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ON $\mathrm{Ext}_k^i(N, G_a)$ FOR FINITE GROUP SCHEMES N OVER NOT NECESSARILY PERFECT BASE FIELDS k.

by

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Contents

1. INTRODUCTION

Let k be a field and G_k the category of commutative algebraic group schemes over k. One can define the extension groups $\mathrm{Ext}_k^i(-,-)$ in G_k by means of Yoneda extensions. Let $\xi \in \mathrm{Ext}_k^i(G, G_a)$ be represented by

$$(*) \qquad 0 \to G_a \to G_1 \to G_2 \to \dots \to G_i \to G \to 0$$

$G \in G_k$ and G_a the additive group over k. For every $x \in$ k let $\lambda_x : G_a \to G_a$ be the homomorphism "multiplication with x". We define $x\xi$ as the element of $\mathrm{Ext}_k^i(G, G_a)$ represented by the push-out of $(*)$ along λ_x. This turns $\mathrm{Ext}_k^i(G, G_a)$ into a left vectorspace over k. Using this and base change k - K we get a natural homomorphism for all i = 0, 1, 2,...

$$\phi^i : K\otimes_k \mathrm{Ext}_k^i(G, G_a) \to \mathrm{Ext}_K^i(G_K, G_{aK})$$

The main theorem of the present note is that ϕ^i is an isomorphism in case G is finite, K/k an algebraic extension with K perfect. As a corollary one then has that $\mathrm{Ext}_k^i(N,G_a) = 0$ for N finite, $i \geq 2$, k any field, which answers a question of [7].

In case k is a perfect field and K/k an algebraic extension, ϕ^i was proved to be an isomorphism in [4]. We are therefore only interested in the case that k is not perfect. In particular we shall always assume that char(k) = p > 0.

As a preliminary to the proof of this theorem we study in section 2 a property of embeddings of abelian categories $\mathcal{C} \hookrightarrow \mathcal{D}$ which guarantees that the natural map $\mathrm{Ext}^i_{\mathcal{C}}(C,C') \to \mathrm{Ext}^i_{\mathcal{D}}(C,C')$, $C,C' \in \mathcal{C}$ is an isomorphism.

We use $\underline{\mathrm{Sch}}_k$ to denote the category of algebraic schemes over k and $\underline{\mathrm{Sch}}_k(X,Y)$ denotes the set of morphisms from X to Y. We write $\mathrm{Hom}_k(G,H)$ for the group of homomorphisms of the group scheme G into the group scheme H, G, H $\in G_k$ and $\mathrm{Ext}^i_k(G.H)$ for the i-th extension group in the category G_k.

2. THE PROPERTIES ESC AND ESD

In any abelian category $\mathcal{C}$ one can define the functors $\mathrm{Ext}^n_{\mathcal{C}}(X,Y)$, $X,Y \in \mathcal{C}$ by means of Yoneda extensions. If $i : \mathcal{C} \to \mathcal{D}$ is an exact embedding of abelian categories there are natural maps i^n: $\mathrm{Ext}^n_{\mathcal{C}}(X,Y) \to \mathrm{Ext}^n_{\mathcal{D}}(X,Y)$. In this section we study two conditions on embeddings i which guarantee that the i^n are isomorphisms. By looking at the cases n = 0 and n = 1 one sees that i must in any case be a full embedding and that it must be (more or less) a complete embedding.

2.1. Definitions.

Let $i : \mathcal{C} \to \mathcal{D}$ be an embedding of abelian categories.

(i) i is exact if exact sequences in $\mathcal{C}$ remain exact under i.

(ii) i is full if $\mathrm{Hom}_{\mathcal{C}}(X,Y) \overset{\sim}{\to} \mathrm{Hom}_{\mathcal{D}}(X,Y)$ for all $X,Y \in \mathcal{C}$.

(iii) i is complete if i is exact, full and if for every short exact sequence

$$0 \to C' \to D \to C'' \to 0$$

in D with $C',C'' \in \mathcal{C}$ it follows that D is in $\mathcal{C}$.

(iv) i is said to satisfy the property ESC (enough small codomains) if i is complete and if for every monomorphism $C \to D$ in $\mathcal{D}$, $C \in \mathcal{C}$, $D \in \mathcal{D}$ there exists a morphism $D \to C'$, $C' \in \mathcal{C}$ such that the composed morphism $C \to D \to C'$ is a monomorphism (in $\mathcal{C}$ or in $\mathcal{D}$).

(v) is said to satisfy the property ESD (enough small domains) if i is complete and if for every epimorphism $D \to C$, $D \in \mathcal{D}$, $C \in \mathcal{C}$ there exists a morphism $C' \to D$, $C' \in \mathcal{C}$ such the composed map $C' \to D \to C$ is an epimorphism.

2.2. Remarks.

1. It suffices for our purposes to require under (iii) (definition of complete) that there be an object in $\mathcal{C}$ isomorphic to D.
2. In (v) the morphism C' → D need not be a monomorphism, and D → C' in (iv) need not be an epimorphism.

2.3. Example.

Let $\mathcal{C}$ be the category of finite abelian groups and $\mathcal{D}$ the category of finitely generated abelian groups. The natural embedding $i : \mathcal{C} \to \mathcal{D}$ has ESC. Indeed, let $C \to D$ be a monomorphism, let n be such that $nC = 0$. Then $D/nD \in \mathcal{C}$ and $C \to D \to D/nD$ is a monomorphism; one easily checks that i is complete.

More generally let R be a ring, $\mathcal{C}$ the category of finite R-modules and $\mathcal{D}$ the category of R-modules which are finitely generated as abelian group. Then $i : \mathcal{C} \to \mathcal{D}$ has ESC.

2.4. Theorem.

Let $i : \mathcal{C} \to \mathcal{D}$ be a complete embedding of abelian categories which satisfies ESC or ESD. The natural map

$$i^n(X,Y) : \mathrm{Ext}^n_{\mathcal{C}}(X,Y) \to \mathrm{Ext}^n_{\mathcal{D}}(iX,iY)$$

is then an isomorphism for all $n = 0, 1, 2, \ldots$; $X,Y \in \mathcal{C}$

To prove this we first prove a lemma.

