$\mathrm{Ext}_k^i(G_1, G_2)$ is a torsion group for all $G_1, G_2 \in G_k$ and $i \geq 2$.

Proof.

It suffices to prove this in the nine cases $\mathrm{Ext}^i(U_1, U_2)$, $\mathrm{Ext}^i(U_1, M)$, $\mathrm{Ext}^i(U, A)$, $\mathrm{Ext}^i(M, U)$, $\mathrm{Ext}^i(M_1, M_2)$, $\mathrm{Ext}^i(M, A)$, $\mathrm{Ext}^i(A, U)$, $\mathrm{Ext}^i(A, M)$, $\mathrm{Ext}^i(A_1, A_2)$, where $U, U_1, U_2 \in \underline{\mathrm{Un}}_k$, $M, M_1, M_2 \in \underline{\mathrm{Mult}}_k$, A, A_1, A_2 abelian varieties. Cases 1, 2, 3, 4, 7 follow from the fact that a unipotent group is killed by some power of $p = \mathrm{char}(k)$. Cases 6, 9 follow from the fact that $\mathrm{Ext}(G, A)$ is torsion if A is an abelian variety (cf. [7], [8], section 7.4). It remains to deal with case 5 and case 8.

Case 5.

The functor $M \mapsto \mathrm{Hom}_k(M, G_m)$ is an antiequivalence of $\underline{\mathrm{Mult}}_k$ with the category $\underline{\mathrm{Mod}}\ \mathfrak{g}$ = Gal $(\bar{k}/k)$-modules which are finitely generated as abelian groups and on which $\mathfrak{g} = \mathrm{Gal}(\bar{k}/k)$ acts continuously. Because $\mathbb{Q}[\mathfrak{g}/\mathfrak{h}]$ is semisimple, $\mathfrak{h} \not\subset \mathfrak{g} = \mathrm{Gal}(\bar{k}/k)$ an open subgroup of finite index there is for every situation

$$\begin{array}{ccccc} & & D & & \\ & & \downarrow g & & \\ D_1 & \xrightarrow{f} & D_2 & \rightarrow & 0 \end{array}$$

in $\underline{\mathrm{Mod}}\,\mathfrak{g}$, an integer n such that $D \xrightarrow{n} D \xrightarrow{g} D_2$ lifts to D_1. This shows that the $\mathrm{Ext}^i_{\mathrm{Mod}(\mathfrak{g})}(\ ,\ )$ are torsion, $i > 1$ and hence that the $\mathrm{Ext}^i_{\underline{\mathrm{Mult}}_k}(\ ,\ )$ are torsion for $i \geq 1$ which implies that the $\mathrm{Ext}_k^i(M_1, M_2)$ are torsion for $i \geq 1$ because $\underline{\mathrm{Mult}}_k \subset \underline{G}_k$ has ESD (cf. [5]).

Case 8.

Let $i \geq 2$ and let

$$0 \rightarrow M \rightarrow G_1 \rightarrow G_2 \rightarrow \ldots \rightarrow G_i \rightarrow A \rightarrow 0$$

represent an element ξ of $\mathrm{Ext}^i(A, M)$. Let G be the kernel of $G_i \rightarrow A$.

There is an exact sequence $0 \to L \xrightarrow{\alpha} G \xrightarrow{\beta} B \to 0$, $L \in \underline{Lin_k}$, B an abelian variety. The element ξ is the product of the elements ξ_1 and ξ_2 represented by $0 \to M \to G_1 \to \dots \to G_{i-1} \to G \to 0$ and $0 \to G \to G_i \to A \to 0$. The image of ξ_2 in Ext (A, B) is torsion. Therefore there exists an $n \in N$ such that $n\xi_2$ becomes zero in Ext (A, B).
There is therefore an $\xi_2' \in$ Ext (A, L) such that $n\xi_2 = \alpha\,\xi_2'$. We now have $n\xi = \xi_1 . n\xi_2 = \xi_1 . \alpha_* \xi_2' = \alpha^* \xi_1 . \xi_2'$. But $\alpha^* \xi_1 \in \mathrm{Ext}^{i-1}$ (L, M) is torsion because of cases 4, 5, which proves that $n\xi$ and hence also ξ is torsion.

