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# On Norm Maps for One Dimensional Formal Groups I: The Cyclotomic $\Gamma$-Extension*

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## 1. INTRODUCTION

Let $K$ be a finite extension of $\mathbf{Q}_p$, the field of $p$-adic numbers. Let $L/K$ be a galois extension. Local class field theory studies the cokernel of the norm map $N_{L/K}: L^* \to K^*$. Let $A_L$, $A_K$ be the ring of integers of $L$, $K$ and let $U(L)$, $U(K)$ be the group of units of $A_L$, $A_K$. The most difficult part of the determination of $N_{L/K}(L^*)$ is the determination of the image (or cokernel) of $N_{L/K}$: $U(L) \to U(K)$. This map can also be viewed as follows. Let $\mathbf{G}_m$ be the multiplicative group. Then $\mathbf{G}_m(A_L) = U(L)$, $\mathbf{G}_m(A_K) = U(K)$ and the map $N_{L/K}$ is: $N_{L/K}(x) =$ sum of all the conjugates of $x$ in $\mathbf{G}_m(L)$.

The following generalization is now natural and also interesting for various reasons (cf., [7, Section 4]). Let $G$ be an arbitrary commutative group scheme over $A_K$. Define Norm$(x) =$ sum in $G(A_L)$ of all the conjugates of $x$, for $x \in G(A_L)$. Problem. Determine the cokernel of Norm: $G(A_L) \to G(A_K)$. As in the case of $\mathbf{G}_m$ an important step is to calculate the cokernel of the induced map $\hat{G}(A_L) \to \hat{G}(A_K)$ where $\hat{G}$ is the formal completion of $G$; $\hat{G}$ is a formal group over $A_K$.

In the following we study the cokernel of Norm: $F(A_L) \to F(A_K)$ where $F$ is a one-dimensional formal group over $A_K$. In case the height of $F$ is equal to 1 the answer is up to a twist given by local class field theory (cf., [7]). Important is the fact that Norm: $F(A_{\widehat{L_{ur}}}) \to F(A_{\widehat{K_{ur}}})$ is surjective if height $(F) = 1$, where $L_{ur}$, $K_{ur}$ is the maximal unramified extension of $L$, $K$. The picture changes drastically as soon as height $(F) > 1$. It is then not true in general that Norm $(F(L)) = F(K)$ if $L/K$ is a finite galois extension and the residue field of $K$ is algebraically closed.

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The main part of this paper is devoted to the precise determination of the cokernel of $F(L) \to F(K)$ for one special class of extensions $L/K$. We take $K = \mathbf{Q}_p$, the $p$-adic numbers. Let $L_\infty$ be the extension of $\mathbf{Q}_p$ obtained by adjoining all $p^r$-th roots of unity. $\text{Gal}(L_\infty/\mathbf{Q}_p) \simeq U(\mathbf{Q}_p) \simeq \Delta \times \mathbf{Z}_p$ where $\Delta$ is the torsion subroup of $U(\mathbf{Q}_p)$. Let $K_\infty$ be the invariant field of $\Delta$. $\text{Gal}(K_\infty/\mathbf{Q}_p) \simeq \mathbf{Z}_p$, i.e., $K_\infty/\mathbf{Q}_p$ is a $\Gamma$-extension. Let $K_n$ be the invariant field of $p^n \text{ Gal}(K_\infty/\mathbf{Q}_p)$. We determine $\text{Im}(F(K_n) \to F(\mathbf{Q}_p))$, where $F$ is any formal group over $\mathbf{Z}_p$ of height $(F) \geqslant 2$.

The results and proofs turn out to be generalizable to some extent (cf., [3]).

The motivation to study precisely $\Gamma$-extensions came from [7].

It remains for me to thank the reviewer who thoroughly criticized an earlier version of this note.

## 2. Generalities on Formal Groups

### 2.1. *Some Notations and Definitions*

$K$ will always denote a local field of characteristic 0 and residue characteristic $p > 0$; $A_K$ is its ring of integers; $\pi_K$ is a uniformizing element and $v_K$ is the normalized exponential valuation on $K$ (i.e., $v_K(\pi_K) = 1$); $\mathfrak{M}_K$ is the maximal ideal of $A_K$.

A one dimensional formal group over $A_K$ is a formal power series in two variables over $A_K$ of the form

$$F(X, Y) = X + Y + \sum_{i,j=1}^{\infty} a_{ij}X^iY^j, \qquad a_{ij} \in A_K, \tag{2.1.1}$$

which satisfies

$$F(X, F(Y, Z)) = F(F(X, Y), Z). \tag{2.1.2}$$

All formal groups considered in this paper will be one dimensional. A one dimensional formal group over $A_K$ is automatically commutative; i.e., it satisfies $F(X, Y) = F(Y, X)$ (cf. [4]).

### 2.2. *Points and Norm Maps*

Let $L$ be a finite extension of $K$. One can use a formal group over $A_K$ to define an abelian group structure on the set $\mathfrak{M}_L$. In fact one simply sets

$$x +_F y = F(x, y), \qquad x, y \in \mathfrak{M}_L. \tag{2.2.1}$$

(The series $F(x, y)$ converges in $\mathfrak{M}_L$.) This group is denoted $F(L)$. If

$x, y \in \mathfrak{M}_L{}^t = \pi_L{}^t A_L$, $t = 1, 2, \ldots$, then $x +_F y \in \mathfrak{M}_L{}^t$. The group $F(L)$ therefore has a natural filtration by subgroups $F^t(L)$ where the underlying set of $F^t(L)$ is $\pi_L{}^t A_L$.

Because $F(X, Y) \equiv X + Y \bmod(\text{degree } 2)$, cf. (2.1.1), we have

$$F^t(L)/F^{t+1}(L) \simeq l^+, \tag{2.2.2}$$

where $l^+$ is the underlying additive group of the residue field $l$ of $L$.

Now let $L/K$ be a galois extension with galois group $G = \mathrm{Gal}(L/K) = \{\sigma_1, \ldots, \sigma_r\}$. We define a norm map $F$-Norm: $F(L) \to F(K)$ by the formula

$$F\text{-Norm}: F(L) \to F(K), \qquad x \mapsto \sigma_1 x +_F \sigma_2 x +_F \cdots +_F \sigma_r(x). \tag{2.2.3}$$

(The $F$-sum of the conjugates of $x$ is in $K$ because it is invariant under $G$.)

EXAMPLES. If $F = \hat{\mathbf{G}}_a$, the additive group, given by $\hat{\mathbf{G}}_a(X, Y) = X + Y$, then $F(L) = \mathfrak{M}_L$ (with its original additive group structure) and $F(K) = \mathfrak{M}_K$. The norm map, $\mathbf{G}_a$-Norm, is equal to $\mathrm{Tr}_{L/K}$, the trace map.

If $F = \hat{\mathbf{G}}_m$, the multiplicative group given by $\hat{\mathbf{G}}_m(X, Y) = X + Y + XY$, then $F(L) = U_L{}^1$, the group of units congruent to 1 mod $\pi_L$ of $A_L$. The norm map, $\hat{\mathbf{G}}_m$-Norm, becomes the ordinary norm map $U_L{}^1 \to U_K{}^1$ under the isomorphisms $F(L) \simeq U_L{}^1$ and $F(U) \simeq U_K{}^1$.

## 2.3. *Height of a Formal Group*

Let $F$ be a formal group over $A_K$. We define inductively

$$\begin{aligned} F_2(X_1, X_2) &= F(X_1, X_2), \ldots, F_{n+1}(X_1, \ldots, X_{n+1}) \\ &= F(F_n(X_1, \ldots, X_n), X_{n+1}), \ldots . \end{aligned} \tag{2.3.1}$$

Because $F$ is associative and commutative, one has that $F(X_1, \ldots, X_n) = F(X_{\sigma(1)}, \ldots, X_{\sigma(n)})$ for every permutation of $\{1, 2, \ldots, n\}$.

Let $p$ be the residue characteristic of $A_K$. One defines $[p]_F(X)$ as $[p]_F(X) = F_p(X, X, \ldots, X)$. We consider $[p]_F(X) \bmod \pi_K$. There are two possibilities (cf., [1, 4]).