2.5. Lemma.

Let $i : \mathcal{C} \to \mathcal{D}$ satisfy ESC. Then for every exact sequence $0 \to Y \to D_1 \to D_2 \to \ldots \to D_n \to X \to 0$, $X,Y \in \mathcal{C}$ there exists a commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & D_1 \arrow[r] \arrow[d] & D_2 \arrow[r] \arrow[d] & \ldots \arrow[r] & D_n \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r] & C_1 \arrow[r] & C_2 \arrow[r] & \ldots \arrow[r] & C_n \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

with $C_i \in \mathcal{C}$, $i = 1, 2, \ldots, n$

(If $i : \mathcal{C} \to \mathcal{D}$ has ESD there exists such a diagram with the vertical arrows reversed).

Proof. The lemma holds for $n = 1$ because i is complete. Assume by induction that the lemma holds for $n-1 > 1$. Because i has ESC there is a morphism $D_1 \to C_1$ such that $Y \to D_1 \to C_1$ is a monomorphism. Let D_1' be the cokernel of $Y \to D_1$ and C_1' the cokernel of $Y \to C_1$. Note that $C_1' \in \mathcal{C}$. Now push out $0 \to D_1' \to D_2 \to \dots \to D_n \to X$ along $D_1' \to C_1'$ and apply the induction hypothesis to the resulting exact sequence $0 \to C_1' \to D_2' \to D_3' \to \dots \to D_n' \to X \to 0$ ($D_2' = D_2 \underset{D_1'}{\Sigma} C_1'$, $D_3' = D_3, \dots, D_n' = D_n$). This gives us commutative diagrams with exact rows

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & D_1 \arrow[r] \arrow[d] & D_1' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Y \arrow[r] & C_1 \arrow[r] & C_1' \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & D_1' \arrow[r] \arrow[d] & D_2 \arrow[r] \arrow[d] & \dots \arrow[r] & D_n \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C' \arrow[r] \arrow[d, equal] & D_2' \arrow[r] \arrow[d] & \dots \arrow[r] & D_n' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C_1' \arrow[r] & C_2 \arrow[r] & \dots \arrow[r] & C_n \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

Now put together the top and bottom rows of these diagrams to obtain the desired result.

2.6. <u>Proof of the theorem</u>

The surjectivity of the maps $i^n(X,Y)$ for $n > 1$ follows directly from the lemma. For $n = 0$, $i^0(X,Y)$ is an isomorphism for all $X,Y \in \mathcal{C}$ because i is full. And for $n = 1$, $i^1(X,Y)$ is an isomorphism because i is complete. By induction we can assume that $i^{n-1}(X,Y)$ is an isomorphism for all $X,Y \in \mathcal{C}$, $n - 1 > 1$. Let $\xi \in \mathrm{Ext}^n_{\mathcal{C}}(X,Y)$ and suppose that $i^n(X,Y)(\xi) = 0$. Let $0 \to Y \xrightarrow{f} C_1 \to \dots \to C_n \to X \to 0$, $C_i \in \mathcal{C}$ represent ξ. Consider the following diagram induced by the exact sequence $0 \to Y \to C_1 \to E \to 0$

$$\begin{tikzcd}
\mathrm{Ext}^{n-1}(X,C_1) \arrow[r] \arrow[d, "i^{n-1}", "\wr"'] & \mathrm{Ext}^{n-1}(X,E) \arrow[r] \arrow[d, "i^{n-1}", "\wr"'] & \mathrm{Ext}^{n}(X,Y) \arrow[r, "f_*"] \arrow[d, "i^{n}"] & \mathrm{Ext}^{n}(X,C_1) \arrow[d, "i^{n}"] \\
\mathrm{Ext}^{n-1}(X,C_1) \arrow[r] & \mathrm{Ext}^{n-1}(X,E) \arrow[r] & \mathrm{Ext}^{n}(X,Y) \arrow[r] & \mathrm{Ext}^{n}(X,C_1)
\end{tikzcd}$$

We have that $f_*(\xi) = 0$ in $\mathrm{Ext}^n(X,C_1)$. Using the fact that $i^{n-1}(X,E)$ and $i^{n-1}(X,C_1)$ are isomorphisms, it is now easy to show that $\xi = 0$. (The same argument was used in [5], Prop (3.3). This concludes the proof.

3. EXAMPLES OF ESD AND ESC

3.1. The natural embedding of an abelian category into its procategory has ESC.

Let $\mathcal{C}$ be an abelian category and $\mathrm{Pro}(\mathcal{C})$ its procategory. An object $(X_\alpha)_{\alpha\in A}$ of $\mathrm{Pro}(\mathcal{C})$ is isomorphic to the zero object if and only if for every $\alpha\in A$ there exists an $\alpha' > \alpha$ such that $X_{\alpha'} \to X_\alpha$ is the zero map.

The embedding $\mathcal{C} \to \mathrm{Pro}(\mathcal{C})$ is full by definition of $\mathrm{Pro}(\mathcal{C})$. Now let

$$0 \to X \overset{j}{\to} (Y_\alpha) \overset{p}{\to} Z \to 0$$

be an exact sequence with $X,Z \in \mathcal{C}$. The map j consists of a coherent system of maps $X \to Y_\alpha$; $\alpha \in A$; the kernel of j is the pro-object (K_α) where $K_\alpha = \mathrm{Ker}(X \to Y_\alpha)$ (cf. e.g. [4], §3); using the remark above we see that there exists an index α_0 such that $X \to (Y_\alpha) \to Y_\beta$ is monomorphic for all $\beta > \alpha_0$. This proves the ESC property. It remains to show that $\mathcal{C} \to \mathrm{Pro}(\mathcal{C})$ is complete. Let p be represented by $Y_{\alpha_1} \to Z$. The cokernel of p is the pro-object $(C_\alpha)_{\alpha\geqslant\alpha_1}$, where $C_\alpha = \mathrm{Coker}\ (Y_\alpha \to Z)$, $\alpha \geqslant \alpha_1$; using the same remark as above we see that there exists an index α_2 such that $Y \to Z$ is epimorphic for $\beta \geqslant \alpha_2$. Take $\gamma > \alpha_1,\alpha_2$. One now shows easily that $(Y_\alpha) \simeq Y_\gamma$ in $\mathrm{Pro}(\mathcal{C})$.

