Some Examples Concerning Linear Continuity of Solutions to Programming Problems

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In this note we construct some counterexamples concerning upper semicontinuity and linear upper and lower semicontinuity of the solution sets and $\varepsilon$-solution sets of nonlinear programming programs: \( \max f(x), \) subject to \( g(x) \leq b \). These examples answer some of the questions in a recent paper by Stern and Topkis.

We consider the programming problem \( \max f(x), g(x) \leq b \), where \( f \) is a function \( \mathbb{R}^n \to \mathbb{R} \), \( g \) a function \( \mathbb{R}^n \to \mathbb{R}^m \), and \( b \) an \( m \)-vector. Here \( g(x) \leq b \) means \( g_j(x) \leq b_j \) for all \( j = 1, \ldots, m \). We define \( S_b = \{ x \in \mathbb{R}^n | g(x) \leq b \} \) and for all \( \varepsilon \geq 0 \) we define the \( \varepsilon \)-solution set

\[
S^\varepsilon_b = \{ x \in \mathbb{R}^n | x \in S_b \; \text{ and } \; f(x) \geq -\varepsilon + \max_{x \in S_b} f(z) \}.
\]

We are interested in upper and lower semicontinuity and linear upper and lower semicontinuity of \( S^\varepsilon_b \) as \( b \) varies. For a definition of these notions see [2]. We will construct some examples that answer some of the questions asked in [2].

1. Examples

Example 1. In this example \( g_j \) is linear for all \( j \), \( -f \) is a convex differentiable function, \( S_b \) is compact, but \( S^\varepsilon_b \) is not lower semicontinuous.
Let \( f \) be the function defined by \( f(x_1, x_2) = x_2^2/x_1 \) on the open halfspace of \( \mathbb{R}^2 \) where \( x_1 < 0 \). The function \(-f\) is convex on \( \{ x \in \mathbb{R}^2 | x_1 < 0 \} \).

Now define \( g_1(x_1, x_2) = x_1, g_2(x_1, x_2) = -x_1, g_3(x_1, x_2) = x_2, g_4(x_1, x_2) = -x_2 \). Let \( b(0) \) be the vector \((-\frac{1}{2}, 1, 0, 1)\) and \( b(\delta) \) the vector \((-\frac{1}{2}, 1, -\delta, 1)\). Then \( S_{b(0), \delta}^* = \{(x_1, x_2) | -1 \leq x_1 \leq -\frac{1}{2}, -1 \leq x_2 \leq -\delta \} \) and the solution sets \( S_{b(0), \delta}^* \) of the problem \( \max f(x), \ g(x) \leq b(\delta) \) are \( S_{b(0), \delta}^* = \{(x_1, x_2) | x_2 = 0, -1 \leq x_1 \leq -\frac{1}{2} \} \) and \( S_{b(0), \delta}^* = \{(-1, -\delta)\} \) for all \( \delta > 0 \). Therefore, \( S_{b, 0}^* \) is not lower semicontinuous with respect to \( b \) in \( b(0) \).

**Example 2.** In this example \( g_j \) is linear for all \( j \), \(-f \) is a convex differentiable function, \( S_0 \) is compact, but \( S_{b, 0}^* \) is not lower linearly semicontinuous. Let \( D \subset \mathbb{R}^2 \) be the region \( x_1^2 + x_2^2 \leq 1 \). Let \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) be the function defined by \( \sqrt{d(x)} = (\sqrt{d(x_1, x_2)} \), where \( d(x) \) is the distance of \( x = (x_1, x_2) \) to \( D \).

A formula for \( f(x) \) is \( f(x) = -(\max\{0, 2x_1 + 2x_2\})^2 \). The function \( f \) is continuously differentiable. Further, \(-f \) is a convex function. Now define \( g_1(x) = x_1, g_2(x) = x_1, g_3(x) = -x_2, g_4(x) = -x_1 \). Let \( b(0) \) be the vector \((2, 1, -1, 1)\) and \( b(\delta) = (2, 1, -1+\delta, 1) \), \( \delta > 0 \). Then we have \( S_{b(0), \delta}^* = \{(0, 1)\} \) and \( S_{b(0), \delta}^* = \{(x_1, x_2) | x_1^2 + x_2^2 \leq 1, x_2 \leq 1-\delta \} \).