(i) There exists a number $h \in \mathbf{N}$ such that $[p]_F(X) \equiv g(X^{p^h}) \bmod \pi_K$ where $g(Z) = b_1 Z + b_2 Z^2 + \cdots$ is a power series over $A_K$ with $b_1 \not\equiv 0 \bmod \pi_K$. The number $h = h(F)$ is called the *height* of $F$.

(ii) $[p]_F(X) \equiv 0 \bmod \pi_K$. In this case one defines $h = h(F)$, the height of $F$, as $h = \infty$.

2.4. *Lemma on* $F$-Norm

Let $F_r$ be a formal group over $A_K$. If $M$ is a monomial in $X_1, \ldots, X_n$, e.g., $M = X_1^{r_1} \cdots X_n^{r_n}$, we define

$$\mathrm{Tr}(M) = X_1^{r_1} \cdots X_n^{r_n} + X_2^{r_1} \cdots X_n^{r_{n-1}} X_1^{r_n} + \cdots + X_n^{r_1} X_1^{r_2} \cdots X_{n-1}^{r_n} .$$

We write $N^i(X)$ for $X_1{}^i \cdots X_n{}^i$. Using these notations one has the following.

Lemma 2.4.1.

$$F_n(X_1, \ldots, X_n) = \mathrm{Tr}(X_1) + \sum_{i=1}^{\infty} a_i N^i(X) + \sum_M a_M \mathrm{Tr}(M),$$

*where* $a_i$, $a_M \in A_K$, *and* $M$ *runs through a set of monomials of total degree* $\geqslant 2$ *which are not of the form* $N^i(X)$. *If moreover* $n = p$, *the residue characteristic of* $K$, *then* $v_K(a_i) \geqslant 1$ *unless* $i = kp^{h-1}$, $k = 1, 2, \ldots$, *and* $v_K(a_i) = 0$ *if* $i = p^{h-1}$, *where* $h$ *is the height of* $F$. (*If* $h = \infty$, $v(a_i) \geqslant 1$ *for all* $i$ *if* $p = n$.)

*Proof.* The first statement follows from the fact that $F(X, Y) \equiv X + Y$ mod (degree 2) and the fact that $F(X_1, \ldots, X_n)$ is invariant under permutations of the $X_1, \ldots, X_n$. The second part of the lemma follows from the first part and (2.3), because substituting $X$ for the $X_i$ in $\mathrm{Tr}(M)$ results in something $\equiv 0 \bmod p$ if $M$ is not of the form $N^i(X)$. Q.E.D.

Now let $L/K$ be a cyclic galois extension of degree $n$. Let $\mathrm{Tr}_{L/K}$ and $N_{L/K}$ denote the trace and norm maps. We write $N_{L/K}(x)$ for $(N_{L/K}(x))^i$. From the definition of $F$-Norm and (2.4.1) one then immediately obtains the following.

Corollary 2.4.2.

$$F\text{-Norm}_{L/K}(x) \equiv \mathrm{Tr}_{L/K}(x) + \sum_{i=1}^{\infty} a_i N^i_{L/K}(x) \bmod \mathrm{Tr}_{L/K}(x^2 A_L)$$

*for all* $x \in F(L)$. *If* $n = p$ *one has the same statements on the valuations of the* $a_i$ *as in* (2.4.1).

## 3. Unramified and Tamely Ramified Extensions

In the case of an unramified or tamely ramified extension $L/K$, the image of $F$-Norm: $F(L) \to F(K)$ is very easy to calculate.

Proposition 3.1. *Let* $L/K$ *be a tamely ramified galois extension, then* $F$-Norm: $F(L) \to F(K)$ *is surjective.*

*Proof.* First suppose that $L/K$ is unramified. $F$-Norm maps $F^s(L)$ into $F^s(K)$ and for every $y \in F(K)$ of valuation $v_K(y) = s$, there exists an $x \in F^s(L)$ such that

$$F\text{-Norm}(x) \equiv y \bmod(\pi_K^{2s}).$$

Indeed, according to (2.2.1) and (2.2.3) we have

$$F\text{-Norm}_{L/K}(x) \equiv \mathrm{Tr}_{L/K}(x) \bmod(\pi_K^{2s}),$$

and it thus suffices to select an $x \in F^s(L)$ such that $\mathrm{Tr}_{L/K}(x) = y$ which can be done because $L/K$ is unramified. It follows that the induced map $F^s(L) \to F^s(K)/F^{s+1}(K)$ is surjective and this proves the proposition in this case according to Lemma (3.2) below.

Now let $L/K$ be totally and tamely ramified. Because $\mathrm{Gal}(L/K)$ is cyclic of order prime to $p$ (cf. [8], Chapter IV, Section 2]), it suffices to treat tamely and totally ramified extensions of prime degree $l$, $(l, p) = 1$. For such extensions one has

$$\mathrm{Tr}_{L/K}(\pi_L{}^t A_L) = \pi_K{}^r A_K, \qquad r = [((l-1)+t)/l] \qquad (3.1.1)$$

where $[s/l]$ denotes the entier of $s/l$. (cf. [8, Chapter V, Section 3].) It follows that for every $s \in \mathbf{N}$ there exists a number $t_s$ such that

(i) $t_s > s$,

(ii) $v_K(\mathrm{Tr}_{L/K}(x)) > s$ if $v_L(x) > t_s$,

(iii) $v_K(\mathrm{Tr}_{L/K}(x)) = s$ if $v_L(x) = t_s$.

It follows from this and (2.4.2) that

$$F\text{-Norm}_{L/K}(zx) \equiv z\,\mathrm{Tr}_{L/K}(x) \bmod(\pi_K^{s+1}A_K)$$

if $v_L(x) = t_s$, $z \in A_K$. Using this, (2.2.2) and (iii) above we see that the induced map $F^{t_s}(L) \to F^s(K)/F^{s+1}(K)$ is surjective, which proves the proposition in this case.

Finally let $L/K$ be tamely ramified. The extension $L/K$ can be decomposed into a tower $K \subset L_{ur} \subset L$, where $L_{ur}/K$ is unramified and $L/L_{ur}$ is totally and tamely ramified. As $F\text{-Norm}_{L/K} = F\text{-Norm}_{L_{ur}/K} \cdot F\text{-Norm}_{L/L_{ur}}$ we are through. Q.E.D.

For completeness sake we state the lemma which was used twice in the proof above, and which we shall use a few more times in the sections below.

LEMMA 3.2. *Let $A$ and $B$ be abelian groups filtered by subgroups $A = A_1 \supset A_2 \supset B = B_1 \supset B_2 \supset \cdots$ such that $A = \varprojlim A/A_n$, and $\bigcap_n B_n = \{0\}$. Let $u\colon A \to B$ be a homomorphism and suppose that there exist indices $t_1 < t_2 < \cdots$*

*such that* $u(A_{t_i}) \subset B_i$ *and* $u\colon A_{t_i} \to B_i/B_{i+1}$ *is surjective for all* $i = 1, 2, \ldots$ . *Then* $u\colon A \to B$ *is surjective.*

*Proof.* Very easy, cf., e.g., [8, Chapter V, Section 1, Lemma 2].

## 4. The Cyclotomic $\Gamma$-Extension

A $\Gamma$-extension of a field $K$ is an (infinite) galois extension $K_\infty/K$ such that $\mathrm{Gal}(K_\infty/K) \simeq \mathbf{Z}_p$, the $p$-adic integers.

### 4.1. *The Cyclotomic $\Gamma$-Extension of* $\mathbf{Q}_p$

Let $\mathbf{Q}_p$ be the field of $p$-adic numbers. Adjoin to $\mathbf{Q}_p$ all $p^r$-th roots of unity, for all $r$. The result is a totally ramified abelian extension $L_\infty/\mathbf{Q}_p$ of galois group isomorphic to $U(\mathbf{Q}_p)$. Let $\Delta$ be the torsion subgroup of $U(\mathbf{Q}_p)$. If $p > 2$, this is the subgroup of the $(p-1)$-st roots of unity; if $p = 2$ this is the subgroup $\{1, -1\}$. Let $K_\infty$ be the invariant field of $\Delta$. Then $K_\infty/\mathbf{Q}_p$ is a $\Gamma$-extension (associated to the prime $p$). We shall call this extension the cyclotomic $\Gamma$-extension of $\mathbf{Q}_p$. Let $K_n$ be the invariant field of the closed subgroup $p^n \, \mathrm{Gal}(K_\infty/\mathbf{Q}_p)$. We obtain a tower of totally ramified extensions of degree $p$ $\cdots - K_{n+1} - K_n - \cdots - K_2 - K_1 - \mathbf{Q}_p = K$.