3.2. Theorem

For every abelian category $\mathcal{C}$ one has

$$\mathrm{Ext}^i_{\mathcal{C}}\ (X,Y) \simeq \mathrm{Ext}^i_{\mathrm{Pro}(\mathcal{C})}(X,Y)$$

Proof. Combine (3.1) and (2.4). This result generalizes a result of [5].

In order to get some more examples of ESC and ESD we first define some full subcategories of $\underline{G}_k$, the category of commulative algebraic group schemes over a field k.

$\underline{G}_k$, the category of commutative algebraic group schemes over k
$\underline{Lin}_k$, the full subcategory of $\underline{G}_k$ consisting of the linear group schemes
$\underline{Un}_k$, the full subcategory of $\underline{Lin}_k$ consisting of the unipotent group schemes
$\underline{Mult}_k$, the full subcategory of $\underline{Lin}_k$ consisting of the multiplicative group schemes
$\underline{FG}_k$, the full subcategory of finite group schemes in $\underline{G}_k$, $\underline{FG}_k$ is in fact a subcategory of $\underline{Lin}_k$
$\underline{FUn}_k$, the full subcategory of finite unipotent group schemes
$\underline{FMult}_k$, the full subcategory of finite multiplicative group schemes.

Because all embeddings are full we can use Hom_k to denote homomorphisms for all these subcategories simultaneously.

3.3. Some well known properties (Cf. [2], [3])

3.3.1. $\mathrm{Hom}_k(U,M) = \mathrm{Hom}_k(M,U) = 0$ if $M \in \underline{\mathrm{Mult}}_k$, $U \in \underline{\mathrm{Un}}_k$

3.3.2. There is a functor $\underline{\mathrm{Lin}}_k \to \underline{\mathrm{Mult}}_k$, $L \mapsto \mathfrak{M}(L)$ and a functor monomorphism $\mathfrak{M} \to \mathrm{Id}$ such that

$$0 \to \mathfrak{M}(L) \to L \to \mathfrak{U}(L) \to 0$$

is exact with $\mathfrak{U}(L) \in \underline{\mathrm{Un}}_k$ for every $L \in \underline{\mathrm{Lin}}_k$.

3.3.3. All the natural embeddings of the categories defined above in each other are thick; i.e. they are complete and subobjects and quotients of an object of the smaller category are also in the smaller category.

3.4. A diagram of the ESC and ESD properties of various categories of group schemes

$$\begin{tikzcd}
\underline{\mathrm{FUn}}_k \arrow[r, "\underline{\mathrm{ESD}}"] \arrow[d, "\mathrm{ESC}"] & \underline{\mathrm{Un}}_k \arrow[d, "\mathrm{ESC}"] & \\
\underline{\mathrm{FG}}_k \arrow[r, "\underline{\mathrm{ESD}}"] & \underline{\mathrm{Lin}}_k \arrow[r, "\underline{\mathrm{ESD}}"] & \underline{\mathrm{G}}_k \\
\underline{\mathrm{FMult}}_k \arrow[r, "\underline{\mathrm{ESD}}"] \arrow[u, "\mathrm{ESD}"] & \underline{\mathrm{Mult}}_k \arrow[u, "\mathrm{ESD}"] &
\end{tikzcd}$$

(NB. If $\mathfrak{C} \hookrightarrow \mathfrak{C}'$, $\mathfrak{C}' \hookrightarrow \mathfrak{C}''$ are two ESD (resp. ESC) embeddings, then the composed embedding $\mathfrak{C} \hookrightarrow \mathfrak{C}''$ is also ESD (resp. ESC)).

3.5. $\underline{\mathrm{FUn}}_k \hookrightarrow \underline{\mathrm{Un}}_k$ is an ESD embedding

This is proved in several steps.

3.5.1. Lemma (A very special case).

Let $0 \to G_a \to U \to E \to 0$ be exact, $U \in \underline{\mathrm{Un}}_k$, $E \in \underline{\mathrm{FUn}}_k$ an etale group scheme. Then there is a finite subgroup scheme E' of U such that $E' \hookrightarrow U \to E$ is monomorphic.

Proof. After a finite <u>separable</u> (galois) extension k'/k, $E_{k'}$ is constant. Now $G_a = U^0$, the connected component of the identity of U because E is etale and G_a is connected. It follows that over k', $U \to E$ as a map of schemes looks like

$$\mathrm{Spec}(k'[X] \oplus k'[X] \oplus \dots \oplus k'[X]) \to \mathrm{Spec}(k' \oplus k' \oplus \dots \oplus k')$$

This map admits a lift, say ϕ, over k' (ϕ is probably not a homomorphism). Let E" be the finite group scheme generated by $\phi(E_{k'})$; E" is defined over k'. (NB. E" is finite because $p^r U = 0$ for a suitable r ($p = \mathrm{char}(k) > 0$)). Let E' be the sum (in $U_{k'}$) of the k'/k conjugates of E". Then E' is defined over k and $E' \to U \to E$ is epimorphic.

3.5.2. Lemma (A special case)

Let $0 \to G_a \to U \to E \to 0$ be exact, $U \in \underline{Un}_k$, $E \in \underline{FUn}_k$. Then there is a finite subgroup scheme E' of U such that $E' \to U \to E$ is epimorphic.