The point \( B = ((2\delta - \delta^2)^{1/2}, 1-\delta) \) is in \( S_{b(0), \delta}^* \) and has distance \( \geq (2\delta - \delta^2)^{1/2} \) to \( A = (0, 1) \). It follows that \( S_{b, \delta}^* \) is not lower linearly semicontinuous in \( b \) at \( b(0) \).

**Example 3.** In this example \(-f \) is a convex differentiable function, \( S_0 \) is compact and uniformly linearly continuous. For every \( \epsilon \) with \( 0 \leq \epsilon \leq \frac{1}{16} \) there is a \( b(\epsilon, 0) \) such that \( S_{b(\epsilon, 0)}^* \) is not lower linearly semicontinuous in \( b \) at \( b(\epsilon, 0) \).

The function \( f \) in this example is the same as the function \( f \) in example 2 above. There are five restriction functions:

\[
g_1(x) = \begin{cases} 
-\left(\frac{1}{2}x_1 + \frac{1}{2}x_2 + \frac{1}{2}\right) + \frac{1}{2}(\frac{1}{2} - 4)(x_1 - x_2)(x_2 - x_1) \\
-4 \quad \text{if} \quad x_1 - x_2 \leq 0 \quad \text{and} \quad x_2 - x_1 \leq \frac{1}{2} \\
-x_1 \quad \text{if} \quad x_1 - x_2 \geq 0 \\
-x_2 + \frac{1}{2} \quad \text{if} \quad x_2 - x_1 \geq \frac{1}{2}. 
\end{cases}
\]

\( g_2(x) = x_1, g_3(x) = -x_2, g_4(x) = x_2, \) and \( g_5(x) = -x_1 \).

The functions \( g_j \) are all continuously differentiable. The level curves of \( g_1 \) are straight lines joined by a quarter circle of radius \( \frac{1}{2} \) (see Figure 1). Choose \( 0 \leq \epsilon \leq \frac{1}{16} \). Let \( b(\epsilon, 0) \) be the vector \((-\frac{1}{4} - \epsilon^{1/2}, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}, \frac{1}{2})\) and \( b(\epsilon, \delta) = (-\frac{1}{4} - \epsilon^{1/2} + \delta, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}, \frac{1}{2}) \). We then have for \( \delta \geq 0 \) sufficiently small \( S_{b(\epsilon, 0), \delta}^* = \{(x_1, x_2) | x_1^2 + x_2^2 \leq 1 + \epsilon, x_1 \geq \frac{1}{4} + \epsilon^{1/2} - \delta \} \). In Figure 1 the case \( \epsilon = \frac{1}{16}, \delta = \frac{1}{10} \) is shown. \( S_{b(\epsilon, \delta), \epsilon}^* \) consists of the cross-hatched area and the point \( A \) and \( S_{b(\epsilon, \delta), \epsilon}^* \) is the union of the cross-hatched and shaded areas.

It now follows as in example 2 that \( S_{b, \epsilon}^* \) is not linearly upper semicontinuous in \( b \) at \( b(\epsilon, 0) \).
2. REMARKS

Example 4.1. If one changes the function $f$ of example 3 to $\tilde{f}(x) = -(e(x))^2$, where $e(x)$ is the distance of $x$ to the region $\{(x_1, x_2) | x_2 \leq -x_1^2 + 1\}$, one can construct an example similar to example 3 such that for every $\epsilon \geq 0$ there is a $b(\epsilon)$ such that $S_b^{\epsilon}$ is not linearly upper semicontinuous in $b$ at $b(\epsilon, 0)$.