Another way to construct this $\Gamma$-extension of $K = \mathbf{Q}_p$ for $p > 2$ is as follows. Let $f(X) = X^p + pX$; Let $f^{(m)}(X)$ be the $m$-th iterate of $f(X)$, i.e., $f^{(m)}(X) = f^{(m-1)}(f(X))$, $f^{(0)}(X) = X$. Let $(LT)_{n+1}$ (the $(n+1)$-st Lubin–Tate extension of $K$; cf. [6] or [2]) be the extension generated by any root $\lambda_{n+1}$ of $f^{(n+1)}(X)$, which is not a root of $f^{(n)}(X)$. The extension $(LT)_{n+1}/K$ is galois and totally ramified; the galois group is isomorphic to $U(\mathbf{Q}_p)/U^{n+1}(\mathbf{Q}_p)$. (Cf. [6] or [2]; $U(\mathbf{Q}_p) =$ units of $\mathbf{Z}_p$; $U^n(\mathbf{Q}_p) = \{u \in U(\mathbf{Q}_p) \mid u \equiv 1 \bmod p^n\}$). The action of $u \in U(\mathbf{Q}_p)$ is given by $\lambda_{n+1} \mapsto [u]_f(\lambda_{n+1})$, where $[u]_f(X)$ is the unique power series such that $[u]_f(X) \equiv uX \bmod (\text{degree } 2)$ and $[u]_f \circ f = f \circ [u]_f$. Let $\zeta$ be a $(p-1)$-st root of unity, then $[\zeta]_f(X) = \zeta X$, because $(\zeta X)^p + p(\zeta X) = \zeta(X^p + pX)$. The element $\mu_n = \lambda_{n+1}^{p-1}$ is therefore invariant under the action of $\Delta$. The extension $\bigcup_n \mathbf{Q}_p(\mu_n)/\mathbf{Q}_p$ is the cyclotomic $\Gamma$-extension of $\mathbf{Q}_p$. (If $p = 2$ one obtains in this way the whole extension $L_\infty/\mathbf{Q}_2$.)

### 4.2. *The Number* $m(L/K)$

Let $L/K$, be a totally ramified extension of degree $p$. Then there exists a certain number $m(L/K) \in \mathbf{N}$ such that

$$\mathrm{Tr}_{L/K}(\pi_L{}^t A_L) = \pi_K{}^r A_K \quad \text{where} \quad r = [\{(m(L/K) + 1)(p-1) + t\}/p] \tag{4.2.1}$$

(cf. [8, Chapter IV, Section 2]).

### 4.3. *Equations for* $\mu_n$

It is not difficult to find equations for the $\mu_n$ defined in (4.1). Indeed, we can choose $\lambda_1, \lambda_2, \ldots$ inductively such that $\lambda_{n+1}^p + p\lambda_{n+1} = \lambda_n$, $n \geqslant 1$, $\lambda_1^{p-1} = -p$. We have $\mu_n = \lambda_{n+1}^{p-1}$; it follows that $\mu_0 = -p$ and that

$$X(X + p)^{p-1} - \mu_{n-1} \tag{4.3.1}$$

is the minimal polynomial of $\mu_n$ over $K_{n-1}$. (Note that $\mu_n$ is a uniformizing element of $K_n$.)

### 4.4. *The Numbers* $m_n$

Let $m_n = m(K_n/K_{n-1})$, $n = 1, 2, \ldots$. One finds by explicit calculations from Eq. (4.3.1) above that

$$\begin{aligned} \mathrm{Tr}_{n/n-1}(\mu_n) &= -(p-1)p, \\ \mathrm{Tr}_{n/n-1}(\mu_n{}^2) &= (p-1)\,p^2, \\ \mathrm{Tr}_{n/n-1}(\mu_n^{p-1}) &= (-1)^{p-1}(p-1)\,p^{p-1}. \end{aligned} \tag{4.4.1}$$

(We have written $\mathrm{Tr}_{n/n-1}$ for $\mathrm{Tr}_{K_n/K_{n-1}}$). Comparing this with (4.2.1) one finds that

$$m_n = 1 + p + \cdots + p^{n-1}. \tag{4.4.2}$$

In the sections below we shall need to know something about $\mathrm{Tr}_{n/n-1}(\mu_n{}^k)$, especially in the case that $k$ is a multiple of $p$.

TRACE LEMMA 4.5.

$$\mathrm{Tr}_{n/n-1}(\mu_n^{kp+c}) \equiv 0 \bmod \mu_{n-1}^k p^c,\ c = 1, 2, \ldots, p-1;\ k = 0, 1, 2, \ldots,$$

$$\mathrm{Tr}_{n/n-1}(\mu_n^{kp}) \equiv p\mu_{n+1}^k \bmod \mu_{n-1}^{k-1} p^p,\ k = 1, 2, \ldots.$$

*Proof.* The formulas (4.4.1) above take care of the cases $k = 0$, $c = 1, 2, \ldots, p-1$. We have the relation

$$\mu_n{}^p + \binom{p-1}{1}\mu_n^{p-1}p + \cdots + \mu_n\binom{p-1}{p-1}p^{p-1} = \mu_{n-1}. \tag{4.5.1}$$

Applying $\mathrm{Tr}_{n/n-1}$ and using (4.4.1) we see that

$$\mathrm{Tr}_{n/n-1}(\mu_n{}^p) \equiv p\mu_{n-1} \quad \mathrm{mod}(\,p^p). \tag{4.5.2}$$

To prove the lemma for $kp + c > p$, multiply the relation (4.5.1) with $\mu_n^{(k-1)p+c}$ and use induction.

## 5. Some Preliminary Calculations

In this and the following Sections 6, 7, $K = K_0 = \mathbf{Q}_p$, and

$$\cdots - K_n - \cdots - K_1 - K$$

is the tower of extensions constructed in 4.1, $K_n = K_{n-1}(\mu_n)$. If $p > 2$, $\bigcup_n K_n$ is the cyclotomic $\Gamma$-extension: if $p = 2$, $\bigcup_n K_n/K$ has galois group isomorphic to $U(\mathbf{Q}_2) \simeq \mathbf{Z}_2 \times \{1, -1\}$. We write $F$-$\mathrm{Norm}_{n/k}$ or $\mathrm{Norm}_{n/k}$ for $F$-$\mathrm{Norm}_{K_n/K_k}$ and $N_{n/k}$ for $N_{K_n/K_k}$. Further $v_{K_n} = v_n$, $A_{K_n} = A_n$.

Lemma 5.1. *Let $x \in F^t(K_n)$. Then*

$$v_{n-1}(\mathrm{Norm}_{n/n-1}(x) \geqslant \min\{[p^{-1}((m_n+1)(p-1)+t)], p^{h-1}t\}.$$

*Proof.* It follows from (2.4.2) that

$$v_{n-1}(\mathrm{Norm}_{n/n-1}(x) \geqslant \min\{[p^{-1}((m_n+1)(p-1)+t)], v_{n-1}(a_i)+ti\}.$$

Because $v_{n-1}(a_{p^{h-1}}) = 0$, we can omit $v_{n-1}(a_i) + ti$ for $i > p^{h-1}$ without changing the minimum. If $1 \leqslant i < p^{h-1}$, then $p \mid a_i$, and $v_{n-1}(a_i) + ti \geqslant p^{n-1} + ti > p^{-1}((m_n+1)(p-1)+t)$, because $m_n = 1 + p + \cdots + p^{n-1}$ and $t \geqslant 1$, $i \geqslant 1$. Q.E.D.

Lemma 5.1 shows that the numbers $[p^{-1}((m_n+1)(p-1)+t)]$, $p^{h-1}t$ are probably important in the determination of $\mathrm{Norm}_{n/0}(F(K_n))$.

