On norm maps for one dimensional formal groups. II

Γ-extensions of local fields with algebraically closed residue field*)

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Dedicated to Professor Helmut Hasse on his 75th birthday

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1. Introduction

Let K be a mixed characteristic complete discrete valuation field; A_K its ring of integers. Let F be a formal group over A_K; let p be the residue characteristic of K and let K_∞/K be a Γ-extension of K, with layers $\cdots - K_n - K_{n-1} - \cdots$. In this paper we continue our investigation of the image of the norm map

$$F\text{-Norm}_{n/0} : F(K_n) \to F(K)$$

begun in [3]. The main result is Theorem 3. 1, which is proved in Sections 4—11. The proof is, except for some technical complications, the same as the proof of Theorem 6. 1 of [3] given in [3]. Cf. also 12. 1 below.

Most of the notations and conventions of [3] remain in force.

The residue field of K is always supposed to be perfect.

2. Γ-extensions

A Γ-extension of a local field K (associated to the prime q) is a galois extension K_∞/K with galois group isomorphic to $\mathbb{Z}_q$, the q-adic numbers. Let p be the residue characteristic of K. We shall only consider Γ-extensions associated to the prime p. (Other Γ-extensions are not very interesting in view of [3], 3. 1.) Let L be the maximal unramified extension of K contained in K_∞. If $L = K_\infty$ we again know the image of the norm map ([3], 3. 1). If $L \neq K_\infty$ then K_∞/L is a totally ramified Γ-extension. Using the proof of [3], 3. 1 (cf. also [3], 6. 3, and 3. 4 below) we see that it suffices for our purposes to consider only totally ramified Γ-extensions associated to the prime p, where p is the residue characteristic of the local field K. All Γ-extensions occurring below will be assumed to be of this type.

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If L/K is a cyclic extension of prime degree p, let $m(L/K)$ be the number governing the behaviour of $\mathrm{Tr}_{L/K}$, i. e. $m(L/K)$ is such that $\mathrm{Tr}_{L/K}(\pi_L^t A_L) = \pi_k^r A_k$ where

$$r = [p^{-1}((m(L/K) + 1)(p - 1) + t].$$

(If $x \in \mathbb{R}$, $[x]$ denotes the entier of x; (cf. [9], Ch. V, § 3).)

Now let K_∞/K be a (totally ramified) Γ-extension; let K_n be the invariant field of $p^n \mathrm{Gal}(K_\infty/K)$. We write $m_n = m(K_n/K_{n-1})$. Then the following holds

2. 1. Lemma (Tate [10]). *There is a constant m_0 such that*

$$m_n = e_K(1 + p + \cdots + p^{n-1}) + m_0$$

for all sufficiently large n.

Here e_K denotes the absolute index of ramification of K, $e_K = v_K(p)$.

If L/K is a totally ramified extension of degree p, then $m(L/K) \leqq (p-1)^{-1} p e_K$ (cf. e. g. [2], (6. 2 D) and Lemma (6. 3. B)). It follows that

2. 2. Lemma. $\qquad e_K - (p-1)m_0 \geqq 0.$

3. Statement of the theorem. Some remarks as to the proof

The main theorem of this paper is

3. 1. Theorem. *Let K be a mixed characteristic local field with algebraically closed residue field; let F be a one parameter formal group over the ring of integers A_K of K of height h; let $\cdots - K_n - \cdots - K_2 - K_1 - K$ be a Γ-extension of K. Then there exist constants c_1, c_2 such that*

$$F^{\alpha_n}(K) \supset F\text{-}\mathrm{Norm}_{n/0}(F(K_n)) \supset F^{\beta_n}(K)$$

where $\alpha_n = \dfrac{h-1}{h} e_K n - c_1$, $\beta_n = \dfrac{h-1}{h} e_K n + c_2$.

(If $h = \infty$, $(h-1)h^{-1}$ should be interpreted as 1.)

Basically, this theorem is proved by means of the same techniques as used in [3]. There are however some complications.

a) We have no longer a completely regular formula for m_n. This causes difficulties in the calculation of the $\sigma_{n/0}(t)$ and α_n (cf. § 8, § 9). The same fact causes difficulties in the calculation of $\mathrm{Tr}_{k/k-1}(x^{p^t})$ and thus makes necessary the introduction of some extra functions $\varrho_{n/k}(t)$ and $\tau_{n/k}(t)$, to keep track of what is happening. Cf. § 6.

b) It is no longer true that either $\mathrm{Tr}_{k/k-1}$ or $N_{k/k-1}^{p^{h-1}}$ (in the step from level k to level $k-1$) is the most important term in the expansion of F-Norm(x) as

$$F\text{-}\mathrm{Norm}(x) = \mathrm{Tr}(x) + \sum_{i=1}^{\infty} a_i N^i(x) + \sum a_M \mathrm{Tr}(M)$$

(cf. [3]). In fact there is for every $x \in F(K_n)$ a finite number of levels (bounded indepently of n!) in which terms of the form $a_i N^i$, $1 \leqq i < p^{h-1}$ dominate.

c) In the case of the cyclotomic Γ-extension of $\mathbb{Q}_p$ it so happened that, provided we started with elements of particularly nice valuations we could always neglect all except precisely one term of the expansion of F-Norm(x). This is not true in the more general case and it is this fact that makes the assumption "the residue field is algebraically closed" necessary. This also causes us to consider "change points" (cf. § 7).

d) It is no longer true as in the cyclotomic case that the set $\{\sigma_{n/0}(t) \mid t = 1, 2, \ldots\}$ is of the form $\{s \in \mathbb{N} \mid s \geqq s_0\}$. In general there will be "holes" in the sequence $\{\sigma_{n/0}(t)\}$. The difficulties due to these facts can be diminished to some extent by "changing the base field". That is instead of considering K_∞/K we take a (suitable) finite extension L/K such that $L \cdot K_\infty/L$ is still totally ramified (and a Γ-extension) and prove the theorem for the Γ-extension $L \cdot K/L$.

The most difficult part of the theorem then follows also for K_∞/K according to (3. 5) below. To prove this we need some preliminaries.

Let L/K be a finite extension. We define functions $\lambda_{L/K}(t)$ and $\chi_{L/K}(s)$ as follows:

3. 2. Definitions. (i) If L/K is unramified $\lambda_{L/K}(t) = t$.

