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A COMPLEMENT TO A PAPER OF DEBREU

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1. INTRODUCTION

The paper referred to in the title is: G. Debreu. Economies with a finite set of equilibria. Econometrica 38, 1970, 387-392. In it Debreu considers pure exchange economics with ℓ commodities and in consumers whose needs and preferences are fixed and whose resources vary. Let R denote the real numbers and let

$$L = \{w \in R \mid w > 0\}$$

$$P = \{x \in R^{\ell} \mid x = (x_1, \ldots, x_\ell), x_i > 0\}$$

$$S = \{p \in P \mid \Sigma p_i = 1\}$$

$$\bar{P} = \{x \in R^{\ell} \mid x_i \geq 0\}, \text{ the closure of } P$$

$$\bar{S} = \{p \in \bar{P} \mid \Sigma p_i = 1\}, \text{ the closure of } S.$$

The preferences of the i-th consumer are specified by means of his demand function $f_i : S \times L \rightarrow \bar{P}$ which has the property $p.\ f_i(p, w_i) = w_i$, where the dot denotes the inner product. (Given the price vector $p \in S$ and wealth $w_i \in L$, the i-th consumer demands the commodity bundle $f_i(p, w_i)$). The following assumption plays a role.

1.1. Assumption (A): If the sequence (p^q, w_i^q), $q = 1, 2, \ldots$ in $S \times L$ converges to (p^o, w_i^o) in $(\bar{S} \setminus S) \times L$, then $\| f_i(p^q, w_i^q) \|$ goes to $+\infty$.

(Here $\| \ \|$ denotes any norm in R^{ℓ} ; e.g. the usual norm)

An economy is defined by $(f_1, \ldots, f_m, \omega_1, \ldots, \omega_m)$, an m-tuple of demand functions and an m-tuple $\omega = (\omega_1, \ldots, \omega_m)$ of vectors in P. Since Debreu keeps the demand functions (fixed) an economy is actually defined by ω.

Given ω an element p of S is an <u>equilibrium price vector</u> of the economy (f,ω) (or ω) if

$$(1.2) \qquad \sum_{i=1}^{m} f_i(p, p.\omega_i) = \sum_{i=1}^{m} \omega_i$$

Let $W(\omega)$ denote the set of p satisfying this equality. Debreu proves:

1.3. <u>Theorem</u>.

Given m continuously differentiable demand functions $(f_1, \ldots, f_m)$, if some f_i satisfies (A), then the set of $\omega \in P^m$ for which $W(\omega)$ is infinite has null closure.

(A set $F \subset \mathbb{R}^{\ell m}$ is called <u>null</u> if it has Lebesque measure zero).

It is now natural to ask whether something similar is true if one allows both f and ω to vary. (Also in view of the fact that the f_i will almost never be exactly known.) This is in fact the case. However, the space of demand functions has no naturally defined measure on it, but there are several more or less canonically defined topologies on it. It therefore becomes convenient to rephrase Debreu's result in topological terms as

1.4. <u>Theorem</u>.

Given m continuously differentiable demand functions $(f_1, \ldots, f_m)$, if some f_i satisfies (A), then the set of $\omega \in P^m$ for which $W(\omega)$ is finite, contains an open dense subset of P^m.

Let Φ be the set of m-tuples of continuously differentiable demand functions; let ΦA be the subset of Φ consisting of m-tuples $(f_1, \ldots, f_m)$ such that at least one f_i satisfies A.

Let τ be a topology on ΦA (cf. section 3 below), and let ΦA_τ denote the resulting topological space. Give the space of economies $\Phi A \times P^m$ the product topology. Our problem is then:

Is the property "having finitely many equilibrium price vectors" generic in some sense.

More precisely:

Does there exist an open dense subset $\mathcal{U}$ of $\Phi A_\tau \times P^m$ such that all economies in U have only finitely many equilibrium price vectors.

The answer depends to some extent on the topology τ (of course). However, even in the weakest topology considered by us (the C^1-compact-open topology) there is in any case a residual subset U in $\Phi A_\tau \times P^m$ of economies with only finitely many equilibrium price vectors. In any case one can therefore say that:

Most economies have only finitely many equilibrium price vectors.