### 5.2. *The Functions $\sigma_{n/k}(t)$ and $\iota_{n/k}(t)$*

We define inductively

$$\begin{aligned} \sigma_{n/n}(t) &= t,\ \sigma_{n/k}(t) = \sigma_{k+1/k}(\sigma_{n/k+1}(t)), \\ \sigma_{n/n-1}(t) &= \min\{[p^{-1}((m_n+1)(p-1)+t)], p^{h-1}t\}. \end{aligned} \tag{5.2.1}$$

It is also convenient to define

$$\begin{aligned} \sigma^0_{n/n-1}(t) &= [p^{-1}((m_n+1)(p-1)+t)], \\ \sigma^1_{n/n-1}(t) &= p^{h-1}t, \end{aligned} \tag{5.2.2}$$

and

$$\begin{aligned} \iota_{n/k}(t) &= -1 \text{ if } \sigma^0_{k/k-1}(\sigma_{n/k}(t)) \leqslant \sigma^1_{k/k-1}(\sigma_{n/k}(t)) \\ &= h-1 \text{ if } \sigma^0_{k/k-1}(\sigma_{n/k}(t)) > \sigma^1_{k/k-1}(\sigma_{n/k}(t)). \end{aligned} \tag{5.2.3}$$

It follows immediately from the definitions that if $k < n$

$$\sigma_{n/k}(t) = \min\{\sigma^0_{k+1/k}(\sigma_{n/k+1}(t)),\ \sigma^1_{k+1/k}(\sigma_{n/k+1}(t))\}. \qquad (5.2.4)$$

The function $\iota_{n/k}(t)$ indicates whether it is the value of $\sigma^0_{k/k-1}$ or $\sigma^1_{k/k-1}$ which determines $\sigma_{n/k-1}(t)$, or in other words whether in the step from $K_k$ to $K_{k-1}$ (having started in $K_n$ with an element of valuation $t$), it is $\mathrm{Tr}_{k/k-1}(\mathrm{Norm}_{n/k}(x))$ or $N^{p^{h-1}}_{k/k-1}(\mathrm{Norm}_{n/k}(x))$ for which the lower bound on the valuation is sharpest.

LEMMA 5.3.

$$\mathrm{Norm}_{n/0}(F^t(K_n)) \subset F^{\sigma_{n/0}(t)}(K).$$

This follows immediately from (5.2.1) and Lemma 5.1.

We now proceed to calculate the functions $\sigma_{n/0}(t)$. In case $h = 1$, the functions $\sigma_{n/0}(t)$ are determined by the Herbrand functions $\psi_{K_n/K}(s)$. Indeed $\psi_{K_n/K}(s) < t \leqslant \psi_{K_n/K}(s+1)$ is equivalent to $\sigma_{n/0}(t) = s + 1$.

LEMMA 5.4.

$$\iota_{n/n}(t) = -1 \leftrightarrow t \geqslant (p^n - 1)/(p^h - 1).$$

*Proof.* $\iota_{n/n}(t) = -1$ is equivalent to $\sigma^0_{n/n-1}(t) \leqslant \sigma^1_{n/n-1}(t)$; i.e., $\iota_{n/n}(t) = -1$ iff

$$\begin{aligned}
&[p^{-1}((1 + p + \cdots + p^{n-1} + 1)(p - 1) + t)] \leqslant p^{h-1}t \\
&\leftrightarrow p^{-1}((1 + p + \cdots + p^{n-1} + 1)(p - 1) + t) \leqslant p^{h-1}t + (p - 1)/p \\
&\qquad \leftrightarrow (p^n - 1) + (p - 1) + t \leqslant p^h t + (p - 1) \\
&\qquad\qquad \leftrightarrow t \geqslant (p^n - 1)/(p^h - 1).
\end{aligned}$$

LEMMA 5.5. *If $k \geqslant 2$ and $\iota_{n/k}(t) = -1$, then $\iota_{n/k-1}(t) = -1$.*

*Proof.* Let $s = \sigma_{n/k}(t)$. Then $\iota_{k/k}(s) = -1$. Let $s' = \sigma_{k/k-1}(s) = \sigma^0_{k/k-1}(s)$. We must show that $\iota_{k-1/k-1}(s') = -1$. We know that

$$s \geqslant (p^k - 1)/(p^h - 1).$$

Hence

$$\begin{aligned}
s' &= \left[\frac{(m_k + 1)(p - 1) + s}{p}\right] \geqslant \frac{m_k(p - 1)}{p} + \frac{p^k - 1}{(p^h - 1)p} \\
&= \frac{p^k - 1}{p} + \frac{p^k - 1}{(p^h - 1)p} \geqslant \frac{p^{k-1} - 1}{p^h - 1}.
\end{aligned}$$

Using (5.4), (5.5) and (4.2) it is not difficult to calculate $\sigma_{n/0}(t)$ for large enough $t$. We find the following.

LEMMA 5.6.

$$\frac{p^n-1}{p^k-1} \leqslant t \leqslant \frac{p^n-1}{p-1} \rightarrow \sigma_{n/0}(t) = n$$

$$\frac{p^n-1}{p-1} + kp^n < t \leqslant \frac{p^n-1}{p-1} + (k+1)\,p^n \rightarrow \sigma_{n/0}(t) = n+k+1.$$

Let $j_n(t)$ be the number of indices $k = n, n-1,..., 2, 1$ such that $\iota_{n/k}(t) = h-1$. In view of (5.5) we have $j_n(t) = s \geqslant 1 \leftrightarrow \iota_{n/n}(t) = h-1,..., \iota_{n/n-s+1}(t) = h-1$, $\iota_{n/n-s}(t) = -1,..., \iota_{n/1}(t) = -1$.

LEMMA 5.7.

$$j_n(t) = s \geqslant 1 \leftrightarrow \frac{p^{n-sh} - p^{s-sh}}{p^h-1} \leqslant t < \frac{p^{n-(s-1)h} - p^{(s-1)-(s-1)h}}{p^h-1},$$

$$j_n(t) = 0 \leftrightarrow \frac{p^n-1}{p^h-1} \leqslant t.$$

*Proof.* The second formula follows from (5.4) and (5.5). As to the first:

$$j_n(t) = s \geqslant 1 \leftrightarrow p^{s(h-1)}t < \left[\frac{(m_{n-s+1}+1)(p-1) + p^{(s-1)(h-1)}t}{p}\right]$$

$$\text{and } p^{(s+1)(h-1)}t \geqslant \left[\frac{(m_{n-s}+1)(p-1) + p^{s(h-1)}t}{p}\right].$$

(Use (5.5) and the fact that $\iota_{k/k}(t') = -1$ if $\iota_{k/k}(t'') = -1$ and $t' \geqslant t''$ (cf. (5.4)) and $p^{m(h-1)}t \geqslant \sigma_{n/n-m}(t)$.) The same calculations as in (5.4) now prove (5.7). Q.E.D.

PROPOSITION 5.8. *Write $n = lh + r$, with $1 \leqslant r \leqslant h$. Then we have*

$$1 \leqslant t \leqslant \frac{p^r-1}{p-1} \rightarrow \sigma_{n/0}(t) = n-l,$$

$$\frac{p^r-1}{p-1} < t \leqslant \frac{p^{r+h}-1}{p-1} \rightarrow \sigma_{n/0}(t) = n-l+1,$$

$$\frac{p^{r+kh}-1}{p-1} < t \leqslant \frac{p^{r+kh+h}-1}{p-1} \rightarrow \sigma_{n/0}(t) = n-l+k+1, \quad k = 0, 1,..., l-1,$$

$$\frac{p^n-1}{p-1} < t \leqslant \frac{p^h-1}{p-1} + p^n \rightarrow \sigma_{n/0}(t) = n+1,$$

$$\frac{p^n-1}{p-1} + kp^n < t \leqslant \frac{p^n-1}{p-1} + (k+1)\,p^n \rightarrow \sigma_{n/0}(t) = n+k+1, \quad k = 1, 2,... .$$

5.9. *Remark*

These formulas are also true if $h = \infty$; take $l = 0$, $r = n$.

COROLLARY 5.10.

$$F\text{-Norm}_{n/0}(F(K_n)) \subset F^{\alpha_n}(K),$$

*where* $\alpha_n = n - [(n-1)/h]$.