(ii) If L/K is tamely and totally ramified of prime degree $l \neq p$ we define $\lambda_{L/K}(t) = [l^{-1}(l - 1 + t)]$.

(iii) If L/K is totally and wildly ramified of prime degree p we define

$$\lambda_{L/K}(t) = [p^{-1}((m(L/K) + 1)(p - 1) + t)].$$

(iv) If $M - L - K$ is a tower we define $\lambda_{M/K}$ as $\lambda_{M/K} = \lambda_{L/K} \circ \lambda_{M/L}$.

This defines $\lambda_{L/K}$ for all L/K because every L/K decomposes as a tower of extensions of the types considered under (i), (ii), (iii), as K is a mixed characteristic local field with perfect residue field.

For each integer $s \geqq \lambda_{L/K}(1)$ we define the number $\chi_{L/K}(s)$ as the largest integer t such that $\lambda_{L/K}(t) = s$. (Note that such a t always exists if $s \geqq \lambda_{L/K}(1)$.)

3. 3. Lemma. (i) $\mathrm{Tr}_{L/K}(\pi_L^t A_L) = \pi_K^s A_K$ *with* $s = \lambda_{L/K}(t)$.

(ii) *If* $t = \chi_{L/K}(s)$, *and* L/K *is totally ramified, then* $v_K(\mathrm{Tr}_{L/K}(x)) = s$, *if* $v_L(x) = t$.

(This also shows that the definition of $\lambda_{L/K}$ does not depend on the decomposition of L/K as a tower.)

3. 4. Lemma. *Let* L/K *be a finite extension and* F *a one parameter formal group defined over* A_K. *Then there exists a constant* t_0 *such that for* $t \geqq t_0$

$$F\text{-}\mathrm{Norm}_{L/K}(F^t(L)) = F^{\lambda_{L/K}(t)}(K).$$

Proof. In case L/K is unramified this follows from the proof of [3], 3. 1. In case L/K is totally ramified of prime degree we have an expansion of $\mathrm{Norm}(x)$ as

(3. 4. 1) $$\mathrm{Norm}(x) = \mathrm{Tr}_{L/K}(x) + \sum_{i=1}^{\infty} a_i N_{L/K}^i(x) + \sum_M a_M \mathrm{Tr}_{L/K}(M).$$

Now because $m(L/K) \leqq (p-1)^{-1} p e_K$ we have that

$$v_K(a_i N_{L/K}^i(x)) > \lambda_{L/K}(v_L(x)) \quad \text{if} \quad v_L(x) \geqq p e_K/(p-1).$$

Hence we have that $\mathrm{Norm}(x) \equiv \mathrm{Tr}_{L/K}(x) \bmod \pi_K^{\lambda_{L/K}(t)+1}$ if $v_L(x)$ is of the form $\chi_{L/K}(s)$ for some $s \geqq \lambda_{L/K}((p-1)^{-1} p e_K)$. This proves the lemma in this case (cf. 3. 3 above and [3], 3. 2). Finally let $M/L/K$ and suppose the lemma holds for M/L and L/K, then it also holds for M/K because $\lambda_{M/K} = \lambda_{L/K} \circ \lambda_{M/L}$, $\lambda_{L/K}(t') \geqq \lambda_{L/K}(t)$ if $t' \geqq t$ and $\lim_{t \to \infty} \lambda_{M/L}(t) = \infty$.

We are now in a position to prove

3. 5. Proposition. *Let*

$$\begin{array}{ccccccc} L_\infty & \cdots - & L_n & - \cdots - & L_1 & - & L \\ | & & | & & | & & | \\ K_\infty & \cdots - & K_n & - \cdots - & K_1 & - & K \end{array}$$

be a diagram of field extensions such that L_∞/L and K_∞/K are Γ-extensions and L/K is finite. Let F be a one dimensional formal group over A_K of height h and suppose that there exists a constant c' such that

$$\mathrm{Norm}_{L_n/L}(F(L_n)) \supset F^{\beta'_n}(L) \quad \textit{where} \quad \beta'_n = \frac{h-1}{h} n e_L + c'.$$

Then there exists a constant c such that

$$\mathrm{Norm}_{K_n/K}(F(K_n)) \supset F^{\beta_n}(K) \quad \textit{where} \quad \beta_n = \frac{h-1}{h} n e_K + c.$$

Proof. Let $t_0 \in \mathbb{N}$ be such that Lemma 3. 4 applies to L/K. For n sufficiently large $\beta'_n \geqq t_0$ and then

$$\mathrm{Norm}_{K_n/K}(F(K_n)) \supset \mathrm{Norm}_{L/K}(\mathrm{Norm}_{L_n/L}(F(L_n)) \supset F^{\lambda_{L/K}(\beta'_n)}(K).$$

It now suffices to remark that

$$\lambda_{L/K}(t) = \frac{t e_K}{e_L} + e_t \tag{3. 5. 1}$$

where the e_t are bounded independently of t. (This follows directly from the definition of $\lambda_{L/K}(t)$.) q. e. d.

4. A trace lemma

As in [3] we shall need to know something of $\mathrm{Tr}_{L/K}(x^{pk})$ for totally ramified extensions of degree p (cf. [3], § 4. 5).

4. 1. Proposition. *Let L/K be a totally ramified galois extension of degree p; let $m = m(L/K)$ and $r = [p^{-1}((m+1)(p-1)+1)]$; let π_L be a uniformizing element of L and let $\pi_K = (-1)^{p-1} N_{L/K}(\pi_L)$. Then we have*

$$\mathrm{Tr}_{L/K}(\pi_L^{kp}) \equiv p \pi_K^k \bmod \pi_K^{2r+k-1}.$$

Proof. The element $\pi_L \in L$ satisfies an equation of type

$$\pi_L^p + a_1 \pi_L^{p-1} + \cdots + a_{p-1}\pi_L = \pi_K \tag{4. 1. 1}$$

where the a_i are equal to $a_i = (-1)^i \sigma_i(\tau_1\pi_L, \ldots, \tau_p\pi_L)$, where σ_i is the i-th elementary symmetric function in p variables and $\tau_1\pi_L, \ldots, \tau_p\pi_L$ are the conjugates of π_L.