Proof. If $G \in \underline{G}_k$ is any commutative group scheme, we denote with $F : G \to G^{(p)}$ the Frobenius homomorphism (cf. [2], Ch.II, §7, 1.1; [3], Exp XVII, App.II; [6], (I.1)). For every $G \in \underline{G}_k$ there is an index n_0 such that $G/\mathrm{Ker}(F^n)$ is reduced for all $n \geqslant n_0$. (cf. [3], exposé XVII, App.II). We shall denote $\mathrm{Ker}(G \xrightarrow{F^n} G^{(p^n)})$ with $I^n(G)$. We have a commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & I^n(G_a) \arrow[r] \arrow[d] & I^n(U) \arrow[r] \arrow[d] & I^n(E) \arrow[d] & \\
0 \arrow[r] & G_a \arrow[r] \arrow[d, "F^n"] & U \arrow[r] \arrow[d, "F^n"] & E \arrow[r] \arrow[d, "F^n"] & 0 \\
0 \arrow[r] & G_a^{(p^n)} \arrow[r] & U^{(p^n)} \arrow[r] & E^{(p^n)} \arrow[r] & 0
\end{tikzcd}$$

It follows by the snake lemma that $I^n(U) \to I^n(E)$ is epimorphic, because $F^n : G_a \to G_a^{(p^n)}$ is epimorphic. We obtain an exact diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & I^n(G_a) \arrow[r] \arrow[d] & I^n(U) \arrow[r] \arrow[d] & I^n(E) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G_a \arrow[r] \arrow[d] & U \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G_a/I^n(G_a) \arrow[r] \arrow[d] & U/I^n(U) \arrow[r] \arrow[d] & E/I^n(E) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

For n large enough $E/I^n(E)$ is étale; also $G_a/I(G_a) \simeq G_a/\alpha_n \simeq G_a$ (cf.[3], Ch.IV, §2, no.1). By (3.5.1) there exists a subgroupscheme E'' of $U/I^n(U)$ such that $E'' \to U/I^n(U) \to E:I^n(E)$ is epimorphic. Let $E' \subset U$ be the inverse image of E'' in U; E' is finite because $I^n(U)$ and E'' are finite. It is clear that $E' \to U \to E$ is epimorphic.

3.5.3. Proposition.

Let $U \to E$ be an epimorphism in with $U \in \underline{Un_k}$, $E \in \underline{FUn_k}$, then there exists a finite subgroupscheme E' of U such that $E' \to U \to E$ is epimorphic.

Proof. There exists a sequence of subgroups of U, $U \supset U_1 \supset U_2 \supset \dots \supset U_n = \{e\}$ such that the successive quotients U_i/U_{i+1} are isomorphic to G_a or to a finite subgroupscheme of G_a. ([2], Ch.IV, §2, no.2) We shall prove the ESD property for epimorphisms $U/U_i \to E_i$ by induction on i; $i = 1,2,3,\dots,n$. Let $i = 1$ and let $U/U_1 \to E_1$ be an epimorphism. If U/U_1 is finite there is nothing to prove; if $U/U_1 \simeq G_a$, $E_1 = \{0\}$ because all subgroupschemes of G_a unequal to G_a are finite. Now suppose the proposition proved for $U/U_i \to E_i$, and let $U/U_{i+1} \to E_{i+1}$ be an epimorphism. We have an exact diagram

$$\begin{tikzcd}
0 \arrow[r] & U_i/U_{i+1} \arrow[r] \arrow[d] & U/U_{i+1} \arrow[r] \arrow[d] & U/U_i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & D \arrow[r] & E_{i+1} \arrow[r] & E_i \arrow[r] & 0
\end{tikzcd}$$

where D is the image of U_i/U_{i+1} in E_{i+1}, and $E_i = E_{i+1}/D$. Let $E_i' \subset U/U_i$ be a finite subgroupscheme such that $E_i' \to U/U_i \to E_i$ is epimorphic. Let

(3.5.3.1) $$0 \to U_i/U_{i+1} \to E''_{i+1} \to E_i' \to 0$$

be the pullback of the toprow of the diagram above along $E_i' \to U/U_i$. The homomorphism $E''_{i+1} \to U/U_{i+1} \to E_{i+1}$ is epimorphic. If U_i/U_{i+1} is finite then E''_{i+1} is finite and we are through. If $U_i/U_{i-1} \simeq G_a$, then $E''_{i+1} \to E_{i+1}$ factorizes through E'_i because G_a is connected and reduced and E_{i+1} is finite, so that it suffices to apply (3.5.2) to the exact sequence (3.5.3.1).

3.6. Proposition

$\underline{FMult}_k \subset \underline{Mult}_k$ is an ESD embedding.

Proof. Let $M \to N$ be an epimorphism with $M \in \underline{Mult}_k$, $N \in \underline{FMult}_k$. For each $M \in \underline{Mult}_k$ let $D(M) = Hom_k(M, G_{m,k})$. The galois group $Gal(\bar{k}/k)$, where $\bar{k}$ is the algebraic closure of k, acts on D(M). Thus we obtain a functor $D : \underline{Mult}_k \to$ (category of finitely generated $G(\bar{k}/k)$ modules on which $G(\bar{k}/k)$ acts continuously). If M is a finitely generated $G(\bar{k}/k)$ module on which $G(\bar{k}/k)$ acts continuously (discrete topology on M) it is also finitely generated as an abelian group because the orbit of every $m \in M$ under $G(\bar{k}/k)$ is a finite set because $G(\bar{k}/k)$ acts continuously. This fundor is an antiequivalence (cf.[3], Exp.X, Prop. 1.4). The embedding (finite $Gal(\bar{k}/k)$ modules) $\to$ ($G(\bar{k}/k)$ modules which are finitely generated as abelian groups) has ESC (cf. (2.3)). Therefore $\underline{FMult}_k \to \underline{Mult}_k$ has ESD.

3.7. Corollary(of the proof)

If $M \to N$ is an epimorphism in $\underline{Mult}_k$, $N \in \underline{FMult}_k$, then there is an integer n such that ${}_nM \to M \to N$ is epimorphic, where ${}_nM$ is the kernel of multiplication with n in M.