5.11. *Proof of Proposition* 5.8

Let $j_n(t) = s \geqslant 1$. Then according to (5.7)

$$\frac{p^{n-sh} - p^{s-sh}}{p^h - 1} \leqslant t < \frac{p^{n-sh+h} - p^{s-1-sh+h}}{p^h - 1}.$$

Further

$$\frac{p^{n-s} - 1}{p^h - 1} \leqslant \sigma_{n/n-s}(t) = p^{s(h-1)}t < \frac{p^{n+h-s} - p^{h-1}}{p^h - 1}.$$

We have $\iota_{n-s/n-s}(\sigma_{n/n-s}(t)) = -1$ (because $j_n(t) = s$) and we can therefore now calculate $\sigma_{n-s/0}(\sigma_{n/n-s}(t)) = \sigma_{n/0}(t)$ by means of Lemma 5.6. The result is

$$\frac{p^{n-sh} - p^{s-sh}}{p^h - 1} \leqslant t \leqslant \frac{p^{n-sh} - p^{s-sh}}{p - 1} \rightarrow \sigma_{n/0}(t) = n - s \qquad (5.11.1)$$

and

$$\frac{p^{n-sh} - p^{s-sh}}{p - 1} < t < \frac{p^{n+h-sh} - p^{s-1-sh+h}}{p^h - 1} \rightarrow \sigma_{n/0}(t) = n - s + 1. \qquad (5.11.2)$$

Because $h \geqslant 1$, we have that $0 < p^{s-sh} \leqslant 1$ for all $s = 0, 1, 2, \ldots$. It follows that

$$t \geqslant \frac{p^{n-sh} - p^{s-sh}}{p - 1} \leftrightarrow \frac{p^{n-sh} - 1}{p - 1} \leqslant t. \qquad (5.11.3)$$

Now put the formulas (5.11.2) and (5.11.1) for $s = 1, 2, \ldots, l$ together (note that $s = l + 1$ gives nothing if $n = lh + h$); use (5.11.3) and combine this with the result of (5.6). The result is Proposition 5.8.

## 6. STATEMENT OF THE THEOREM AND OUTLINE OF THE PROOF

THEOREM 6.1. *Let $F$ be a formal group over $\mathbf{Z}_p$. Let*

$$\cdots - K_n - K_{n-1} - \cdots - K_1 - K = \mathbf{Q}_p$$

*be the tower of extensions constructed in Section* 4. *(If $p > 2$, $\bigcup_n K_n$ is the* *cyclotomic $\Gamma$-extension of $\mathbf{Q}_p$: if $p = 2$ it is a slightly larger abelian totally ramified extension). Let $h = h(F) \geqslant 2$. Then we have ($n \geqslant 1$)*

$$F\text{-Norm}_{n/0}(F(K_n)) = F^{\alpha_n}(K),$$

*where $\alpha_n$ is equal to $\alpha_n = n - [(n-1)/h]$.*

*Remark* 6.2. The theorem is also true for $h = \infty$; $(n-1)/h = 0$.

6.3. *Proof of Theorem* 6.1 *in case $h = \infty$.* For each $s \geqslant n$, let

$$t_s = (p^n - 1)/(p-1) + (s-n)p^n.$$

It is not difficult to calculate $\sigma_{n/k}(t_s)$ and $\sigma_{n/k}(t_s+1)$ for $k = n-1$, $n-2, \ldots, 2, 1, 0$. One finds

$$\sigma_{n/k}(t_s) = (p^k - 1)/(p-1) + (s-n)\,p^k + (n-k)\,p^k \quad \text{for } k \geqslant 1 \text{ and } \sigma_{n/0}(t_s) = s \tag{6.3.1}$$

$$\sigma_{n/k}(t_s+1) = (p^k - 1)/(p-1) + (s-n)\,p^k + (n-k)\,p^k + 1 \quad \text{for } k \geqslant 1 \text{ and } \sigma_{n/0}(t_s+1) = s+1.$$

It is now easy to check that

$$\sigma^0_{k/k-1}(\sigma_{n/k}(t_s)) < \sigma^1_{k/k-1}(\sigma_{n/k}(t_s)). \tag{6.3.2}$$

It follows from this, (2.4.2) and (6.3.1) that the induced map

$$\text{Norm}_{k/k-1} : F^{\sigma_{n/k}(t_s)}(K_k)/F^{\sigma_{n/k}(t_s)+1}(K_k) \to F^{\sigma_{n/k-1}(t_s)}(K_{k-1})/F^{\sigma_{n/k-1}(t_s)+1}(K_{k-1}) \tag{6.3.3}$$

is equal to the map

$$\text{Tr}_{k/k-1} : \pi_k^{\sigma_{n/k}(t_s)} A_k / \pi_k^{\sigma_{n/k}(t_s)+1} A_k \to \pi_{k-1}^{\sigma_{n/k-1}(t_s)} A_{k-1} / \pi_{k-1}^{\sigma_{n/k-1}(t_s)+1} A_{k-1}\,. \tag{6.3.4}$$

This last map is surjective because

$$\sigma^0_{k/k-1}(\sigma_{n/k}(t_s) + 1) = \sigma^0_{k/k-1}(\sigma_{n/k}(t_s)) + 1,$$

and $K_k/K_{k-1}$ is totally ramified (cf. 4.2.1). It follows from this and the fact that (6.3.3) and (6.3.4) are the same maps that the map

$$\text{Norm}_{n/0} : F^{t_s}(K_n) \to F^{\sigma_{n/0}(t_s)}(K)/F^{\sigma_{n/0}(t_s)+1}(K) \tag{6.3.5}$$

is surjective. In view of Lemma 3.2 and Corollary 5.10 this concludes the proof in case $h = \infty$ because $\sigma_{n/0}(t_n) = n = \alpha_n$ if $h = \infty$.

6.4. *Idea of the proof of Theorem* 6.1 *in case $h < \infty$*. A first step in the proof of Theorem 6.1 is to show that for every $s \geqslant n - [(n-1)/h]$ there exists a $t_s$, and an element $x_s \in \pi_n A_n$ such that $v_0(\mathrm{Norm}_{n/0}(x_s)) = s$. For $s \geqslant n$ one can take $t_s = (p^n - 1)/(p-1) + (s-n)p^n$ (cf. (6.3)). Let $l = [(n-1)/h]$. For $n - l \leqslant s < n$ a natural choice of $t_s$ is

$$t_s = \frac{p^{n-(n-s)h} - 1}{p - 1}. \tag{6.4.1}$$

Then $j_n(t_s) = n - s$ (cf. (5.7)). It is easy to calculate $\sigma_{n/k}(t_s)$ for $k = n-1, n-2, \ldots, 1, 0$. The result is

$$\begin{aligned}
\sigma_{n/m}(t_s) &= p^{(n-m)(h-1)}t_s \qquad \text{for} \quad n \geqslant m \geqslant s,\\
\sigma_{n/m}(t_s) &= p^{(n-s)(h-1)}p^{-(s-m)}t_s + (s-m)p^m, \quad s \geqslant m \geqslant n-(n-s)h,\\
\sigma_{n/n-(n-s)h}(t_s) &= t_s + (n-s)(h-1)p^{n-(n-s)h},\\
\sigma_{n/m}(t_s) &= (p^m - 1)/(p-1) + (s-m)p^m, \quad n-(n-s)h \geqslant m \geqslant 0,\\
\sigma_{n/0}(t_s) &= s.
\end{aligned} \tag{6.4.2}$$

As in (6.3) it is useful to calculate also $\sigma_{n/k}(t_s + 1)$. Because $h > 1$, also $j_n(t_s + 1) = n - s$. Let $\alpha_{n/k}(t_s)$ be defined by

$$\begin{aligned}
\alpha_{n/k}(t_s) &= (n-m)(h-1), && \text{for} \quad n \geqslant m \geqslant s,\\
\alpha_{n/k}(t_s) &= (n-s)(h-1) - (s-m), && \text{for} \quad s \geqslant m \geqslant n-(n-s)h,\\
\alpha_{n/k}(t_s) &= 0, && \text{for} \quad n-(n-s)h \geqslant m \geqslant 0.
\end{aligned} \tag{6.4.3}$$