If $S = \{\tau_{i_1}, \ldots, \tau_{i_s}\}$ is a subset of $G(L/K)$, let $\tau S = \{\tau\tau_{i_1}, \ldots, \tau\tau_{i_s}\}$, $\pi_L^S = \prod_{\tau \in S} \tau\pi_L$ and $|S| =$ the number of elements of S. With these notations

$$a_i = (-1)^i \sum_{|S|=i} \pi_L^S. \tag{4. 1. 2}$$

Now if $|S| \neq 0, p$, then $\tau S \neq S$ if $\tau \neq \mathrm{id}$ because $G(L/K)$ is cyclic of prime order p. Hence each a_i is of the form

$$a_i = \mathrm{Tr}_{L/K}(b_i) \quad b_i \in \pi_L^i A_L. \tag{4. 1. 3}$$

Therefore $v_K(a_i) \geqq r$. Now apply $\mathrm{Tr}_{L/K}$ to the relation (4. 1. 1) to obtain

$$\mathrm{Tr}_{L/K}(\pi_L^2) \equiv p\pi_K \bmod \pi_K^{2r}. \tag{4. 1. 4}$$

This proves the proposition for $k = 1$. For $k > 1$ multiply the relation (4. 1. 1) above with $\pi_L^{(k-1)p}$, apply $\mathrm{Tr}_{L/K}$ to the result, use induction, and use

$$v_K(\mathrm{Tr}_{L/K}(\pi_L^s)) \geqq k - 1 + r$$

if $s \geqq (k-1)p + 1$. Cf. 3. 3.

5. Some functions

Let F be a formal group over A_K and K_∞/K a Γ-extension of K. Let

$$F(X_1, \ldots, X_p) = \mathrm{Tr}(X_1) + \sum_{i=1}^{\infty} a_i N^i(X) + \sum_M a_M \mathrm{Tr}(M)$$

(cf. [3], 2. 4). We write $\mathrm{Norm}_{n/k}$ for the norm map $F(K_n) \to F(K_k)$. In [3] we used a number of functions $\sigma_{n/k}$, $\iota_{n/k}$, $d_{n/k}$ to keep track of what was happening and to calculate $\sigma_{n/k}$ we used some auxiliary functions j_n and l_n. All these functions and some more will be needed again. All of them will be defined in this section.

We use $v = v_K$ for the normalised exponential valuation on K and $v_k = v_{K_k}$ for the normalised exponential valuation on K_k.

5. 1. The function $\sigma_{n/k}(t)$. Let $d_i = v(a_i) = v_K(a_i)$, where a_i is the coefficient of $N^i(X)$ in the expansion of $F(X_1, \ldots, X_p)$. We define for all $t \in \mathbb{R}$, $t \geqq 1$,

$$\begin{aligned} \sigma^0_{k/k-1}(t) &= \left[\frac{(m_k+1)(p-1)+t}{p}\right], \\ \sigma^i_{k/k-1}(t) &= d_i p^{k-1} + it, \end{aligned} \qquad i = 1, 2, \ldots \tag{5. 1. 1}$$

(note that the $\sigma^1_{k/k-1}(t)$ of [3] is equal to $\sigma^{p^{h-1}}_{k/k-1}(t)$ as defined here).

Using the $\sigma^i_{k/k-1}$ we define

$$\sigma_{k/k-1}(t) = \min_{i=0,1,p,p^2,\ldots,p^{h-1}} \{\sigma^i_{k/k-1}(t)\}. \tag{5. 1. 2}$$

Remark. Let t be an integer. Then the smallest integer i_0 such that

$$\sigma^{i_0}_{k/k-1}(t) = \min_{i=0,1,2,\ldots} \{\sigma^i_{k/k-1}(t)\} = s$$

is necessarily equal to 0, 1 or a power of p. Indeed $\mathrm{Norm}_{k/k-1}$ induces a homomorphism ($\bar{K}$ is the residue field of K)

$$\bar{K}^+ \cong F^t(K_k)/F^{t+1}(K_k) \to F^s(K_{k-1})/F^{s+1}(K_{k-1}) = \bar{K}^+.$$

And if $i_0 > 0$ this homomorphism is given by a polynomial $a_{i_0} z^{i_0}$ + higher degree terms. It follows that $i_0 = 1$ or a power of p.

A corollary of this remark is that

$$\mathrm{Norm}_{k/k-1}(F^t(K_k)) \subset F^{\sigma_{k/k-1}(t)}(K_{k-1}), \qquad t \in \mathbb{N}. \tag{5. 1. 3}$$

We now define $\sigma_{n/k}(t)$ for $k \leqq n$ inductively by

$$\sigma_{n/n}(t) = t, \ \sigma_{n/k}(t) = \sigma_{k+1/k}(\sigma_{n/k+1}(t)). \tag{5. 1. 4}$$

It follows from (5. 1. 3) that

(5. 1. 5) $$\mathrm{Norm}_{n/k}(F^t(K_n)) < F^{\sigma_{n/k}(t)}(K_k).$$

We define

(5. 1. 6) $$\begin{aligned}\iota_{k/k}(t) &= -1 \text{ if } \sigma_{k/k-1}(t) = \sigma^0_{k/k-1}(t),\\ \iota_{k/k}(t) &= r \quad \text{ if } \sigma_{k/k-1}(t) = \sigma^{p^r}_{k/k-1}(t) < \sigma^j_{k/k-1}(t), \qquad j = 0, 1, p, \ldots, p^{r-1}.\end{aligned}$$

Using this, we also define

(5. 1. 7) $$\iota_{n/k}(t) = \iota_{k/k}(\sigma_{n/k}(t)), \qquad k = 1, 2, \ldots, n,$$

(5. 1. 8) $$\alpha_{n/n}(t) = 0;\ \alpha_{n/k}(t) = \alpha_{n/k+1}(t) + \iota_{n/k+1}(t),$$