3.8. Proposition

$\underline{FG}_k \to \underline{Lin}_k$ is an ESD embedding.

Proof. Let $L \to N$ be an epimorphism in $\underline{Lin}_k$ with $F \in \underline{FG}_k$. Applying (3.3.2) we get an exact diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & \\
(L) \arrow[r] \arrow[d] & (N) \arrow[r] \arrow[d] & 0 \\
L \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
(L) \arrow[r] \arrow[d] & (N) \arrow[r] \arrow[d] & 0 \\
0 & 0 &
\end{tikzcd}$$

There is a number $n \in N$ such that a) ${}_n\mathfrak{M}(L) \to \mathfrak{M}(\ ) \to \mathfrak{M}(N)$ is epimorphic (cf. (3.7)) and b) ${}_n\mathfrak{U}(L) = 0$. Let ${}_nL$ be the kernel of $L \xrightarrow{xn} L$. The homomorphism ${}_nL \to L \to N$ is then surjective. We have an exact diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & \\
{}_n\mathfrak{M}(L) \arrow[r] \arrow[d] & \mathfrak{M}(N) \arrow[r] \arrow[d] & 0 \\
{}_nL \arrow[r] \arrow[d, "q"] & N \arrow[r] \arrow[d] & 0 \\
\mathfrak{U}(L) \arrow[r] \arrow[d] & \mathfrak{U}(N) \arrow[r] \arrow[d] & 0 \\
0 & 0 &
\end{tikzcd}$$

Because <u>FUn_k</u> → <u>Un_k</u> has ESD there is a finite group scheme $E' \subset \mathfrak{U}(L)$ such that $E' \to \mathfrak{U}(L) \to \mathfrak{U}(N)$ is epimorphic. Let N' be the inverse image of E' under q; N' is finite because ${}_n\mathfrak{M}(L)$ is finite (cf. proof of (3.6)) and E' is finite. It is clear that $N' \to {}_nL \to N$ is epimorphic.

3.9. <u>Proposition</u>

The embedding <u>$Mult_k$</u> → <u>Lin_k</u> is ESD. The embedding <u>Un_k</u> → <u>Lin_k</u> is ESC.

Proof. These statements are proved in a similar way. Let $U \to L$ be a monomorphism in <u>Lin_k</u> with $U \in$ <u>Un_k</u>. Applying (3.3.2) and (3.3.1) we obtain an exact diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
 & 0 \arrow[r] \arrow[d] & \mathfrak{M}(L) \arrow[d] \\
0 \arrow[r] & U \arrow[r] \arrow[d, equal] & L \arrow[d] \\
 & U \arrow[r] \arrow[d] & \mathfrak{U}(L) \arrow[d] \\
 & 0 & 0
\end{tikzcd}$$

3.10. <u>Remark</u>

It is not true in general that <u>$Mult_k$</u> → <u>Lin_k</u> is ESD or that Un_k → <u>Lin_k</u> is

ESD. Indeed, there are over nonperfect fields exact sequences $0 \to M \to L \to U \to 0$, $M \in \underline{\mathrm{Mult}}_k$, $U \in \underline{\mathrm{Un}}_k$ which do not split (Cf. [3], Exp.XVII), and if $\underline{\mathrm{Mult}}_k \to \underline{\mathrm{Lin}}_k$ were ESC or $\underline{\mathrm{Un}}_k \subset \underline{\mathrm{Lin}}_k$ ESD these sequences would split. (Use snake lemma and (3.3))

To prove the last one of the ESD properties we listed in (3.4), we need a lemma.

3.11. Lemma.

For every $G \in \underline{G}_k$ there exists an exact sequence

$$0 \to L \to G \to A \to 0$$

with $L \in \underline{\mathrm{Lin}}_k$ and A an abelian variety.

Proof. Let $I^n(G)$ be the kernel of the iterated Frobenius homomorphism $F^n : G \to G^{(p^n)}$. For n sufficiently large $G_1 = G/I^n(G)$ is reduced. Let G_1^0 be the connected component of the identity of G_1. Let $N = G_1/G_1^0$ and $n \in N$ such that $nN = 0$; let A be the Albanese variety of G_1^0. $A = G_1^0/L'$ where L' is the maximal reduced connected linear subgroup of G_1^0 (cf.[1]). We have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & G_1^0 \arrow[r] \arrow[d, "\times n"] & G_1 \arrow[r] \arrow[d, "\times n"] \arrow[dl, dashed] & N \arrow[r] \arrow[d, "\times n = 0"'] & 0 \\
0 \arrow[r] & G_1^0 \arrow[r] \arrow[d] & G_1 \arrow[r] & N \arrow[r] & 0 \\
& A & & &
\end{tikzcd}$$

because $\times n$ is zero on N we have a lift of $G_1 \xrightarrow{\times n} G_1$ as indicated. The composed map $G_1 \to G_1^0 \to A$ is epimorphic because $\times n : A \to A$ and $G_1^0 \to A$ are epimorphic. The kernel of $G_1 \to G_1^0$ is ${}_nG_1$ so that $\mathrm{Ker}(G_1 \to A)$ is an extension of a subgroup of $L' = \mathrm{Ker}(G_1^0 \to A)$ with ${}_nG_1$ which shows that $\mathrm{Ker}(G_1 \to A)$ is linear. The kernel of $G \to G/I^n(G) = G_1 \to A$ is an extension of $\mathrm{Ker}(G_1 \to A)$ with $I^n(G)$ and hence also linear. This proves the lemma.

3.12. Corollary

The embedding $\underline{\mathrm{Lin}}_k \to \underline{G}_k$ is ESD

Proof. Let $G \to L$ be an epimorphism in $\underline{G}_k$ with $L \in \underline{\mathrm{Lin}}_k$. Applying (3.11) we get an exact diagram

$$\begin{tikzcd}
0 \arrow[r] & L' \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L'' \arrow[r] & L \arrow[r] \arrow[d] & L/L'' \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd}$$

where L'' is the image of L' in L. The map $A \to L/L''$ is epimorphic because $G \to L \to L/L''$ is epimorphic; but because A is an abelian variety and L/L'' is linear it follows that $(A \to L/L'') = 0$, so that $L/L'' = 0$, i.e. $\mathrm{Im}(L' \to G \to L) = L$. q.e.d.