One then has

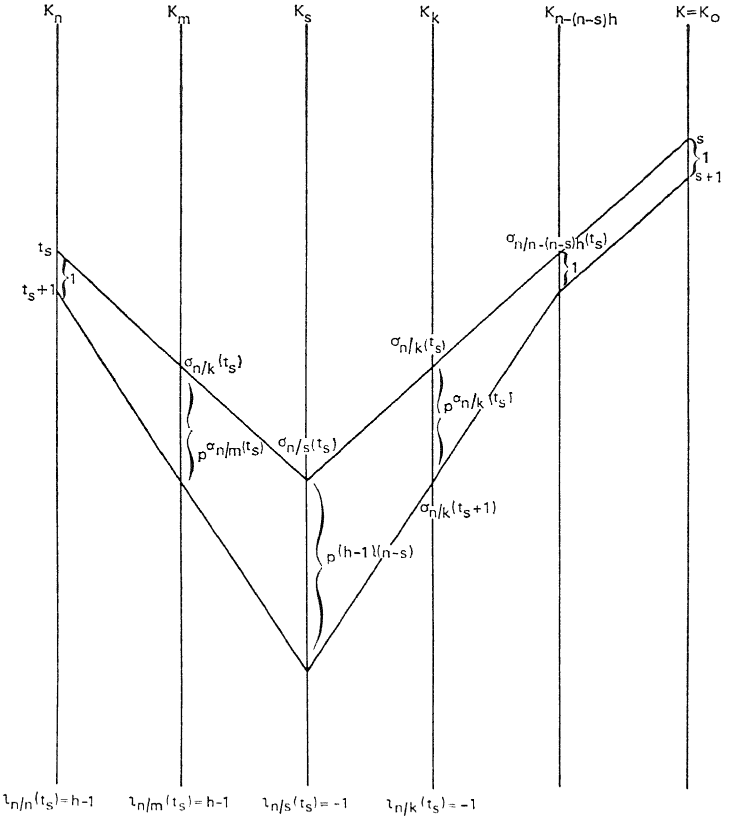
$$\sigma_{n/k}(t_s + 1) = \sigma_{n/k}(t_s) + p^{\alpha_{n/k}(t_s)}. \tag{6.4.4}$$

(In all these calculations the simple fact $\sigma^0_{k/k-1}(rp) = r + p^{k-1}$, $k \geqslant 2$ is very usefull. It follows immediately from $m_k = (1 + p + \cdots + p^{k-1})$).

A convenient picture of $\sigma_{n/k}(t_s)$ and $\sigma_{n/k}(t_s + 1)$ is sketched below.

According to Lemma 6.5, to calculate $\mathrm{Norm}_{n/k}(x) \bmod \pi_k^{\sigma_{n/k}(t_s+1)}$, where $x$ has valuation $v_n(x) = t_s$, we can disregard for all $m$, where $n \geqslant m \geqslant k$ all terms of $\mathrm{Norm}_{n/m}(x)$ of which the valuation falls below the lower line in the picture above. In Section 7 below we shall show that in fact for $x \in \pi_m A_m$

$$\mathrm{Norm}_{m/m-1}(x) \equiv N^{p^{h-1}}_{m/m-1}(x) \bmod \pi_{m-1}^{\sigma_{n/m-1}(t_s+1)}, \text{ if } v_m(x) = \sigma_{n/m}(t_s)$$
$$\text{and} \quad n \geqslant m > s \tag{6.4.5}$$




and for $x \in \pi_k A_k$

$$\mathrm{Norm}_{k/k-1}(x) \equiv \mathrm{Tr}_{k/k-1}(x) \bmod \pi_{k-1}^{\sigma_{n/k-1}(t_s+1)} \quad \text{if} \quad v_k(x) = \sigma_{n/k}(t_s)$$

$$\text{and} \quad s \geqslant k > 0. \qquad (6.4.6)$$

LEMMA 6.5. *Let* $t > t' \geqslant 1$, $\sigma_{m/k}(t) = s$, $\sigma_{m/k}(t') = s'$. *If* $x, y \in \pi_m A_m$, $v_m(x) = t$, $v_m(y) = t'$, *then*

$$\mathrm{Norm}_{m/k}(x + y) \equiv \mathrm{Norm}_{m/k}(x) \bmod \pi_k^{s'}.$$

*Proof.* Because $A_m$ is complete and (2.1.1), (2.2.1), there is an $y' \in \pi_m^{t'} A_m$ such that $x + y = x +_F y'$. Now

$$\mathrm{Norm}_{m/k}(x +_F y') = \mathrm{Norm}_{m/k}(x) +_F \mathrm{Norm}_{m/k}(y').$$

Because $\sigma_{m/k}(t') = s'$, $\mathrm{Norm}_{m/k}(y') \equiv 0 \bmod \pi_k^{s'}$. Another appeal to (2.1.1) concludes the proof.

## 7. Proof of Theorem 6.1

Proposition 7.1. *Let $F$, $K_n$, $h$, $\alpha_n$ be as in Theorem 6.1. In this section we take the uniformizing element $\pi_n$ of $K_n$ equal to $\mu_n$. Then for every $s \geqslant \alpha_n = n - [(n-1)/h]$, there is a $t_s$ such that*

(i) *$F$-$\mathrm{Norm}_{n/0}$ maps $F^{t_s}(K_n)$ into $F^s(K)$.*

(ii) *$F$-$\mathrm{Norm}_{k/0}$ maps $F^{\sigma_{n/k}(t_s+1)}(K_k)$ into $F^{s+1}(K)$ for all $0 < k \leqslant n$.*

(iii) *The induced map*

$$F^{t_s}(K_n) \to F^s(K)/F^{s+1}(K)$$

*is surjective.*

*Proof.* Let $n = lh + r$, $1 \leqslant r \leqslant h$. For $s \geqslant n$ take

$$t_s = (p^n - 1)/(p-1) + (s-n)p^n.$$

For $n - l \leqslant s < n$ take $t_s = (p^{n-(n-s)h} - 1)/(p-1)$. Parts (i) and (ii) of the proposition then follow from (6.4.2)–(6.4.4). For $s \geqslant n$ (iii) follows from (6.3) (the proof for $h = \infty$) and (6.5). Now let $n - l \leqslant s < n$. We shall first establish (6.4.5) and (6.4.6).

Let $n \geqslant j > s$. To prove (6.4.5) we must show that

$$\sigma^0_{j/j-1}(\sigma_{n/j}(t_s)) \geqslant \sigma_{n/j-1}(t_s) + p^{\alpha_{n/j-1}(t_s)}, \tag{7.1.1}$$

$$v_{j-1}(a_i) + i\sigma_{n/j}(t_s) \geqslant \sigma_{n/j-1}(t_s) + p^{\alpha_{n/j-1}(t_s)}, \quad i \in \mathbf{N}, \quad i \neq p^{h-1}, \tag{7.1.2}$$

where $a_i$, $i = 1, 2, \ldots$ are the coefficients appearing in formula (2.4.2) for $F$-Norm.

Now

$$\begin{aligned}\sigma^0_{j/j-1}(\sigma_{n/j}(t_s)) &= \left[\frac{(m_j+1)(p-1) + p^{(n-j)(h-1)}\,t_s}{p}\right] \\ &\geqslant p^{j-1} + p^{-1}p^{(n-j)(h-1)}\,\frac{p^{n-(n-s)h} - 1}{p-1} - \frac{1}{p},\end{aligned}$$

and

$$\begin{aligned}\sigma_{n/j-1}(t_s) + p^{\alpha_{n/j-1}(t_s)} &= p^{(n-j+1)(h-1)} \cdot \frac{p^{n-(n-s)h} - 1}{p-1} + p^{(n-j+1)(h-1)} \\ &= p^{h+sh-jh+j-2} + \cdots + p^{(n-j+1)(h-1)+1} + 2p^{(n-j+1)(h-1)} \\ &\leqslant p^{j-1}\end{aligned}$$

because $j-1 \geqslant (h+sh-jh+j-2)+1$ (as $j \geqslant s+1$). This proves (7.1.1). If $i$ is not a multiple of $p^{h-1}$, $v_{j-1}(a_i)+i.\ t \geqslant \sigma^0_{j/j-1}(t)$ for all $t \in \mathbf{N}$, this proves (7.1.2) for those $i \neq p^{h-1}$, which are not a multiple of $p^{h-1}$. Finally if $i \geqslant 2p^{h-1}$, then

$$\begin{aligned} v_{j-1}(a_i) + i\sigma_{n/j}(t_s) &\geqslant p^{2(h-1)}\sigma_{n/j}(t_s) \\ &\geqslant 2p^{h-1}\sigma_{n/j}(t_s) \geqslant p^{h-1}\sigma_{n/j}(t_s) + p^{h-1}\cdot p^{\alpha_{n/j}(t_s)} \\ &= p^{h-1}\sigma_{n/j}(t_s) + p^{\alpha_{n/j-1}(t_s)} \end{aligned}$$

because $\sigma_{n/j}(t_s) \geqslant p^{\alpha_{n/j}(t_s)}$. This proves (7.1.2).