(5. 1. 9) $$\begin{cases}\tau_{n/n}(t) = t\\ \tau_{n/k}(t) = v_k(a_{p^r}) + p^r\tau_{n/k+1}(t) & \text{if } \iota_{n/k+1}(t) = r \geqq 0,\\ \tau_{n/k}(t) = v_k(p) + p^{-1}\cdot\tau_{n/k+1}(t) & \text{if } \iota_{n/k+1}(t) = -1,\ \alpha_{n/k+1}(t) \geqq 1,\\ \tau_{n/k}(t) = \dfrac{(m_{k+1}+1)(p-1) + \tau_{n/k+1}(t)}{p} & \text{if } \iota_{n/k+1}(t) = -1,\ \alpha_{n/k+1}(t) \leqq 0,\end{cases}$$

(5. 1. 10) $$\begin{cases}\varrho_{n/n}(t) = t,\\ \varrho_{n/k}(t) = v_k(a_{p^r}) + p^r\cdot\varrho_{n/k+1}(t) & \text{if } \iota_{n/k+1}(t) = r \geqq 0,\\ \varrho_{n/k}(t) = v_k(p) + p^{-1}\cdot\varrho_{n/k+1}(t) & \text{if } \iota_{n/k+1}(t) = -1.\end{cases}$$

Further we define

(5. 1. 11) $l_n(t) = 0$ if $\alpha_{n/0}(t) \geqq 0$,
$l_n(t)$ = smallest natural number i such that $\alpha_{n/i}(t) = 0$ and $\iota_{n/i}(t) = -1$, if $\alpha_{n/0}(t) < 0$,

(5. 1. 12) $k_n(t)$ = largest integer i such that $\iota_{n/i}(t) = -1$,

(5. 1. 13) $j_n(t)$ = number of different indices i such that $\iota_{n/i}(t) = h - 1$.

The last function we define is

(5. 1. 14) $$\begin{aligned}d_{n/k}(t) &= \tau_{n/k}(t) + p^{\alpha_{n/k}(t)}(e_1 - m_0) && \text{if } \alpha_{n/k}(t) \geqq 0,\ e_1 = (p-1)^{-1}e_K,\\ &= \tau_{n/k}(t) + 1 && \text{if } \alpha_{n/k}(t) < 0.\end{aligned}$$

6. Some elementary properties of the functions $\sigma_{n/k}, \rho_{n/k}, \tau_{n/k}, \iota_{n/k}, l_n, j_n, k_n, \alpha_{n/k}$

Let K_∞/K be a Γ-extension and let F be a formal group over A_K. In this section we assume that the Γ-extension K_∞/K is such that

$$m_n = (1 + p + \cdots + p^{n-1})e_K + m_0$$

for all $n = 1, 2, 3, \ldots$ (cf. Lemma 2. 1).

6. 1. Lemma. $\iota_{n/k}(t) \leqq \iota_{n/k+1}(t)$ *for all* t, $1 \leqq k < n$.

Proof. Let $s = \sigma_{n/k+1}(t)$ and $s' = \sigma_{n/k}(t)$. Suppose that $\iota_{n/k+1}(t) = r \geqq 0$, i. e.

(6. 1. 1) $$p^k d_i + is \geqq p^k d_{p^r} + p^r s \quad \text{for } i = p^{r+1}, p^{r+2}, \ldots, p^{h-1}.$$

Then $s' = p^k d_{pr} + p^r s$ and we must show that

$$(6.1.2)\quad p^{k-1}d_i + is' \geqq p^{k-1}d_{pr} + p^r s' \quad \text{for } i = p^{r+1}, p^{r+2}, \dots, p^{h-1}$$

which follows from (6.1.1) because $s' \geqq p^{-1}s$. Now suppose that $\iota_{n/k+1}(t) = -1$. This means that

$$s' = \left[\frac{(m_{k+1}+1)(p-1)+s}{p}\right] \leqq p^k d_i + is \quad \text{for } i = 1, p, p^2, \dots, p^{h-1}$$

which is equivalent to

$$(6.1.3)\quad s \geqq \frac{m_{k+1}(p-1) - p^{k+1}d_i}{pi-1}, \quad i = 1, p, p^2, \dots, p^{h-1}.$$

And we must show that

$$(6.1.4)\quad s' \geqq \frac{m_k(p-1) - p^k d_i}{pi-1}, \qquad i = 1, p, p^2, \dots, p^{h-1}.$$

It suffices to show that $s' \geqq m_k$. We have

$$\begin{aligned} s' &= [p^{-1}((m_{k+1}+1)(p-1)+s)] \geqq p^{-1}(p-1)m_{k+1} \\ &= p^{-1}(p-1)(1+p+\cdots+p^k)e_K + p^{-1}(p-1)m_0 \\ &= p^k e_K - p^{-1}e_K + p^{-1}(p-1)m_0. \end{aligned}$$

And this is greater or equal to $m_k = (1+p+\cdots+p^{k-1})e_K + m_0$ because

$$p^k \geqq 1 + p + \cdots + p^{k-1} + 1$$

and $e_K \geqq (p-1)m_0$. q. e. d.

Note that we have also shown that

$$(6.1.5)\quad \iota_{k/k}(t) = -1 \quad \text{if } t \geqq (p-1)^{-1}p^k e_K$$

(this result holds because $m_k \leqq (p-1)^{-1}p^k e_K$ and is independent of the assumption on K_∞/K).

6.2. Some properties of $\iota_{n/k}$, $\sigma_{n/k}$, $\varrho_{n/k}$, $\tau_{n/k}$, $\alpha_{n/k}$. Directly from the definition of $\iota_{k/k}$ one sees that

$$(6.2.1)\quad \iota_{k/k}(t') \leqq \iota_{k/k}(t) \quad \text{if } t' \geqq t.$$

This is obvious if $\iota_{k/k}(t) \geqq 0$. If $\iota_{k/k}(t) = -1$, then writing b_i for $d_i p^{k-1}$, and c for $(m_k+1)(p-1)$ we have

$$\left[\frac{c+t}{p}\right] \leqq b_i + it, \qquad i = 1, 2, \dots, p^{h-1}.$$

Now suppose that there is a $t' > t$ such that $\left[\frac{c+t'}{p}\right] > b_i + it'$ for a certain i then $\left[\frac{c+t'}{p}\right] > \left[\frac{c+t}{p}\right]$; let t'' be such that $\left[\frac{c+t'}{p}\right] = \frac{c+t''}{p}$, then also $\left[\frac{c+t''}{p}\right] > b_i + it''$ and $t'' > t$ and t'' is an integer. Then $\frac{c+t''}{p} \geqq b_i + it'' + 1$ because $\frac{c+t''}{p}$ and $b_i + it''$ are integers, and $\frac{c+t}{p} < b_i + it + 1$ while $t'' > t$ which is a contradiction.