4. $\mathrm{Ext}_k^i(N,G_a)$ FOR FINITE GROUP SCHEMES N

4.1. Lemma

If $(P_\alpha) \in \mathrm{Pro}(\underline{FG}_k)$ is projective then (P_α) is also projective in $\mathrm{Pro}(\underline{G}_k)$

Proof. It suffices to show that for every exact diagram

$$\begin{tikzcd}
 & (P_\alpha) \arrow[d, "g"] & \\
G_2 \arrow[r, "q"] & G_1 \arrow[r] & 0
\end{tikzcd}$$

there exists a lift of g. (Cf.[8],§ 3, Prop.2) Let g be represented by $P_\alpha \to G_1$ and let N_1 be the image of $P_\alpha \to G_1$. Then $N_1 \in \underline{FG}_k$, because $P_\alpha \in \underline{FG}_k$. Let G_2' be the inverse image of N_1 under g. Because $\underline{FG}_k \to \underline{G}_k$ is ESD there exists an $N_2 \to G_2'$ such that $N_2 \to G_2' \to N_1$ is epimorphic. There exists a lift of $(P_\alpha) \to N_1$ to a homomorphism $(P_\alpha) \to N_2$ because (P_α) is projective in $\mathrm{Pro}(\underline{FG}_k)$.

$$\begin{tikzcd}
 & & (P_\alpha) \arrow[d] \arrow[dll, dashed] & \\
N_2 \arrow[r] & G_2' \arrow[r] \arrow[d] & N_1 \arrow[d] & \\
 & G_2 \arrow[r] & G_1 \arrow[r] & 0
\end{tikzcd}$$

4.2. The Weil restriction functor (cf.[2], Ch.I, §1, no.(6.6))

Let l/k be a finite field extension. Then there exists a right adjoint to the base change fundor $\underline{Sch}_k \to \underline{Sch}_l$ which takes group objects to group objects. We have a canonical isomorphism

$$Hom_k(T,W(S)) \overset{\sim}{\to} Hom_l(T_l,S) \qquad S \in \underline{G}_l,\ T \in \underline{G}_k$$

Substituting T = W(S) we get a canonical morphism

$$j_S : W(S)_l \to S$$

and substituting S = T_l we get a canonical morphism

$$i_T : T \to W(T_l)$$

The composed map $j_{T_l} \circ (i_T)_l$ is the identity on T_l. If l/k is separable then $j_S : W(S)_l \to S$ is epimorphic (cf. e.g. [4]). Being a right adjoint functor W is automatically left exact.

4.3. Lemma

Let k_i/k be an algebraic extension such that k_i is perfect. For every $N \in \underline{FG}_{k_i}$ there exists an $N' \in \underline{FG}_k$ and an epimorphism $N'_{k_i} \to N$.

Proof. Let $N \in \underline{FG}_{k_i}$; N can be written as a direct sum $N = N_m \oplus N_{um} \oplus N_{uu}$, where N_m is multiplicative, N_{um} is unipotent and N^D_{um} multiplicative and N_{uu} is unipotent and N^D_{uu} is unipotent (cf. [6]) (If $N \in \underline{FG}$, N^D denotes the linear dual.) It suffices therefore to prove the lemma for finite group schemes of these three types.

Let N be defined over l/k, [l:k] < ∞ and let $k \subset l_s \subset l$ be a decomposition of $k \subset l$ into a separable part $k \subset l_s$ and a purely inseparable part $l_s \subset l$.
1) Let N be multiplicative (cf.[ ], exp.). Then N is isomophic to an group scheme N'_l where N' is defined over l_s. Let W denote the Weil restriction functor $\underline{Sch}_{l_s} \longrightarrow \underline{Sch}_k$. There is a natural map $W(N')_{l_s} \to N'$ which is epimorphic because l_s/k is separable. (Cf. e.g. [4])
2) Let N^D be multiplicative. Then N^D is isomorphic to a group scheme N'_l where N' is defined over L_s, so that N is isomorphic to $(N'^D)_l$. Now use W again.

3) N is unipotent and N^D is unipotent. By enlarging l if necessarily we can assume that there exists an embedding over l

$$N^D \to W_n^s$$

where W_n is the ring of Witt vectors of length n,([9]). Let F be the Frobenius morphism. W_n and F are both defined over k. Let $N'_r = \mathrm{Ker}(F^r : W_n^s \to W_n^s) \in \underline{FG}_k$. Because N^D is local there exists an r such that $N^D \subset (N'_r)_l$. Dualizing we get the desired epimorphism also in this case.

4.4. Lemma

Let $P \in \mathrm{Pro}(\underline{FG}_k)$ be a profinite projective object; and let k_i/k be an algebraic extension with k_i perfect. Then $P_{k_i} \in \mathrm{Pro}(\underline{G}_{k_i})$ is projective.

Proof. Using (4.1) and (4.2) we see that it suffices to prove that for every exact diagram

$$\begin{tikzcd} & P_{k_i} \arrow[d, "g"] & \\ N_{k_i} \arrow[r, "q"] & N_1 \arrow[r] & 0 \end{tikzcd}$$

where $N \in \underline{FG}_k$, there exists a lift of g. Let l be a finite extension of k such that q, g and N_1 are defined over l. By enlarging l if necessarily we can also assume that there exists an embedding $N_1 \to W_n^s \times G_m^t$. Let W be the Weil restriction functor $\underline{Sch}_l \to \underline{Sch}_k$. We have a diagram

$$\begin{tikzcd} & P \arrow[d, "i_P"] \\ & W(P_l) \arrow[d, "W(g)"] \\ W(N_l) \arrow[r, "W(q)"] & W(N_1) \end{tikzcd}$$