To prove (6.4.6) we must show that (cf., (2.4.2)) for $s \geqslant j > 0$

$$\sigma^1_{j/j-1}(\sigma_{n/j}(t_s)) \geqslant \sigma_{n/j-1}(t_s) + p^{\alpha_{n/j-1}(t_s)}, \tag{7.1.3}$$

$$v_{j-1}(a_i) + i\cdot t_s \geqslant \sigma_{n/j-1}(t_s) + p^{\alpha_{n/j-1}(t_s)},\ i = 1, 2, 3, \ldots, \tag{7.1.4}$$

$$\sigma^0_{j/j-1}(2\sigma_{n/j}(t_s)) \geqslant \sigma_{n/j-1}(t_s) + p^{\alpha_{n/j-1}(t_s)}. \tag{7.1.5}$$

First let $s \geqslant j > n-(n-s)h$. Then $\alpha_{n/j}(t_s) > 0$ and $p$ divides $\sigma_{n/j}(t_s)$ (cf. (6.4.2)). It follows that

$$\sigma^0_{j/j-1}(2\sigma_{n/j}(t_s)) = \sigma_{n/j-1}(t_s) + p^{-1}\sigma_{n/j}(t_s).$$

As $\sigma_{n/j}(t_s) \geqslant p^{\alpha_{n/j}}(t_s)$, and $\alpha_{n/j-1}(t_s) = \alpha_{n/j}(t_s) - 1$, this proves (7.1.5) for $s \geqslant j > n-(n-s)h$. If $n-(n-s)h \geqslant j > 0$, then $\sigma_{n/j}(t_s) \geqslant p$ and hence

$$\begin{aligned} \sigma_{j/j-1}(2\sigma_{n/j}(t_s)) &\geqslant \sigma_{j/j-1}(\sigma_{n/j}(t_s)) + 1 \\ &= \sigma_{n/j-1}(t_s) + 1 = \sigma_{n/j-1}(t_s) + p^{\alpha_{n/j-1}(t_s)}. \end{aligned}$$

This proves (7.1.5). As to (7.1.4), let $i$ be not divisible by $p^{h-1}$. Then $v_{j-1}(a_i) \geqslant p^{j-1}$ and we have

$$\begin{aligned} \left[\frac{(m_j+1)(p-1)+2t}{p}\right] &\leqslant \frac{(m_j+1)(p-1)+2t}{p} \\ &= p^{j-1} + \frac{p-1}{p} + \frac{2t}{p} \leqslant p^{j-1} + it \leqslant v_{j-1}(a_i) + it \end{aligned}$$

provided $p > 2$ and $t \geqslant p$. If $p = 2$ then

$$\left[\frac{(m_j+1)(p-1)+2t}{p}\right] = 2^{j-1} + t \leqslant v_{j-1}(a_i) + it$$

for all $t \geqslant 1$. As $\sigma_{n/j}(t_s) \geqslant p$ for all $s \geqslant j > 0$ this shows that (7.1.5) implies (7.1.4) for those $i$ which are not divisible by $p^{h-1}$. If $i$ is divisible by $p^{h-1}$ (7.1.4) follows from (7.1.3) (which is the case $i = p^{h-1}$ of (7.1.4)). It therefore remains to prove (7.1.3). We have

$$\sigma^1_{j/j-1}(\sigma_{n/j}(t_s)) = p^{h-1} \cdot p^{(n-s)(h-1)} \cdot p^{-(s-j)} t_s + (s-j) p^j \cdot p^{h-1}$$

and

$$\begin{aligned}\sigma_{n/j-1}(t_s) &+ p^{\alpha_{n/j-1}(t_s)} \\ &= p^{(n-s)(h-1)} \cdot p^{-(s-j+1)} \cdot t_s + (s-j+1) p^{j-1} + p^{(n-s)(h-1)} p^{-(s-j+1)}.\end{aligned}$$

If $s > j > n-(n-s)h$, we have $(s-j)p^j p^{h-1} - (s-j+1)p^{j-1} \geqslant 3p^{j-1}$ and $(n-s)(h-1) - (s-j+1) \leqslant j-1$ because $s \geqslant n-l > n-(n/h)$. This proves (7.1.3) in this case. If $s > n-(n-s)h \geqslant j > 0$, then $\alpha_{n/j-1}(t_s) = 0$ and $(s-j)p^j p^{h-1} \geqslant (s-j+1)p^{j-1} + 1$. It remains to prove (7.1.3) in the case $s = j$. We have to prove that

$$p^{h-1} \cdot p^{(n-s)(h-1)} \cdot t_s \geqslant p^{-1} \cdot p^{(n-s)(h-1)} \cdot t_s + p^{s-1} + p^{-1} \cdot p^{(n-s)(h-1)}$$

or equivalently

$$t_s \geqslant (p^{n-(n-s)h} + 1)/(p^h - 1)$$

as $t_s = (p-1)^{-1}(p^{n-(n-s)h} - 1)$, this follows from the fact that

$$(p-1)^{-1}(p^f - 1) \geqslant (p^h - 1)^{-1}(p^f + 1)$$

if $f \geqslant 1$, and $h \geqslant 2$ and the fact that $n-(n-s)h = n - nh + sh \geqslant n - nh + (n-l)h = n - lh = r \geqslant 1$ because $s \geqslant n-l$ and $n = lh + r$, $1 \leqslant r \leqslant h$. This concludes the proof of (6.4.6).

Let $a = a_{p^{h-1}}$, the coefficient of $N^{p^{h-1}}$ in (2.4.2). Let $z \in A_0 = A_K = \mathbf{Z}_p$. According to (6.4.5) and (5.5) we have

$$\mathrm{Norm}_{n/s}(z\mu_n^{t_s}) \equiv \pm z^{p^{(n-s)h}} a^{n-s} \cdot \mu_{n-s}^{\sigma_{n/s}(t_s)} \bmod \mu_{n-s}^{\sigma_{n/s}(t_s+1)} \qquad (7.1.6)$$

(the sign is $+$ if $p > 2$, and $(-1)^{n-s}$ if $p = 2$).

For $k \leqslant s$, it is $\mathrm{Tr}_{k/k-1}$ which is the most important part of $F$-$\mathrm{Norm}_{k/k-1}$ according to (6.4.6). We wish to apply (4.5) and shall therefore need to show that for $s \geqslant k > n-(n-s)h$

$$\mathrm{Tr}_{k/k-1}(p^{s-k}\mu_k^{p^{-(s-k)}\sigma_{n/s}(t_s)}) \equiv p^{s-k+1}\mu_{k-1}^{p^{-(s-k+1)}\sigma_{n/s}(t_s)} \bmod \mu_{k-1}^{\sigma_{n/k-1}(t_s+1)}. \qquad (1.1.7)$$

(Note that $v_k(p^{s-k}) + p^{-(s-k)}\sigma_{n/s}(t_s) = \sigma_{n/k}(t_s)$ for $s \geqslant k \geqslant n-(n-s)h$; furthermore, $n-(n-s)h \geqslant r \geqslant 1$, and for $k \leqslant n-(n-s)h$, $\sigma_{n/k}(t_s)$

contains no factors $p$ so that we cannot apply (the second formula of) Lemma 4.5 for $k \leqslant n - (n-s)h$).

If $s \geqslant k > n - (n-s)k$, there is a factor $p$ in $p^{-(s-k)}\sigma_{n/s}(t_s)$ so that we can apply the second formula of Lemma 4.5. The result is that formula (7.1.7) holds modulo

$$p^{s-k} \cdot p^p \cdot \mu_{k-1}^{p^{-(s-k+1)}\sigma_{n/s}(t_s)} \cdot \mu_{k-1}^{-1} .$$

We must show that the valuation of this is larger than or equal to $\sigma_{n/k-1}(t_s)$. But $v_{k-1}(p^{s-k+1}) + p^{-(s-k+1)}\sigma_{n/s}(t_s) = \sigma_{n/k-1}(t_s)$ so that it suffices to show that

$$v_{k-1}(p^{p-1}\mu_{k-1}^{-1}) \geqslant p^{\alpha_{n/k-1}(t_s)}. \tag{7.1.8}$$

We have

$$v_{k-1}(p^{p-1}\mu_{k-1}^{-1}) \geqslant p^{k-1} - 1; \quad \alpha_{n/k-1}(t_s) = (n-s)(h-1) - (s-k+1).$$

(7.1.8) follows from this because $(k-1) - \{(n-s)(h-1) - (s-k+1) = -nh + sh + n \geqslant -nh + (n-l)h + n = n - lh = r \geqslant 1$. This proves (7.1.7).