As a corollary to Lemma 6.1 we get

$$(6.2.2)\quad \sigma_{n/k}(t) = \varrho_{n/k}(t) = \tau_{n/k}(t) \quad \text{if } \iota_{n/k+1}(t) \geqq 0.$$

Using this and (6. 2. 1) we see that

(6. 2. 3) $\sigma_{n/k}(t') \geqq \sigma_{n/k}(t), \varrho_{n/k}(t') \geqq \varrho_{n/k}(t), \tau_{n/k}(t') \geqq \tau_{n/k}(t)$ if $t' \geqq t$

and this and (6. 2. 1) gives

(6. 2. 4) $\alpha_{n/k}(t') \leqq \alpha_{n/k}(t), \iota_{n/k}(t') \leqq \iota_{n/k}(t)$ if $t' \geqq t$.

The functions $\tau_{n/k}$, $\sigma_{n/k}$ and $\varrho_{n/k}$ do not differ much. Indeed we have

(6. 2. 5) $\sigma_{n/k}(t) \leqq \tau_{n/k}(t) \leqq \varrho_{n/k}(t) \leqq \sigma_{n/k}(t) + (p-1)^{-1}e_K - m_0$.

This is proved by induction on k. The first two inequalities are immediate. As to the last one if $\iota_{n/m}(t) \geqq 0$ then $\varrho_{n/k}(t) = \sigma_{n/k}(t)$, for $k \geqq m-1$. It remains to show that if $\varrho_{n/k+1}(t) \leqq \sigma_{n/k+1}(t) + (p-1)^{-1}e_K - m_0$, and $\iota_{n/k+1}(t) = -1$ then

$$\varrho_{n/k}(t) \leqq \sigma_{n/k}(t) + (p-1)^{-1}e_K - m_0.$$

We have

$$\sigma_{n/k}(t) = \left[\frac{(m_{k+1}+1)(p-1) + \sigma_{n/k+1}(t)}{p}\right] \geqq p^{-1}\sigma_{n/k+1}(t) + p^{-1}(p-1)(m_{k+1})$$
$$\geqq p^{-1}\sigma_{n/k+1}(t) + p^k e_K - p^{-1}e_K + \frac{p-1}{p} m_0.$$

On the other hand

$$\varrho_{n/k}(t) = e_K \cdot p^k + p^{-1}\varrho_{n/k+1}(t) \leqq e_K p^k + \frac{\sigma_{n/k+1}(t)}{p} + \frac{e_K}{p(p-1)} - \frac{m_0}{p}$$
$$\leqq \sigma_{n/k}(t) + \frac{e_K}{p(p-1)} - \frac{m_0}{p} + \frac{e_K}{p} - \frac{p-1}{p} m_0$$
$$= \sigma_{n/k}(t) + (p-1)^{-1}e_K - m_0.$$

6. 3. Some properties of j_n, k_n, l_n.

(6. 3. 1) $l_n(t') \geqq l_n(t), k_n(t') \geqq k_n(t), j_n(t') \leqq j_n(t)$ if $t' \geqq t$.

Further we have as a consequence of 6. 1, (6. 2. 1), (6. 2. 4):

(6. 3. 2) If $l_n(t') = l_n(t) > 0$ then for all $0 < k \leqq n$ we have

$$\iota_{n/k}(t) = \iota_{n/k}(t') \quad \text{and} \quad \alpha_{n/k}(t) = \alpha_{n/k}(t').$$

Finally we have

(6. 3. 3) $\tau_{n/k}(t') - \tau_{n/k}(t) = t' - t$ if $0 < k = l_n(t) = l_n(t')$.

(6. 3. 4) $t \equiv \tau_{n/k}(t) \bmod p^k$ if $0 < k = l_n(t)$.

(6. 3. 5) If $l_n(t) = b > 0$, then $\alpha_{n/i}(t) = i - b$ for $b \leqq i \leqq k_n(t)$ and $\alpha_{n/i}(t) < i - b$ for $k_n(t) < i \leqq n$.

(6. 3. 6) For $s \geqq l_n(t)$, $\varrho_{n/s}(t) = \tau_{n/s}(t)$.

The properties (6. 3. 3)—(6. 3. 6) follow directly from the definitions of the various functions involved.

7. Change points and nice pairs $(F, K_\infty/K)$

Let F be a formal group over A_K and K_∞/K a Γ-extension of K. Let

$$F(X_1, \ldots, X_p) = \mathrm{Tr}(X_1) + \sum_{i=1}^{\infty} a_i N^i(X) + \sum_M a_M \mathrm{Tr}(M)$$

(cf. [3], 2. 4). We write $\mathrm{Norm}_{k/k-1}$ for the norm map $F(K_k) \to F(K_{k-1})$. It may happen that for certain $x \in F(K_k)$ there occur several terms of the same (minimal) valuation in the expansion

$$\mathrm{Norm}_{k/k-1}(x) = \mathrm{Tr}_{k/k-1}(x) + \sum_{i=1}^{\infty} a_i N_{k/k-1}(x) + \sum_M a_M \mathrm{Tr}_{k/k-1}(M).$$

The valuations $v_k(x)$ of elements x at which this is to be expected will be called *change points*.

More precisely, the *smallest* number t such that $\sigma^i_{k/k-1}(t) = \sigma^j_{k/k-1}(t)$, $0 \leqq i \neq j \leqq p^{h-1}$, $k = 1, 2, \ldots, n$ will be called an *(i, j)-level k-change point*, and will be denoted $c_{ij}(k)$. The change point $c_{ij}(k)$ is called *actual* if moreover

$$\sigma_{k/k-1}(c_{ij}(k)) = \sigma^i_{k/k-1}(c_{ij}(k)) = \sigma^j_{k/k-1}(c_{ij}(k)).$$

7. 1. Lemma. $c_{ij}(k) = \dfrac{d_i - d_j}{j - i} p^{k-1}$ *if* $0 < i < j$,

$$\frac{(p-1)(m_k+1) - p^k d_i}{pi - 1} - \frac{p}{pi-1} < c_{i0}(k) \leqq \frac{(p-1)(m_k+1) - p^k d_i}{pi-1} \quad \textit{if} \quad 0 < i.$$

Proof. The first part is obvious. As to the second if $t = c_{i0}(k)$ then

(7. 1. 1) $$\left[\frac{(m_k+1)(p-1)+t}{p}\right] = p^{k-1} d_i + it$$

which means

(7. 1. 2) $$\frac{(m_k+1)(p-1)+t}{p} = p^{k-1} d_i + it + \varepsilon, \quad 0 \leqq \varepsilon < 1,$$
$$t = \frac{(p-1)(m_k+1) - p^k d_i}{pi-1} - \frac{p\varepsilon}{pi-1}.$$

This proves the lemma.