Assertion: $(W(g) \circ i_P)(P) \subset \mathrm{Im}(W(q))$. It suffices to prove this after base change k - l. We have a diagram

$$\begin{tikzcd}
 & P_1 \arrow[d, shift right, "i_P"'] \arrow[dddr, bend left=60, "h"] & \\
 & W(P_1)_1 \arrow[u, shift right, "j_P"'] \arrow[d, "W(g)_1 = g'"] \arrow[dr, "h'"] & \\
W(N_1)_1 \arrow[r, "q' = W(q)_1"] \arrow[d, shift right, "j_N"'] & W(N_1)_1 \arrow[r, "\phi'"] \arrow[d, "j_{N_1}"] & W(G_1)_1 \arrow[d, shift right, "j_G"'] \\
N_1 \arrow[u, shift right, "i_N"'] \arrow[r, "q"] & N_1 \arrow[r, "\phi"] & G \arrow[u, shift right, "i_G"']
\end{tikzcd}$$

Here $G = W_n^s \times G_m^t \in \underline{G}_k$, and we have sometimes written i_G for $(i_G)_1$ if $G \in \underline{G}_k$, j_T for j_{T_1} if $T \in \underline{G}_k$. Let $h = \phi \circ g$, $h' = W(h)_1$. We have $h' \circ i_P = i_G \circ h$. Now

$$\phi' q' i_N(N_1) = i_G \phi q(N_1) \supset i_G \phi g(P_1)$$

further

$$i_G \phi g(P_1) = i_G h(P_1) = h' i_P(P_1) = \phi' g' i_P(P_1)$$

which gives us

$$\phi' q' i_N(N_1) \supset \phi' g' i_P(P_1)$$

and hence because ϕ' is monomorphic

$$q' i_N(N_1) \supset g' i_P(P_1)$$

and hence certainly

$$q'(W(N_1)_1) \supset g' i_P(P_1)$$

which proves the assertion.

There exists therefore a homomorphism $f : P \to W(N_1)$ which lifts $W(g) \circ i_P$. It is now not difficult to check that the composed map

$$j_N \circ f_1 : P_1 \to W(N_1)_1 \to N_1$$

lifts g.

4.5. $K \otimes_k \mathrm{Hom}_k(G,G_a) \simeq \mathrm{Hom}_K(G_K,G_{a,K})$

For any $G \in \underline{G}_k$ the group $\mathrm{Hom}_k(G,G_a)$, where G_a is the additive group over k has a natural left-k-vectorspace structure because $k \to \mathrm{Hom}_k(G_a,G_a)$. Let $\underline{\mathrm{Sch}}_k(G,G_a)$ denote the set of all scheme morphisms over k from G to G_a. Let A be k-algebra of global sections of G. Then $\underline{\mathrm{Sch}}_k(G,G_a) \simeq \mathrm{Alg}_k(k[X],A) \simeq A$. Using $k \to \mathrm{Hom}_k(G_a,G_a) \to \underline{\mathrm{Sch}}_k(G_a,G_a)$ one also finds a k-vectorspace structure on $\underline{\mathrm{Sch}}_k(G,G_a)$ and this structure coincides with the k-vectorspace structure of A under the isomorphism $\underline{\mathrm{Sch}}_k(G,G_a) \xrightarrow{\sim} A$. The multiplication $G \times G \to G$ induces a comultiplication $m : A \to A \otimes_k A$. The elements of $\mathrm{Hom}_k(G,G_a) \subset \underline{\mathrm{Sch}}_k(G,G_a)$ correspond to those elements $f \in A$ for which $mf = 1 \otimes f + f \otimes 1$ (i.e. the group like elements of the coalgebra A).

Now let K be any extension of k. Similarly we have K-vectorspace structures on $\underline{\mathrm{Sch}}_K(G_K,G_{a,K})$ and $\mathrm{Hom}_K(G_K,G_{a,K})$. We find a natural diagram

$$\begin{tikzcd}
K \otimes_k A \arrow[r, phantom, "\simeq"] \arrow[d, "\|"] & K \otimes_k \mathrm{Sch}_k(G,G_a) \arrow[d, "\wr"] & K \otimes_k \mathrm{Hom}_k(G,G_a) \arrow[l, hook'] \arrow[d, "\phi"] \\
K \otimes_k A \arrow[r, phantom, "\simeq"] & \mathrm{Sch}_K(G_K,G_{aK}) & \mathrm{Hom}_K(G_K,G_{aK}) \arrow[l, hook']
\end{tikzcd}$$

It follows that ϕ is injective. Let $\{e_\alpha \mid \alpha \in I\}$ be a basis of K over k. Then $A \otimes_k K$ is a free A-module with basis $\{1 \otimes e_\alpha\}$.

If one identifies $(K \otimes_k A) \otimes_K (K \otimes_k A)$ with $K \otimes_k (A \otimes_k A)$ in the natural way, then the comultiplication m_K on $K \otimes_k A$ deduced from the multiplication on G_K looks like $1 \otimes_k m$.

Let $f \in \mathrm{Hom}_K(G_K,G_{aK})$; we can write $f = \sum e_\alpha \otimes f_\alpha$, $f_\alpha \in A$. We then have, using the identification mentioned above,

$$\Sigma(e_\alpha \otimes 1 \otimes f_\alpha + e_\alpha \otimes f_\alpha \otimes 1) = 1 \otimes \Sigma e_\alpha \otimes f_\alpha + \Sigma e_\alpha \otimes f_\alpha \otimes 1 =$$

$$= 1 \otimes f + f \otimes 1 = m_K(f) = m_K(\Sigma e_\alpha \otimes f_\alpha) = \sum_\alpha e_\alpha \otimes (mf_\alpha)$$

It follows that $1 \otimes f_\alpha + f_\alpha \otimes 1 = mf_\alpha$ for all f_α involved in $f = \Sigma e_\alpha \otimes f_\alpha$ because $(A \otimes_k A) \otimes_k K$ is a free $A \otimes_k A$ algebra with basis $\{1 \otimes e_\alpha\}$. This shows that $f_\alpha \in \mathrm{Hom}_k(G,G_a)$ and we have therefore shown that the map ϕ in the diagram above is an isomorphism.