Using (6.4.6), (7.1.6), (7.1.7), and (6.5) we now obtain, writing $l(s)$ for $n-(n-s)h$,

$$\mathrm{Norm}_{n/l(s)}(z\mu_n^{t_s}) \equiv \pm z^{p^{(n-s)h}} \cdot a^{n-s} \cdot p^{s-l(s)} \cdot \mu_{l(s)}^{t_s} \bmod \mu_{l(s)}^{\sigma_{n/l(s)}(t_s+1)} \tag{7.1.9}$$

(because $p^{-(s-l(s))}\sigma_{n/s}(t_s) = p^{-s-n+nh-sh} \cdot p^{(n-s)(h-1)}t_s = t_s$). Now

$$t_s = \frac{p^{n-(n-s)h} - 1}{p-1} = \frac{p^{l(s)} - 1}{p-1} .$$

It follows from (4.2.1) that

$$v_{l(s)-1}(\mathrm{Tr}_{l(s)/l(s)-1}(\mu_{l(s)}^{t_s})) = (p^{l(s)-1} - 1)/(p-1) + p^{l(s)-1}$$

and (using induction) one finds

$$v_0(\mathrm{Tr}_{l(s)/0}(\mu_{l(s)}^{t_s})) = l(s). \tag{7.1.10}$$

Combining this with (7.1.9) and (6.4.6) we find

$$\mathrm{Norm}_{n/0}(z\mu_n^{t_s}) \equiv \pm z^{p^{(n-s)h}} a^{n-s} p^s b \bmod p^{s+1}, \tag{7.1.11}$$

where $b$ is some elemnt of $\mathbf{Z}_p$ of valuation $v(b) = l(s)$. Part (iii) of Proposition 7.1 follows because $v(a) = 0$ and we can extract $p$-th roots in $\mathbf{Z}/(p)$.

Q.E.D.

7.2. *Proof of Theorem* (6.1). Combine (7.1) and (5.10) and use the Lemma 3.2 on filtered abelian groups.

COROLLARY 7.3 (of the proof of Theorem 6.1). *Let $L$ be an unramified algebraic extension of $\mathbf{Q}_p$; let $L_n = K_n \cdot L$ where $K_n$ is as in Theorem* 6.1. *Then Theorem* 6.1 *also holds with $K_n$ replaced by $L_n$.*

COROLLARY 7.4. *Let $L$ be an unramified algebraic extension of $\mathbf{Q}_p$, and let $\cdots - L_n - \cdots - L_1 - L$ be an extension such that there exists a finite unramified extension $K'$ of $L$ such that $L_n \cdot K' = K' \cdot K_n$. Then Theorem* 6.1 *also holds with $K_n$ replaced by $L_n$.*

*Proof.* Consider the commutative diagram

$$\begin{array}{ccc} F(L_n) & \xleftarrow{\operatorname{Norm}_{K_n'/L_n}} & F(K' \cdot L_n) = F(K_n') \\ \Big\downarrow \operatorname{Norm}_{L_n/L} & & \Big\downarrow \operatorname{Norm}_{K_n'/K'} \\ F(L) & \xleftarrow{\operatorname{Norm}_{K'/L}} & F(K') \end{array}.$$

The map $\operatorname{Norm}_{K_n'/L_n}$ is surjective according to Proposition 3.1. The image of $\operatorname{Norm}_{K_n'/K'}$ is $F^{\alpha_n}(K')$ according to (7.3). The same arguments as used to prove (3.1) in the unramified case show that $\operatorname{Norm}_{K'/L}(F^{\alpha_n}(K')) = F^{\alpha_n}(L)$.

Q.E.D.

## 8. CONCLUDING REMARKS

### 8.1. *A Counter Example*

Let $K_n$ be as in Theorem 6.1. Fix an index $i$ and consider the $\Gamma$-extension $\cdots - K_n - \cdots - K_{i+1} - K_i$ of $K_i$. It is not difficult to check that Theorem 6.1 is not true for this $\Gamma$-extension if $i$ is large enough, even if $F$ is defined over $\mathbf{Z}_p$.

### 8.2. *More General $\Gamma$-Extensions*

Let $K$ be a local field of characteristic 0 and residue characteristic $p$, and let $K_\infty/K$ be a totally ramified extension of galois group $\operatorname{Gal}(K_\infty/K) \simeq Z_p$. Let $K_n$ be the invariant field of $p^n \operatorname{Gal}(K_\infty/K)$. Let $F$ be a formal group of height $h \geqslant 2$ over $K$. For each $n$ we define

$\gamma_n$ is the smallest natural number such that $\operatorname{Norm}_{n/0}(F(K_n)) \subset F^{\gamma_n}(K)$,

$\delta_n$ is the largest natural number such that $F^{\delta_n}(K) \subset \operatorname{Norm}_{n/0}(F(K_n))$.

Then one can prove the following.

If the residue field of $K$ is algebraically closed then the differences

$$\delta_n - \frac{(h-1)}{h} ne_K \quad \text{and} \quad \gamma_n - \frac{(h-1)}{h} ne_K$$

are bounded independently of $n$ (cf. [3]).

*Remark* 8.3. In the case considered in this paper, i.e., the situation $F$-Norm: $F(K_n) \to F(K)$, where $K = \mathbf{Q}_p$, $K_n$ is the $n$-th level of the cyclotomic $\Gamma$-extension of $\mathbf{Q}_p$ and $F$ is a formal group defined over $\mathbf{Z}_p$, the cokernel of $F$-Norm depends only on the height $h$ of $F$ and the extension $K_n/K$.

Now consider the following situation

$$F\text{-Norm}: F(L) \to F(K),$$

where $K$ is the quotient field of the ring of Witt-vectors, $W(\mathbf{F}_q)$, over the finite field of $q$ elements, $\mathbf{F}_q$; where $F$ is defined over $W(\mathbf{F}_q)$ and $L/K$ is a finite (galois) extension. In this situation one can conjecture that the cokernel of $F$-Norm depends only on the reduction $F^*$ over $\mathbf{F}_q$ of $F$ and the extension $L/K$. This is certainly the case if $K = \mathbf{Z}_p$ because two formal groups over $\mathbf{Z}_p$ with isomorphic reductions are isomorphic. Moreover, in the situation described above, one can show that the image of $F$-Norm is necessarily of the form $F^i(K)$, i.e., a filtration subgroup of $F(K)$.

## References


1. A. Fröhlich, "Formal Groups," Lecture notes in mathematics 74, Springer, Berlin/New York, 1968.
2. M. Hazewinkel, Abelian extensions of local fields, Thesis, Amsterdam.
3. M. Hazewinkel, On norm maps for one dimensional formal groups. II: $\Gamma$-extensions of local fields with algebraically closed residue field, Report 7210, Econometric Institute, Rotterdam, *J. Reine Angew. Math.* (1974), to appear.
4. M. Lazard, Sur les groupes de Lie formels à un paramètre, *Bull. Soc. Math. France* **94** (1955), 251–274.
5. J. Lubin, One parameter formal lie groups over p-adic integer rings, *Ann. of Math.* **80** (1964), 464–484.
6. J. Lubin and J. Tate, Formal complex multiplication in local fields, *Ann. of Math.* **81** (1965), 380–387.
7. B. Mazur, Rational points of abelian varieties with values in towers of number fields, *Invent. Math.* 18 (1972), 183–266.
8. J. P. Serre, "Corps Locaux," Hermann, Paris, 1962.
9. J. Tate, "$p$-divisible Groups," Proceedings of a Conference on Local Fields held at Driebergen, ed. by T. A. Springer, Springer, Berlin–New York, 1967, 158–183.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