Remark. It follows from (1.4) that for every $f \in \Phi A_\tau$ there is an open dense subset $V_f \subset \{f\} \times P^m$ of economies with at most finitely many equilibrium vectors. This does not imply that there is an open dense subset of $\Phi A_\tau \times P^m$ of economies with at most finitely many equilibrium vectors. To see this consider the following elementary example in R x R; for every $x \in R$, let $U_x = \{x\} \times R$ if x is irrational and $U_{r/s} = \{x\} \times \{\frac{m}{s} | m \in Z\}$ (Z denotes the integers), if $x = r/s$, $(r,s) = 1$. Then U_x is open and dense in $\{x\} \times R$ for every $x \in R$ (and its complement is closed and has measure zero in $\{x\} \times R$). However if $F_x = \{x\} \times R \setminus U_x$, then $F = \bigcup_x F_x$ is dense in R^2 !

2. PART OF DEBREU'S PROOF.

In this section we recall that part of the proof given by Debreu in [1] which will be needed in the sequel.

One can assume without loss of generality that the (demand function of the) first consumer satisfies condition (A). Let $U = S \times L \times P^{m-1}$; we define a function $F : S \times L \times P^{m-1} \to P^m$ by the formula

$$F(p, w_1, \omega_2, \ldots, \omega_n) = (\omega_1, \omega_2, \ldots, \omega_n) \tag{2.1}$$

$$\omega_1 = f_1(p, w_1) + \sum_{i=2}^{n} f_i(p, p.\omega_i) - \sum_{i=2}^{m} \omega_i$$

The price vector p belongs to $W(f,\omega)$, the set of equilibrium price vectors of the economy (f,ω) if and only if $F(p,p.\omega_1,\omega_2, \ldots, \omega_n) = \omega$, and the points of $W(f,\omega)$ are in a one to one correspondence with $F^{-1}(\omega)$.

Debreu proves:

(2.2) if $K \subset P^m$ is compact, then $F^{-1}(K)$ is compact

Let $g : V \to R^b$ be a continuously differentiable function, V an open set in R^a; let B be a subset of V and let $\omega \in R^b$ be a point. One then says that g is transversal to in B if

(2.3) for every $x \in B$ we have either $g(x) \neq \omega$ or, if $g(x) = \omega$, then the Jacobian matrix of g in x has rank b.

For the purposes of this note we shall call an economy (f,ω) a good economy if the following conditions hold

(2.4) (i) $f \in \Phi A$
(ii) The map F defined above in (2.1) is transversal to ω in $U = S \times L \times P^{m-1}$.

Debreu now proves by means of Sard's theorem (cf.[4]) that

(2.5) given $f \in \Phi A$, the set of ω such that (f,ω) is good is open and dense in P^m.

The result (2.2) implies that a good economy has only finitely many equilibrium points.

3. SOME TOPOLOGIES ON ΦA.

Let V be an open set in R^a. We consider $C^1(V,R^b)$ the set of continuously differentiable function $V \to R^b$. We shall define three topologies on $C^1(V,R^b)$

3.1. The C^1-compact-open topology.

Given a number $\varepsilon > 0$, a compact subset K of V and an element $f \in C^1(V,R^b)$ we define

$$\mathcal{U}_{co}(\varepsilon,K,f) = \{g \in C^1(V,R^b) \mid \forall y \in K, \ ||g(y) - f(y)|| < \varepsilon,$$

$$|\frac{\partial g}{\partial x_i}(y) - \frac{\partial f}{\partial x_i}(y) < \varepsilon \ , \ i = 1, \ldots, a\}$$

The sets $\mathcal{U}_{co}(\varepsilon,K,f)$ for varying ε,K,f form a basis of a topology on $C^1(V,R^b)$ called the C^1-compact-open topology. If M is a subset of $C^1(V,R^b)$, we denote with M_{co} the topological space induced by this topology on $C^1(V,R^b)$.

3.2. <u>The C^1-uniform topology</u>.