5.2. Proposition

$$\mathrm{cohdim}\ (\underline{G_k}) = \mathrm{cohdim}\ (\underline{Lin_k})$$

Proof.

Let cohdim $(\underline{Lin_k}) = d$. Note that $d \geq 1$.
Because $\underline{Lin_k} \hookrightarrow \underline{G_k}$ is an ESD embedding it suffices to show that $\mathrm{Ext}_k^{d+1}(L, A) = 0$, $\mathrm{Ext}_k^{d+1}(A, L) = 0$, $\mathrm{Ext}^{d+1}(B, A)$ where $L \in \underline{Lin_k}$, A, B abelian varieties (cf. [5] (3.11)). Let $\xi \in \mathrm{Ext}^{d+1}$ (L, A) and let $n\xi = 0$, we have an exact sequence

$$\mathrm{Ext}_k^{d+1}\ (L, {}_nA) \to \mathrm{Ext}_k^{d+1}\ (L, A) \xrightarrow{\times n} \mathrm{Ext}_k^{d+1}\ (L, A)$$

where ${}_nA$ is the kernel of $A \xrightarrow{\times n} A$. But Ext_k^{d+1} (L, ${}_nA$) = 0 because L, ${}_nA \in \underline{Lin_k}$ and $\underline{Lin_k} \subset \underline{G_k}$ is an ESD embedding. This shows that $\xi = 0$, and we have proved that Ext_k^{d+1} (L, A) = 0, $L \in \underline{Lin_k}$, A an abelian variety.

To deal with Ext_k^{d+1} (B, A) we need a lemma.

5.3. Lemma

Let codim ($\underline{Lin}_k$) = d, then Ext^d_k (N, A) = 0 if A is an abelian variety and

$$N \in \underline{FG}_k, \text{ or } N \in \underline{Un}_k$$

Proof. Let n ∈ N be such that nN = 0 we have an exact sequence

$$Ext^d_k (N, A) \xrightarrow{xn=0} Ext^d_k (N, A) \to Ext^{d+1}_k (N, {}_nA)$$

It follows that Ext^d_k (N, A) = 0 because $Ext^{d+1}_k (N, {}_nA) = 0$

5.4. Proof that Ext^{d+1}_k (A, B) = 0; A, B abelian varieties

Let $\xi \in Ext^{d+1}_k$ (A, B), $n\xi = 0$. The exact sequence $0 \to {}_nA \to A \to A \to 0$ gives an exact sequence

$$Ext^d_k ({}_nA, B) \to Ext^{d+1}_k (A, B) \xrightarrow{xn} Ext^{d+1}_k (A, B) \to Ext^{d+1}_k ({}_nA, B)$$

Now use 5.3 to conclude that $\xi = 0$.

5.5. Proof that Ext^{d+1}_k (A, L) = 0; A an abelian variety, L ∈ $\underline{Lin}_k$.

It suffices to do this in the cases (i)

(i) L ∈ $\underline{FG}_k$;

(ii) L ∈ $\underline{Un}_k$;

(iii) L ∈ $\underline{Mult}_k$; such that $Hom_k(L, G_m)$ is torsion free (i.e. L a twisted torus).

(Every Gal ($\bar{k}/k$)-module decomposes as $0 \to T_{tor} \to T \to T_{free} \to 0$ where T_{tor} is the torsion subgroup of T and $T/T_{tor} = T_{free}$ is torsion free.)

Cases (i) and (ii). Let n be such that nL = 0. We have an exact sequence

$$Ext^{d+1}_k (A, L) \xrightarrow{xn=0} Ext^{d+1}_k (A, L) \to Ext^{d+1}_k ({}_nA, L)$$

proving that Ext^{d+1}_k (A, L) = 0.