Figure 1

Example 4.2. By repeating infinitely often the trick by which the function $g_1$ of example 3 was constructed, one can construct an example with the following properties: 1) $-f$ is a convex differentiable function; 2) $g_1$ is a continuous differentiable function; 3) $g_2, \ldots, g_6$ are linear functions; 4) $S_b$ is uniformly linearly continuous for $b$ in a suitable hypercube $B$; and 5) for every $b(0)$ in the 4-dimensional subcube of $B$ where $b_1 = -1$ there is an infinite sequence $\{\epsilon(n)\}_n$, $\epsilon(n) > 0$ such that $S_{b, \epsilon(n)}^*$ is not linearly upper semicontinuous in $b$ at $b(0)$ for all $n \in \mathcal{N}$. Moreover, $\lim_{n \to \infty} \epsilon(n) = 0$ and $S_{b, \epsilon}^*$ is not linearly upper semicontinuous in $b$ at $b(0)$. 
In this example $f, g_2, \ldots, g_5$ are as in example 3. The function $g_1$ is constructed as follows. We first define the level curve $g_1(x) = -1$. Draw rays $l_n$ from the origin in $\mathbb{R}^2$ at an angle of $2^{-n-1} \pi$ with the positive $X_1$-axis. Let $A_n$ be the point on $l_n$ at distance $1+\alpha_n$ from the origin. In the points $A_n$,

![Figure 2](image)

draw lines $m_n$ perpendicular to $l_n$. If one takes, e.g., $\alpha_n = 4^{-n-1} \pi^2/2$, then the intersection point of $m_n$ and $m_{n+1}$ is in the angle formed by $l_n$ and $l_{n+1}$. Now join suitable segments of the $m_n$ by “smoothing the corners in the intersection points.” This can be done by means of circle arcs of radius $\frac{1}{2^k}$. The resulting curve $C$ is differentiable and has the property that it is a straight line near every intersection point $C \cap l_n = A_n$, $n \in \mathbb{N}$.
Let $E \subset \mathbb{R}^2$ be the region

$$E = \{ x \in \mathbb{R}^2 | x_1 \geq \frac{1}{2}, x_2 \geq \frac{1}{4} \} \cup \{ x \in \mathbb{R}^2 | x_2 \geq \frac{1}{4} \} \cup \{ x \in \mathbb{R}^2 | x_1 \leq 0, x_2 \geq 0, \|x\| \leq \frac{1}{4} \}.$$

The boundary of $E$ is the dashed line in Figure 2. For each $x \in E$ let $l_x$ be the ray from the origin passing through $x$, $B_x$ be the intersection point $l_x \cap C$, and $r_x$ be the distance of $B_x$ to the origin. We now define $g_1(x) = -\|x\| r_x^{-1}$. This defines a differentiable function $g_1 : E \to \mathbb{R}$, which by the familiar tools of differential topology can be extended to a suitable, continuously differentiable function $g_1 : \mathbb{R}^2 \to \mathbb{R}$, such that $g_1(x) \geq -\frac{1}{2}$ for all $x \in \mathbb{R}^2 \setminus E$ (see [1]).

A suitable hypercube such that property 4 holds is, e.g., defined by the inequalities $-\frac{3}{4} \leq b_1 \leq -2$, $3 \leq b_2 \leq 4$, $3 \leq b_3 \leq 4$, $4 \leq b_4 \leq 5$, $2 \leq b_5 \leq 3$. In the example as described (see Figure 2) property 5 holds for all $b(0)$ in this cube for which $b_1 = -1$. For example, $S_{b,0}$ grows nonlinearly at $b = (-1, b_2, b_3, b_4, b_5)$ as $b_1$ becomes greater than $-1$. Nonlinear growth in the $S_{b,0}$ for $s_n = a_n^2$ at $b = (-1, b_2, b_3, b_4, b_5)$ as $b_1$ becomes greater than $-1$ is caused by the straight-line sections in the curve $g_1(x) = -1$ at the points $A_n$.

The functions $f$ and $g_1$ of examples 2, 3, and 4.2 are continuously differentiable but not twice continuously differentiable. There are similar examples with $f$ and $g_1$ of class $C^\infty$, i.e., $n$ times continuously differentiable for all $n$. This is done by smoothing the functions $g_1$ suitably, using techniques as in [1]. To obtain suitable functions $f$ one needs only a $C^\infty$ function $h$ of one variable $r$ that is convex, $\geq 0$ everywhere and such that $h(r) = 0 \iff 0 \leq r \leq 1$. Such functions $h$ exist, e.g., $h(r) = (r-1)^6 \exp \left( - (r-1)^2 \right)$ if $r \geq 1$, $h(r) = 0$ if $0 \leq r \leq 1$.

REFERENCES