Given a number $\varepsilon > 0$, and an element $f \in C^1(V,R^b)$ we define

$$\mathcal{U}_u(\varepsilon,f) = \{g \in C^1(V,R^b) \mid \forall y \in V, \ ||g(y) - f(y)|| < \varepsilon,$$

$$|\frac{\partial g}{\partial x_i}(y) - \frac{\partial f}{\partial x_i}(y)| < \varepsilon, \ i = 1, \ldots, a\}$$

The sets $\mathcal{U}_u(\varepsilon,f)$ for varying ε,f form a basis of a topology on $C^1(V,R^b)$ which could be called the C^1-uniform topology. If M is a subset of $C^1(V,R^b)$, we denote with M_u the topological space induced on M by this topology on $C^1(V,R^b)$.

3.3. <u>A C^1-mixed-topology</u>.

Given a positive number $\varepsilon > 0$, a compact subset $K \subset V$, and an element $f \in C^1(V,R^b)$ we define

$$\mathcal{U}_m(\varepsilon,K,f) = \{g \in C^1(V,R^b) \mid \forall y \in V, \ ||g(y) - f(y)|| < \varepsilon;$$

$$\forall y \in K, \ |\frac{\partial f}{\partial x_i}(y) - \frac{\partial g}{\partial x_i}(y)| < \varepsilon\}$$

The sets $\mathcal{U}_m(\varepsilon,K,f)$ for varying ε,K,f form a basis of a topology on $C^1(V,R^b)$ which is intermediate between the two previous ones. If M is a subset of $C^1(V,R^b)$ we denote with M_m the topological space induced by this topology on $C^1(V,R^b)$.

The topologies c.o,u,m, are related as follows:

$$u \prec m \prec c.o.$$

where $\prec$ denotes "finer then". E.g. a subset U of $C^1(V,R^b)$ which is open in the m-topology is also open in the u-topology, but not necessarily open in the c.o.-topology.

Taking $V = S \times L$, $b = m$ and for $\mathcal{M}$ the set Φ of demand functions $f = (f_1, \ldots, f_m)\colon S \times L \to \bar{P} \subset R^{\ell m}$, or the set ΦA of demand functions $f = (f_1, \ldots, f_m)$ such that at least one f_i satisfies assumption A we get topological spaces Φ_{co}, Φ_u, Φ_m, ΦA_{co}, ΦA_u, ΦA_m.

Remark. If $f \in \Phi A$ and $g \in \Phi$ and $g \in U_m(\varepsilon,K,f)$ or $g \in U_u(\varepsilon,K,f)$ for certain ε,K then $g \in \Phi A$.

4. MOST ECONOMICS HAVE FINITELY MANY EQUILIBRIUM POINTS.

The precise form of this statement is the theorem below. To be able to state it in a convenient way we define an economy (f,ω) to be good with respect to a compact set $K \subset S \times L$ if the corresponding map $F : S \times L \times P^{m-1} \to P^m$ is transversal to ω in $K \times P^{m-1}$ (cf. (2.1)).

We denote with $\mathcal{E}_{oc}$, $\mathcal{E}_m$, $\mathcal{E}_u$ the topological spaces of economies $\mathcal{E}_{oc} = \Phi A_{oc} \times P^m$, $\mathcal{E}_m = \Phi A_m \times P^m$, $\mathcal{E}_u = \Phi A_m \times P^m$. Note that the underlying sets of these spaces are the same; only their topologies differ.

4.1. Theorem.

a. Given a compact set $K \subset S \times L$, the set of economies good with respect to K is open and dense in $\mathcal{E}_{oc}$.

b. The set of good economies is residual in $\mathcal{E}_{oc}$.

c. The set of good economies is open and dense in $\mathcal{E}_m$.

d. The set of good economies is open and dense in $\mathcal{E}_u$.

Because a good economy has finitely many (none is possible) equilibrium points in virtue of (2.2), we get as a corollary that the set of economies with finitely many equilibrium points contains a residual (resp. open and dense, resp. open and dense) subset of $\mathcal{E}_{oc}$ (resp. $\mathcal{E}_m$, resp. $\mathcal{E}_u$). Another way to say this is as follows.