Case (iii). Let $\xi \in \mathrm{Ext}_k^{d+1}(A, L)$, $L \in \underline{\mathrm{Mult}}_k$ a twisted torus, $n\xi = 0$. Because L is a twisted torus we have an exact sequence $0 \to {}_nL \to L \xrightarrow{xn} L \to 0$. This gives us an exact sequence

$$\mathrm{Ext}_k^{d+1}(A, {}_nL) \to \mathrm{Ext}_k^{d+1}(A, L \xrightarrow{xn} \mathrm{Ext}_k^{d+1}(A, L)$$

which proves that $\xi = 0$ because $\mathrm{Ext}_k^{d+1}(A, {}_nL) = 0$ in virtue of case (i)

5.6. Remarks on cohdim ($\underline{\mathrm{Lin}}_k$) if k is Perfect

If k is perfect every object L of $\underline{\mathrm{Lin}}_k$ decomposes as a direct sum: $L = M_L \oplus U_L$ with $M_L \in \underline{\mathrm{Mult}}_k$, $U_L \in \underline{\mathrm{Un}}_k$, and every morphism $\phi : L \to L'$ decomposes as a direct sum of morphims ϕ_M $M_L \to M_{L'}$, $\phi_U: U_L \to U_{L'}$. It follows that

$$\mathrm{cohdim}\,(\underline{\mathrm{Lin}}_k) = \max\{\mathrm{cohdim}\,(\underline{\mathrm{Un}}_k), \mathrm{cohdim}\,(\underline{\mathrm{Mult}}_k)\}$$

if k is perfect. If k is not perfect this is not necessarily true. Cf. example (5.11) below.

5.7. Remark on cohdim ($\underline{\mathrm{Mult}}_k$)

Using similar arguments as in (5.5), especially case (iii), one easily shows that cohdim ($\underline{\mathrm{Mult}}_k$) = cohdim ($\underline{\mathrm{FMult}}_k$). (Use also that $\underline{\mathrm{FMult}}_k \subset \underline{\mathrm{Mult}}_k$ is an ESD embedding). The category $\underline{\mathrm{FMult}}_k$ is equivalent to the category F-$\underline{\mathrm{Mod}}\,\mathfrak{g}$ of finite $\mathfrak{g} = \underline{\mathrm{Gal}}\,(\bar{k}/k)$-mudules, where $\bar{k}$ is the algebraic dosure of k. The cohomological dimension of this category is related to the cohomological dimension of the profinite group Gal ($\bar{k}$/k) in the sense of galois cohomology by means of a spectral sequence of the "change of rings type", which is obtained as follows. (Cf. [6].

For a fixed finite $\mathfrak{g} = \mathrm{Gal}(\bar{k}/k)$ module M, let α be the functor $N \to Hom(M, N)$, homomorphisms as abelian groups; $Hom(M, N)$ has a natural $\mathfrak{g}$-module structure and is finite if N is finite. Let β be the functor $L \to L^{\mathfrak{g}}$, L a finite $\mathfrak{g}$-module; $L^{\mathfrak{g}}$ its submodule of $\mathfrak{g}$-invariant elements. The composed functor $\beta 0 \alpha$ is precisely the functor $\gamma: N \to Hom_{\mathfrak{g}}(M, N)$, where $Hom_{\mathfrak{g}}$ denotes the $\mathfrak{g}$-homomorphisms. This decomposition of γ gives rise to a spectral sequence (cf. [2], [6]; some properties must of course be verified).

$$H_i^i(\mathfrak{g}, \mathrm{Ext}_Z^j(M, N) \Rightarrow \mathrm{Ext}^{i+j}_{\mathrm{Fmod}(\mathfrak{g})}(M, N)$$

One deduces from this that cohdim $(\mathrm{FMod}(\mathfrak{g})) = 1 + \mathrm{cd}(\mathfrak{g})$, where $\mathrm{cd}(\mathfrak{g})$ is the cohomological dimension of $\mathfrak{g}$ in the sense of galois cohomology.