Corollary. *If* $m_k = (1 + p + \cdots + p^{k-1}) e_K + m_0$ *then*

(7. 1. 3) $$c_{i0}(k) = \frac{(e_K - d_i) p^k}{pi - 1} - \frac{e_K - (p-1)(m_0+1)}{pi-1} - \frac{p\varepsilon}{pi-1}, \quad 0 \leqq \varepsilon < 1.$$

Remark. If $i = 1$, then we see from (7. 1. 1) that $c_{i0}(k) = t$ is an integer; it follows that $\varepsilon \leqq \dfrac{p-1}{p}$, so that

(7. 1. 4) $$-1 + \frac{(p-1)(m_k+1) - p^k d_1}{p-1} \leqq c_{10}(k) \leqq \frac{(p-1)(m_k+1) - p^k d_1}{p-1}.$$

7. 2. Definition. A pair $(F, K_\infty/K)$ consisting of a formal group over A_K and a (totally ramified) Γ-extension will be called *nice* if the following conditions are satisfied:

(i) $m_n = m(K_n/K_{n-1}) = e_K(1 + p + \cdots + p^{n-1}) + m_0$ for some constant m_0 for *all* $n = 1, 2, 3, \ldots$,

(ii) the numbers $\dfrac{d_i - d_j}{j - i}$ are integers for all $1 \leqq i \leqq j \leqq p^{h-1}$,

(iii) the numbers $\frac{e_K - d_i}{pi - 1}$ are integers for all $1 \leqq i \leqq p^{k-1}$,

(iv) e_K is divisible by $p-1$.

In the following section we shall need a few technical results on the position of various change points in the case of nice pairs $(F, K_\infty/K)$.

7. 3. Lemma. *If $i < j$, $t \geqq c_{ij}(k)$ then $\sigma^i_{k/k-1}(t) \leqq \sigma^j_{k/k-1}(t)$.*
If $i < j$, $t < c_{ij}(k)$ then $\sigma^i_{k/k-1}(t) > \sigma^j_{k/k-1}(t)$.

This is not immediately clear only in the case that $i = 0$. In this case one uses the same argument as was used to establish (6. 2. 1); note that $c_{ij}(k)$ is the *smallest* number t such that $\sigma^i_{k/k-1}(t) = \sigma^j_{k/k-1}(t)$.

7. 4. Lemma. *Let $(F, K_\infty/K)$ be a nice pair and suppose $p^r > i > 0$. Let t be an integer, $\iota_{k/k}(t) = r$, and suppose that $c_{i,p^r}(k)$ does not belong to the interval $(t, c_{0,p^r}(k)]$. Then $c_{0,i}(k) \leqq c_{0,p^r}(k)$.*

Proof. We write $j = p^r$. Because $\iota_{k/k}(t) = r$ we have, because $i < p^r = j$, that $\sigma^i_{k/k-1}(t) > \sigma^j_{k/k-1}(t)$ and hence, by (7. 3), $c_{ij}(k) > t$; therefore $c_{ij}(k) > c_{0j}(k)$ and hence, by (7. 3), $\sigma^j_{k/k-1}(c_{0j}(k)) < \sigma^i_{k/k-1}(c_{0j}(k))$ and as $\sigma^j_{k/k-1}(c_{0j}(k)) = \sigma^0_{k/k-1}(c_{0j}(k))$ we have again by (7. 3) that $c_{0j}(k) \geqq c_{0i}(k)$.

7. 5. Lemma. *Let $(F, K_\infty/K)$ be a nice pair and let $e_1 - m_0 \geqq 1$, $p^k > (e_1 - m_0) + 1$. Suppose that $0 < i < j$ and $c_{0i}(k) \leqq c_{0j}(k)$ then $c_{0i}(k-1) \leqq c_{0j}(k-1) + 1$ (here $e_1 = (p-1)^{-1} e_K$).*

Proof. We have (using the fact that $m_k = e_K(1 + p + \cdots + p^{k-1}) + m_0$) according to (7. 1)

$$c_{0,i}(k) = \frac{(e_K - d_i)p^k}{pi-1} - \frac{e_K - (p-1)(m_0+1)}{pi-1} - \frac{p\varepsilon}{pi-1};$$

$$c_{0j}(k) = \frac{(e_K - d_j)p^k}{pj-1} - \frac{e_K - (p-1)(m_0+1)}{pj-1} - \frac{p\varepsilon'}{pj-1}$$

where $0 \leqq \varepsilon, \varepsilon' < 1$. Because $0 < (pi-1)^{-1}(e_K - (p-1)(m_0+1) + p\varepsilon) < p^k$ by hypothesis, we have that $(pi-1)^{-1}(e_K - d_i) \leqq (pj-1)^{-1}(e_K - d_j)$. (Both these numbers are integers because $(F, K_\infty/K)$ is nice.) Therefore

$$c_{0,i}(k-1) \leqq c_{0,j}(k-1) + \frac{p\varepsilon''}{pj-1} < c_{0j}(k-1) + 1$$

because $(pi-1)^{-1}(e_K - (p-1)(m_0+1)) > (pj-1)^{-1}(e_K - (p-1)(m_0+1))$.