4.2. Corollary.

a. The closure of the set of economies with infinitely many equilibrium points in $\mathcal{E}_{oc}$ is a countable union of closed nowhere dense sets.

b. The closure of the set of economies with infinitely many equilibrium points in $\mathcal{E}_m$(resp. $\mathcal{E}_u$) is a closed nowhere dense set in $\mathcal{E}_m$(resp. $\mathcal{E}_u$).

The proof of theorem (4.1) rests on the following fairly elementary transversality lemma (cf. e.g. [3])

4.3. Lemma.

Let $U \subset R^a$ be open, K a compact subset of U, $f : U \to R^b$ a continuously differentiable function which is transversal to $z \in R^b$ in K. Then there is an $\varepsilon > 0$ such that every $g \in U_{co}(\varepsilon,K,f)$ is also transversal to z in K. (for the definition of $U_{co}(\varepsilon,K,f)$ cf. (3.1)).

4.4. Proof of Theorem (4.1).

a. Let (f,ω) be an economy. Let $F : S \times L \times P^{m-1} \to P^m$ be the associated map (cf. (2.1)).

Given $\omega = (\omega_1, \ldots, \omega_m)$, F is entirely determined by its associated map

$$(4.4.1) \qquad F_\omega : S \times L \to P,\ (p,w) \to f_1(p,w) + \sum_{i=2}^{m} f_i(p,p\omega_i) - \sum_{i=1}^{m} \omega_i$$

Indeed

$$(4.4.2) \qquad F(p,w,\omega_2, \ldots, \omega_n) = (F_\omega(p,w) + \omega_1, \omega_2, \ldots, \omega_n)$$

Let K be a compact subset of $S \times L$. It follows from (4.4.2) that F is transversal to $\omega \in P^m$ in $K \times P^{m-1}$ if and only if F_ω is transversal to 0 in K

Given ω,f and K we define $K_1 = K$, $K_i = \{(p,p.\omega_i) | (p,w) \in K\}$ $i = 2, \ldots, m$. Let $K' = \bigcup_{i=1}^{n} K_i$; this is a compact set. Choose $\varepsilon_1 > 0$ such that

$K^* = \{(p,w) \in S' \times R \mid \exists (p',w') \in K'$ such that $||(p,w) - (p',w')|| \leq \varepsilon_1\}$ is contained in $S \times L$. Here $S' = \{x \in R^{\ell} \mid \Sigma x_i = 1\}$. K^* is also compact. Let $\varepsilon_2 > 0$. Now choose $\varepsilon_3 > 0$ such that

(i) $\omega' \in R^m$, $||\omega' - \omega|| \leq \varepsilon_3 \Rightarrow \omega' \in {}^m$

(ii) $||\omega' - \omega|| \leq \varepsilon_3$, $(p,w) \in K \Rightarrow (p,p.\omega_i') \in K^*$ for all $i = 2, \ldots, m$

(iii) $||f_i(p,p.\omega_i') - f_i(p,p.\omega_i)|| < \varepsilon_2$ if $||\omega' - \omega|| < \varepsilon_3$, $(p,w) \in K$, $i = 2, \ldots, m$

(iv) $|\frac{\partial f_i}{\partial p_j}(p,p.\omega_i') - \frac{\partial f_i}{\partial p_j}(p,p.\omega_i)| < \varepsilon_2$ if $||\omega' - \omega|| < \varepsilon_3$, $(p,w) \in K$, $i = 2, \ldots, m$

Such an $\varepsilon_3 > 0$ exists because of the following facts: P^m is open in $R^{\ell m}$; the function $\omega_i' \mapsto p.\omega_i'$ is continuous and the set K_o of $p \in S$ such that $(p,w) \in K$ is compact; f_i is continuous, $\omega_i' \mapsto p.\omega_i'$ is continuous and K_o is compact; $\frac{\partial f_i}{\partial p_j}$ and $\omega_i' \mapsto p.\omega_i'$ are continuous and K_o is compact.