5.8. Proposition

$$\mathrm{cohdim}\,(\underline{\mathrm{Lin}}_k) \leq \mathrm{cohdim}\,(\underline{\mathrm{Un}}_k) + \mathrm{cohdim}\,(\underline{\mathrm{Mult}}_k) + 1$$

Proof.

Let $d_m = \mathrm{cohdim}\,(\underline{\mathrm{Mult}}_k)$, $d_u = \mathrm{cohdim}\,(\underline{\mathrm{Un}}_k)$. Because $\underline{\mathrm{Mult}}_k \subset \underline{\mathrm{Lin}}_k$ is an ESD embedding and $\underline{\mathrm{Un}}_k \subset \underline{\mathrm{Lin}}_k$ is an ESC embedding and the fact that every $L \in \underline{\mathrm{Lin}}_k$ admits an exact séquence $0 \to M \to L \to U \to 0$, $M \in \underline{\mathrm{Mult}}_k$, $U \subset \underline{\mathrm{Un}}_k$, it suffices to prove that

(i) $\mathrm{Ext}^{d_m+d_u+2}_{\underline{\mathrm{Lin}}_k}(M, U) = 0$ and

(ii) $\mathrm{Ext}^{d_m+d_u+2}_{\underline{\mathrm{Lin}}_k}(U, M) = 0$, where $U \in \underline{\mathrm{Un}}_k$ and $M \in \underline{\mathrm{Mult}}_k$

Proof of (i).

We shall show that $\mathrm{Ext}^i(M, U) = 0$ for all $i \geqslant 1$, $U \in \underline{\mathrm{Un}}_k$, $M \in \underline{\mathrm{Mult}}_k$. For $i = 1$ this is clear because every exact sequence $0 \to U \to L \to M \to 0$ splits. We proceed by induction. Let $0 \to U \to L_1 \to L_2 \to \dots \to L_n \to M \to 0$ represent an element ξ in $\mathrm{Ext}^n(M, U)$; let $L = \mathrm{coker}\,(U \to L_1)$ and let ξ_1 and ξ_2 be the elements in $\mathrm{Ext}^1(L, U)$ and $\mathrm{Ext}^{n-1}(M, L)$ represented by $0 \to U \to L_1 \to L \to 0$ and $0 \to L \to L_2 \to \dots \to L_n \to M \to 0$. There is an exact sequence $0 \to M' \xrightarrow{j} L \xrightarrow{q} U' \to 0$ and because

$\mathrm{Ext}^1(M', U) = 0$ there is an $\xi_1' \in \mathrm{Ext}(U', U)$ such that $q^*\xi_1' = \xi_1$.
We have $\xi = \xi_1.\xi_2 = q^*\xi_1'.\xi_2 = \xi_1'.q_*\xi_2$. But $q_*\xi_2 \in \mathrm{Ext}^{n-1}(M, U')$ is zero by induction hypothesis, hence $\xi = 0$.

Proof of (ii).

Let $0 \to M \to L_1 \to \dots \to L_{d_m+1} \to L_{d_m+2} \to \dots \to L_{d_m+d_u+2} \to U \to 0$ represent $\xi \in \mathrm{Ext}^{d_m+d_u+2}(U, M)$. Let $L = \mathrm{Ker}(L_{d_m+2} \to L_{d_m+3})$. And let ξ_1, ξ_2 be the elements represented by the exact sequences

$$0 \to M \to L_1 \to \dots \to L_{d_m+1} \to L \to 0$$

$$0 \to L \to L_{d_m+2} \to \dots \to L_{d_m+d_u+2} \to U \to 0$$

There is an exact sequence $0 \to M' \xrightarrow{i} L \xrightarrow{q} U' \to 0$, $M' \in \underline{\mathrm{Mult}}_k$, $U' \in \underline{\mathrm{Un}}_k$. Because $i^*\xi_1 = 0$ there is an $\xi_1' \in \mathrm{Ext}^{d_m+1}(U', M)$ such that $\xi_1 = q^*\xi_1'$. We have $\xi = \xi_1.\xi_2 = q^*\xi_1'.\xi_2 = \xi_1'.q_*\xi_2$.
But $q_*\xi_2 \in \mathrm{Ext}^{d_u+1}(U, U') = 0$; hence $\xi = 0$.