7. 6. Lemma. *Let $(F, K_\infty/K)$ be a nice pair and suppose that $e_1 - m_0 \geqq 2$, $t \equiv -(e_1 - m_0)p^f \bmod p^{f+g}$, $f \in \mathbb{N} \cup \{0\}$, $(e_1 - m_0) < p^g$, $k > f + g$ and $\iota_{k/k}(t) = r$. Then $c_{ij}(k) < t$ for $i > j$, where $j = p^r$ if $r \geqq 0$, $j = 0$ if $r = -1$, except possibly in the case $f = 0$, $r = -1$, $i = 1$.*

Proof. Because $\iota_{k/k}(t) = r$ we have in any case $c_{ij}(k) \leqq t$ (cf. (7. 3)). Suppose that $r \geqq 0$, then $c_{ij}(k) \equiv 0 \bmod p^{k-1}$ because $(F, K_\infty/K)$ is nice. This makes $c_{ij}(k) = t$ impossible. Now let $r = -1$ then we have

$$(pi-1)^{-1}(e_K - (p-1)(m_0+1) + p) < (e_1 - m_0)p^f$$

if either $f > 0$ or $i > 1$, which makes $c_{i0}(k) = t$ impossible (cf. (7. 1. 3)).

7. 7. Lemma. *Let $(F, K_\infty/K)$ be a nice pair, $t \in \mathbb{N}$, $\iota_{s+1/s+1}(t) = -1$, and $e_K p^s \geqq 2(e_1 - m_0) \geqq 4$. Then*

$$\sigma_{s+1/s}(t) > c_{0,i}(s) + 1 \quad \text{for all} \quad i = 1, 2, \ldots, p^{h-1}.$$

Proof. Let

$$c_{0i}(s) = (pi-1)^{-1}(e_K - d_i)p^s - (pi-1)^{-1}(e_K - (p-1)(m_0+1)) - \frac{\varepsilon p}{pi-1}$$

then

$$c_{0i}(s+1) = (pi-1)^{-1}(e_K - d_i)p^{s+1} - (pi-1)^{-1}(e_K - (p-1)(m_0+1)) - \frac{\varepsilon p}{pi-1}$$

with the same ε (cf. (7. 1. 1)). Because $\iota_{s+1/s+1}(t) = -1$ we have that $t \geqq c_{0i}(s+1)$. It follows that

$$\begin{aligned}
\sigma_{s+1/s}(t) - c_{0i}(s) - 1 &\geqq \left[\frac{(m_{s+1}+1)(p-1)+t}{p}\right] - c_{0i}(s) - 1 \\
&\geqq \frac{m_{s+1}(p-1)}{p} + \frac{t}{p} - c_{0i}(s) - 1 \\
&\geqq \frac{p^{s+1}e_K}{p} - \frac{p-1}{p}(e_1 - m_0) \\
&\quad + \frac{(e_K - d_i)p^{s+1}}{p(pi-1)} - \frac{(p-1)(e_1-m_0)}{p(pi-1)} - \frac{\varepsilon}{pi-1} \\
&\quad - \frac{(e_K-d_i)p^s}{pi-1} + \frac{(p-1)(e_1-m_0)}{pi-1} - \frac{p-1}{pi-1} + \frac{\varepsilon p}{pi-1} - 1 \\
&= p^s e_K - \frac{p-1}{p}(e_1-m_0) - \frac{(p-1)(e_1-m_0)}{p(pi-1)} \\
&\quad + \frac{(p-1)(e_1-m_0)}{pi-1} + \frac{(p\varepsilon+1-p)(p-1)}{p(pi-1)} - 1 \\
&> p^s e_K - \frac{p-1}{p}(e_1-m_0) - 2 \geqq p^s e_K - (e_1-m_0) - 2 \\
&\geqq p^s e_K - 2(e_1-m_0) \geqq 0.
\end{aligned}$$

7. 8. Corollary. *Let $(F, K_\infty/K)$ be a nice pair, $t \in \mathbb{N}$, $\iota_{s+1/s+1}(t) = -1$, and $e_K p^s \geqq 2(e_1 - m_0) \geqq 4$. Then*

$$\sigma^i_{s/s+1}(\sigma_{s+1/s}(t)) > \sigma^0_{s/s-1}(\sigma_{s+1/s}(t)) \quad \text{for all} \quad i = 1, 2, \ldots .$$

This follows directly from Lemma 7. 7 above, and the definition of the σ^i, because if t is an integer then $\sigma^0_{s/s-1}(t+t') \leqq \sigma^0_{s/s-1}(t) + 1 + \dfrac{t'-1}{p}$.

8. The main proposition

The proposition below is our main tool in the proof of Theorem 3. 1. The proof is rather lengthy and involved but not difficult.

8. 1. Proposition. *Let $(F, K_\infty/K)$ be a nice pair such that $e_1 - m_0 \geqq p$, $e_K \geqq (e_1 - m_0)$. Let t be an integer of the form $\chi_{b/0}(t')$, $b \in \mathbb{N}$, $b < n$ such that*

$$(8.1.1) \qquad l_n(t) = b,$$

$$(8.1.2) \qquad t \geqq p^{h+1}(e_1 - m_0) + p^{r_0+1}(e_1 - m_0 + 2)$$

where r_0 *is the smallest natural number such that* $p^{r_0} \geqq (e_1 - m_0)$, *and*

(8. 1. 3) $$p^b \geqq p(e_1 - m_0)(p^h + 3),$$

(8. 1. 4) $$p^{b-r_0} \geqq 2(e_1 - m_0 + 1).$$

Then we have

(i) $\sigma_{n/0}(t) = \tau_{n/0}(t)$, F-$\mathrm{Norm}_{n/0}$ *maps* $F^t(K_n)$ *into* $F^{\tau_{n/0}(t)}(K)$.

(ii) F-$\mathrm{Norm}_{i/b}$ *maps* $F^{d_{n/i}(t)}(K_i)$ *into* $F^{d_{n/b}(t)}(K_b)$ *for all* $b \leqq i \leqq n$.

(iii) F-$\mathrm{Norm}_{i/0}$ *maps* $F^{d_{n/i}(t)}(K_i)$ *into* $F^{\tau_{n/0}(t)+1}(K)$ *for all* $1 \leqq i \leqq n$.