Let $\varepsilon_4 > 0$ and let $g \in U_{co}(\varepsilon_4, K^*, f)$. Construct G and $G_{\omega'}$ from (g,ω') in the same way as we obtained F and F_ω from (f,ω). Then we have if $||\omega' - \omega|| < \varepsilon_3$ and $(p,w) \in K$

$$
\begin{aligned}
||G_{\omega'}(p,w) - F_\omega(p,w)|| &\leq ||g_1(p,w) - f_1(p,w)|| + \\
&+ \sum_{i=2}^{m} ||g_i(p,p.\omega_i') - f_i(p,p.\omega_i')|| \\
&+ \sum_{i=2}^{m} ||f_i(p,p.\omega_i') - f_i(p,p.\omega_i)|| + \\
&+ \sum_{i=1}^{m} ||\omega_i' - \omega_i|| \\
&\leq \varepsilon_4 + (m-1)\varepsilon_4 + (m-1)\varepsilon_2 + m\varepsilon_3
\end{aligned} \tag{4.4.3}
$$

and

$$
\begin{aligned}
|\frac{\partial G_{\omega'}}{\partial p_j}(p,w) - \frac{\partial F_\omega}{\partial p_j}(p,w)| &\leq |\frac{\partial g_1}{\partial p_j}(p,w) - \frac{\partial f_1}{\partial p_j}(p,w)| + \\
&+ \sum_{i=2}^{n} \frac{\partial g_i}{\partial p_j}(p,p.\omega_i') - \frac{\partial f_i}{\partial p_j}(p,p.\omega_i')| +
\end{aligned}
$$

$$
(4.4.4) \qquad \begin{aligned}
&+ \sum_{i=2}^{n} \left|\frac{\partial f_i}{\partial p_j}(p,p.\omega_i') - \frac{\partial f_i}{\partial p_j}(p,p.\omega_i)\right| \\
&+ \sum_{i=2}^{n} \left|\frac{\partial g_i}{\partial w}(p,p.\omega_i')\omega_{ij}' - \frac{\partial f_i}{\partial w}(p,p.\omega_i')\omega_{ij}'\right| \\
&+ \sum_{i=2}^{n} \left|\frac{\partial f_i}{\partial w}(p,p.\omega_i')\omega_{ij}' - \frac{\partial f_i}{\partial w}(p,p.\omega_i)\omega_{ij}'\right| \\
&+ \sum_{i=2}^{n} \left|\frac{\partial f_i}{\partial w}(p,p.\omega_i)\omega_{ij}' - \frac{\partial f_i}{\partial w}(p,p.\omega_i)\omega_{ij}\right| \\
&\leq \varepsilon_4 + (n-1)\varepsilon_4 + (n-1)\varepsilon_2 + (n-1)A\varepsilon_4 + (n-1)A\varepsilon_2 + \\
&\qquad + (n-1)B\varepsilon_3
\end{aligned}
$$

Where ω'_{ij} is the j-th component of ω_i'; $A = \max \{\omega_{ij}' \mid ||\omega'-\omega|| \leq \varepsilon_3\}$;
$B = \max \{\frac{\partial f_i}{\partial w}(p,p.\omega_i) | (p,w) \in K\}$. Note that A decreases as ε_3 decreases and that B is a constant depending only on K, f, ω.
Choose $\varepsilon > 0$ and (f,ω), take any ε_1 and the corresponding K^*. It follows from the above that there exist an ε_3, $\varepsilon_4 > 0$ such that

$$
(4.4.5) \qquad g \in U_{co}(\varepsilon_4, K^*, f),\ ||\omega'-\omega|| \leq \varepsilon_3 \Rightarrow G_{\omega'} \in U_{co}(\varepsilon, K, F_{\omega})
$$

Now let (f,ω) be a good economy with respect to K. Lemma (4.3) and (4.4.5) imply that there is an open neighbourhood of (f,ω) in $\mathcal{E}_{co}$ consisting of economies good with respect to K.