5.9. Corollary

$$\mathrm{cohdim}\,(\underline{G}_k) \leq 3 + \mathrm{cohdim}\,(\underline{\mathrm{Mult}}_k)$$

(and cohdim ($\underline{\mathrm{Mult}}_k$) depends only on the galoisgroup $\mathfrak{g} = \mathrm{Gal}(\bar{k}/k)$, cf. 5.7.

5.10. An Example

Let k be a nonperfect field. Then these exists a nonsplitting exact sequence $0 \to \mu_p \to E \to \alpha_p \to 0$. Cf, e,g, [3]. Exp XVII (6.4); or let $0 \to \alpha_p \to E' \to \mathbb{Z}/(p) \to 0$ be the exact sequence corresponding to an $k[U]/(U^p - U) \to k[X, Y]/(X^p - X), Y^p - aX) \to k[Z]/(Z^p)$, $U \mapsto X$; $X \mapsto 0$, $Y \mapsto Z$, and comultiplications $U \mapsto 1 \otimes U + U \otimes 1$; $X \mapsto 1 \otimes X + X \otimes 1$, $Y \mapsto 1 \otimes Y + Y \otimes 1$; $Z \mapsto Z \otimes 1 + 1 \otimes Z$. This sequence does not split if $a \notin k^p$. Now take the dual of $0 \to \alpha_p \to E' \to \mathbb{Z}/(p) \to 0$.

Further we know that (cf. [3] Exp XVII, (6.1))

$$\mathrm{Ext}_k^1(G_a, M) = 0 \text{ if } M \in \underline{\mathrm{Mult}}_k$$

The exact sequence $0 \to \alpha_p \to G_a \to G_a \to 0$ gives an exact sequence

$$\mathrm{Ext}^1 (G_a, \mu_p) \to \mathrm{Ext}^1 (\alpha_p, \mu_p) \to \mathrm{Ext}^2 (G_a, \mu_p)$$

which shows that $\mathrm{Ext}_k^2 (G_a, \mu_p) \neq 0$ if k is not perfect.

The exact sequence $0 \to \mu_p \to G_m \xrightarrow{xp} G_m \to 0$ gives an exact sequence

$$\mathrm{Ext}^1 (G_a, G_m) \to \mathrm{Ext}^2 (G_a, \mu_p) \to \mathrm{Ext}^2 (G_a, G_m) \xrightarrow{xp=0} \mathrm{Ext}^2 (G_a, G_m) \to$$

$$\mathrm{Ext}^3 (G_a, \mu_p) \to \mathrm{Ext}^3 (G_a, G_m) \xrightarrow{xp=0} \mathrm{Ext}^3 (G_a, G_m) \to \mathrm{Ext}^4 (G_a, \mu_p)$$

Because $\mathrm{Ext}^1 (G_a, G_m) = 0$ we find $\mathrm{Ext}^2 (G_a, \mu_p) \simeq \mathrm{Ext}^2 (G_a, G_m)$. We have an exact sequence

$$0 \to \mathrm{Ext}^2 (G_a, G_m) \to \mathrm{Ext}^3 (G_a, \mu_p) \to \mathrm{Ext}^3 (G_a, G_m) \to 0$$

which shows that

$$\mathrm{Ext}_k^3 (G_a, \mu_p) \neq 0$$

if k is not perfect.

5.11. Remark

As we have seen cohdim ($\underline{\mathrm{Un}}_k$) = 2 for all k. Let k be nonperfect separately closed. Then cohdim ($\underline{\mathrm{Mult}}_k$) = 1 so that $\mathrm{Ext}_k^i(-, -) = 0$ for $i \geq 5$. On the other hand $\mathrm{Ext}^3 (-, -)$ is not necessarily zero (example (5.10)) which would have to be the case if cohdim ($\underline{\mathrm{Lin}}_k$) = max {cohdim $\underline{\mathrm{Un}}_k$, cohdim Mult_k} were true.

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