4.6. Corollary

The natural map $K \otimes_k \mathrm{Hom}_k((G_\alpha), G_a) \overset{\sim}{\rightarrow} \mathrm{Hom}_K((G_{\alpha K}), G_{aK})$ is an isomorphism for all proobjects $(G_\alpha) \in \mathrm{Pro}(\underline{G}_k)$.

(This follows from (4.5) because inductive limits commute with tensorproducts.)

4.7. Theorem

Let K/k be an algebraic field extension such that K is perfect and let $N \in \underline{FG}_k$ or $\mathrm{Pro}(\underline{FG}_k)$. The natural map

$$K \otimes_k \mathrm{Ext}_K^n (N,G_a) \rightarrow \mathrm{Ext}_K^n (N_K,G_{aK})$$

is then an isomorphism for all n = 0, 1, 2, ...

Proof. Let

$$\dots \to X_n \to X_{n-1} \to \dots \to X_1 \to X_0 \to N \to 0$$

be a profinite projective resolution of N in $\mathrm{Pro}(\underline{G}_k)$. Such a resolution exists by (4.1). After base change k - K we obtain a projective resolution

$$\dots \to X_{n,K} \to X_{n-1,K} \to \dots \to X_{0,K} \to N_K \to 0$$

of N_K. The $\mathrm{Ext}_K^n (N_K, G_{aK})$ are equal to the homology of the complex

$$0 \to \mathrm{Hom}_K(X_{0,K}, G_{a,K}) \to \dots \to \mathrm{Hom}_K(X_{n,K}, G_{a,K}) \to \dots$$

The $\mathrm{Ext}_k^n (N,G_a)$ are equal to the homology groups of

$$0 \to \mathrm{Hom}_k(X_0,G_a) \to \dots \to \mathrm{Hom}_K(X_n,G_a) \to \dots$$

Now use (4.6) and the exactness of $K \otimes_k$ - to obtain the theorem.

4.8. Remark

The left-k-vectorspace structure on $\mathrm{Ext}_k^n (N,G_a)$ can be obtained in two ways: 1) by viewing $\mathrm{Ext}_k^n (N,G_a)$ as the n-th homology group of a complex of left k-vectorspaces; 2) by means of Yoneda extensions as follows:

let $\xi \in \mathrm{Ext}^n(N,G_a)$ be represented by

$$0 \to G_a \to G_1 \to \dots \to G_n \to N \to 0$$

then for $x \in k, x\xi$ is represented by the pushout of this sequence along $G_a \xrightarrow{x} G_a$.

If $0 \to N_1 \to N \to N_2 \to 0$ is exact, then all the maps in the long exact sequence of the $\mathrm{Ext}(-,G_a)$ groups are k-vectorspace maps.

4.9. Corollary

$\mathrm{Ext}_k^i\ (N,G_a) = 0$ for $i \geqslant 2$ for all fields k. $N \in \underline{FG_k}$.

This answers a question of [7].

5. CONCLUDING REMARKS

5.1. $\mathrm{Ext}^i(M,G_a)$, $M \in \underline{\mathrm{Mult}_k}$

Let $M \in \underline{\mathrm{Mult}_k}$ be such that its corresponding galois module $\mathrm{Hom}_k(M,G_m)$ is torsion free. The map $M \xrightarrow{xp} M$ is then epimorphic, let ${}_pM$ be the kernel. Because xp is zero on G_a we get exact sequences

$$0 \to \mathrm{Hom}({}_pM,G_a) \to \mathrm{Hom}(M,G_a) \to 0$$

$$0 \to \mathrm{Hom}(M,G_a) \to \mathrm{Ext}^1({}_pM,G_a) \to \mathrm{Ext}^1(M,G_a) \to 0$$

$$\cdots\cdots$$

$$0 \to \mathrm{Ext}^i(M,G_a) \to \mathrm{Ext}^{i+1}({}_pM,G_a) \to \mathrm{Ext}^{i+1}(M,G_a) \to 0$$

From this one easily shows by induction that $K \otimes_k \mathrm{Ext}_k^i\ (M,G_a) \simeq \mathrm{Ext}_K^i\ (M_K,G_{aK})$ for such multiplicative groups and hence for all multiplicative groups because for every multiplicative group M there exists an exact sequence

$$0 \to M' \to M \to M'' \to 0$$

with $M'' \in \underline{\mathrm{FMult}_k}$, M' such that D(M') is torsion free.

5.2. $\mathrm{Ext}^i(A,G_a)$ for abelian varieties A

The same argument as above shows that $K \otimes_k \mathrm{Ext}_k^i(A,G_a) \simeq \mathrm{Ext}_K^i(A_K,G_{aK})$ for abelian varieties.

5.3. Remark

Because every group scheme over k can be built up from finite group schemes, multiplicative group schemes, abelian varieties and copies of G_a it only remains to see whether $K \otimes_k \mathrm{Ext}^i_k(G_a,G_a) \simeq \mathrm{Ext}^i_K(G_{aK},G_{aK})$ to prove $K \otimes_k \mathrm{Ext}^i_k(G,G_a) \simeq \mathrm{Ext}^i_K(G_K, G_a)$ for all $G \in \underline{G}_k$. This will be dealt with in a future note.

5.4. Remark

Let L_{nm} be the kernel of $F^m : W_n \to W_n$ and let $L_{\infty,\infty}$ be the pro-object consisting of the L_{nm}; $L_{\infty,\infty}$ is defined over every field k and is projective if k is perfect. But if k is not perfect then one sees fairly easily using the exact sequence $0 \to \alpha_p \to G_a \to G_a \to 0$ and (4.7) that $\mathrm{Ext}(L_{\infty,\infty},\alpha_p)$ is very large (a countably infinite sum of copies of k/k^p), showing some difference for Ext groups over non perfect fields, as compared to Ext groups over perfect fields.

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