It follows that the set of economies good with respect to K is open in $\mathcal{E}_{co}$. It remains to show that this set is also dense in $\mathcal{E}_{co}$. This follows from (2.5). This concludes the proof of a.

b. S x L can be written as a countable union of compact subsets. It follows therefore from a) that the set of good economies is a countable intersection of open dense sets in $\mathcal{E}_{co}$ and it is therefore residual in $\mathcal{E}_{co}$.

c. Let (f,ω) be a good economy. The map $F : S \times L \times P^{m-1} \to P^m$ is than transversal at ω in all of $S \times L \times P^{m-1}$, and there are finitely many, say k, elements in $F^{-1}(\omega)$. It follows from this and (2.2) (cf. also the remark on p. 390 of [1])that in a small neighbourhood V of ω there are k continuously differentiable functions $g_1, \ldots, g_k : V \to S \times L \times P^{m-1}$ such that the elements of $F^{-1}(\omega')$,

$\omega' \in V$ are $g_1(\omega'), \ldots, g_k(\omega')$. The solutions of $F_{\omega'}(p,w) = \eta$ are in 1-1 correspondence (in a differentiable way) with the elements of $F^{-1}(\omega_1' + \eta, \omega_2', \ldots, \omega_n')$. It follows that there are positive numbers $\varepsilon_1, \varepsilon_2$ and k continuously differentiable functions $h_1, \ldots, h_k$ such that

$$(4.4.6)\quad ||\eta|| \leq \varepsilon_1,\ ||\omega - \omega'|| \leq \varepsilon_2 \Rightarrow F_{\omega'}^{-1}(\eta) = \{h_1(\omega_1'+\eta, \omega_2', \ldots, \omega_n'), \ldots, h_k(\omega_1'+\eta, \omega_2', \ldots, \omega_n')\}$$

Let K be compact set in S x L containing $F_{\omega'}^{-1}(\eta)$ for all η, ω' such that $||\eta|| \leq \varepsilon_1$, $||\omega-\omega'|| \leq \varepsilon_2$. It follows from (4.4.3) that there exists an $\varepsilon_4 > 0$ such that

$$(4.4.7)\quad g \in U_m(\varepsilon_4, K^*, f) \Rightarrow ||F_{\omega'}(p,w) - G_{\omega'}(p,w)|| < \varepsilon_1$$

Now let (p^*, w^*) be a solution of $G_{\omega'}(p,w) = 0$ where $||\omega'-\omega|| \leq \varepsilon_2$. Then $F_{\omega'}(p^*, w^*) = \eta$ with $||\eta|| \leq \varepsilon_1$ and it follows that $(p^*, w^*) \in K$. Every solution of $G_{\omega'}(p,w) = 0$ must therefore lie in K. An economy (g,ω) with $g \in U_m(\varepsilon_4, K^*, f)$ and $||\omega-\omega'|| \leq \varepsilon_2$ is therefore good if and only if it is good with respect to K.

Now repeat the argument of a) to find an $\varepsilon_5, \varepsilon_6 > 0$ such that

$$(4.4.8)\quad ||\omega-\omega'|| < \varepsilon_5,\ g \in U_{co}(\varepsilon_6, K, f) \Rightarrow (g,\omega') \text{ is good with respect to } K$$

Take $\varepsilon_7 = \min\{\varepsilon_2, \varepsilon_5\}$, $\varepsilon_8 = \min\{\varepsilon_4, \varepsilon_6\}$. Then $||\omega-\omega'|| < \varepsilon_7$ and $g \in U_m(\varepsilon_8, K, f)$ guarantee that the economy (g,ω') is also good. This proves that the set of good economies is open in $\mathcal{E}_m$. This set is dense in $\mathcal{E}_m$ because of (2.5).

d. The openness of the set of good ec onomies follows from c because every set open in $\mathcal{E}_m$ is also open in $\mathcal{E}_u$. The denseness follows once more from (2.5).

This concludes the proof of the theorem.

4.5. Remark.

Given $(f,\omega) \in \mathcal{E}$, there exists a g arbitrarily close to f with respect to any one of the topologies introduced in 3 such that the economy (g,ω) is good. This requires another result on transversality and the use of partitions of unity. The same techniques were used in [2]; they are wellknown.

4.6. Remark.

Given an economy (f,ω) such that an f_i satisfies condition A, there is no guarantee that there exist an equilibrium point. However, Debreu proves that if all the f_i, i = 1, ..., n satisfy condition (A) then there does exist at least one equilibrium price vector. Let $\mathcal{E}A$ be the set of economies (f,ω) such that all the f_i satisfy assumption (A). Theorem (4.1) remains true if one replaces $\mathcal{E}$ by $\mathcal{E}A$.

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