(iv) *The induced map* $F^t(K_n) \to F^{\tau_{n/0}(t)}(K)/F^{\tau_{n/0}(t)+1}(K)$ *is surjective if the residue field of* K *is algebraically closed.*

Proof. To prove (i) we must show that $\lambda_{l_n(t)/0}(\tau_{n/l_n(t)}(t)) = \sigma_{n/0}(t)$ (because $\iota_{n/k}(t) = -1$ if $k \leqq l_n(t)$; cf. the definitions of $\tau_{n/k}$ and $\sigma_{n/k}$). Now

$$\tau_{n/l_n(t)}(t) = \tau_{n/b}(t) \geqq t$$

and by (6. 3. 4), $\tau_{n/b}(t) \equiv t \bmod p^b$. Because t is of the form $\chi_{b/0}(s)$ it follows that $\tau_{n/b}(t)$ is also in the image of $\chi_{b/0}$. And we have therefore

(8. 1. 5) $$\lambda_{b/0}(t') = \lambda_{b/0}(\tau_{n/b}(t)) \quad \text{if} \quad \tau_{n/b}(0) - p^b < t' \leqq \tau_{n/b}(0).$$

Now $\sigma_{n/b}(t) \leqq \tau_{n/b}(t) \leqq \sigma_{n/b}(t) + (e_1 - m_0)$ by (6. 2. 5) and $(e_1 - m_0) < p^b$ by condition (8. 1. 3). This proves the first part of (i), the second part follows immediately (cf. (5. 1. 5)).

To prove (ii) we use induction on i. Let $k = k_n(t) =$ largest integer for which $\iota_{n/b}(t) = -1$. Let $i \leqq k$. Then we have

(8.1.6) $$v_b(\mathrm{Tr}_{i/b}(x)) \geqq \tau_{n/k}(t) + 1 \quad \text{if} \quad v_i(x) \geqq \tau_{i/b}(t) + (p^{i-b} - 1)(e_1 - m_0) + 1.$$

This will be proved in (8. 4) below. This proves (ii) for $i \leqq k$ because $\tau_{n/b}(t) + 1 = d_{n/b}(t)$, and $d_{n/i}(t) = p^{i-b}(e_1 - m_0) + \tau_{n/i}(t)$ $(\alpha_{n/i}(t) = i - b$ because $l_n(t) = b$ and $\iota_{n/i}(t) = -1)$.

Now let $i \geqq k + 1 = k_n(t) + 1$. Let t' be the smallest actual level i-change point (cf. 7) such that $t' \geqq \tau_{n/i}(t)$. There are three possibilities

1) $t' - \tau_{n/i}(t) > (e_1 - m_0)p^{\alpha_{n/i}(t)}$.

2) $t' - \tau_{n/i}(t) = (e_1 - m_0)p^{\alpha_{n/i}(t)}$.

3) $t' - \tau_{n/i}(t) < (e_1 - m_0)p^{\alpha_{n/i}(t)}$.

In the first case as $\sigma_{n/i}(t) = \tau_{n/i}(t)$ we have $\iota_{i/i}(d_{n/i}(t)) = \iota_{i/i}(\tau_{n/i}(t)) = r \geqq 0$ and hence by the definition of $d_{n/k}$ and $\tau_{n/k}$

$$\begin{aligned}\sigma_{i/i-1}^{p^r}(d_{n/i}(t)) &= p^{i-1}d_{p^r} + p^r(\tau_{n/i}(t) + p^{\alpha_{n/i}(t)}(e_1 - m_0)) \\ &= \tau_{n/i-1}(t) + p^{\alpha_{n/i-1}(t)}(e_1 - m_0) = d_{n/i-1}(t)\end{aligned}$$

which implies

$$\mathrm{Norm}_{i/i-1}(F^{d_{n/i}(t)}(K_i)) \subset F^{d_{n/i-1}(t)}(K_{i-1})$$

which proves the induction step in this case.

In case 2) we have that $d_{n/i}(t) = t'$, let $t' = c_{p^r j}(i)$, $j < p^r$. Then $\iota_{i/i}(d_{n/i}(t)) < r$ and $\sigma_{i/i-1}(d_{n/i}(t)) = \sigma^j_{i/i-1}(d_{n/i}(t))$. But because $d_{n/i}(t) = t' = c_{p^r j}$ we have that

$$\sigma^j_{i/i-1}(d_{n/i}(t)) = \sigma^{p^r}_{i/i-1}(d_{n/i}(t))$$

and the same calculation as above proves the induction step also in this case.

Now suppose that case 3) applies. Because $(F, K_\infty/K)$ is a nice pair we have that $(p-1) \mid e_K$ and as t is of the form $\chi_{b/0}(t'')$ for some t'' we have according to 8. 3, $t \equiv -(e_1 - m_0) \bmod p^b$. It follows that

$$\tau_{n/i}(t) \equiv -(e_1 - m_0)p^{\alpha_{n/i}(t)} \bmod p^{b+\alpha_{n/i}(t)} \qquad k < i \leqq n \tag{8. 1. 7}$$

(cf. (6. 3. 5)). Let $\iota_{n/i}(t) = r \geqq 0$, then t' must be a $c_{p^r s}(i)$ with $s < p^r$ because t' is the next largest actual change point. Now if $s > 0$ then

$$c_{p^r s}(i) \equiv 0 \bmod p^{i-1} \tag{8. 1. 8}$$

because $(F, K_\infty/K)$ is nice. Furthermore $\alpha_{n/i}(t) < i - b$ (cf. (6. 3. 5)) so that

$$c_{p^r s}(i) \equiv 0 \bmod p^{b+\alpha_{n/i}(t)};$$

further $(e_1 - m_0) < p^b$ (condition (8. 1. 3)). It follows that case 3) can only occur if t' is the actual change point $t' = c_{p^r 0}(i)$. We then have (cf. (7. 1. 3) and (7. 1. 4))

$$-(e_1 - m_0)p^{\alpha_{n/i}(t)} + \frac{p^i(e_K - d_{p^r})}{p^{r+1}-1} - \frac{p-1}{(p^{r+1}-1)}(e_1 - m_0 - 1) - \frac{p\varepsilon}{p^{r+1}-1},$$
$$< \tau_{n/i}(t) \leqq \frac{p^i(e_K - d_{p^r})}{p^{r+1}-1} - \frac{p-1}{p^{r+1}-1}(e_1 - m_0 - 1) - \frac{p\varepsilon}{p^{r+1}-1} = c_{p^r 0}(i)$$

where $0 \leqq \varepsilon < 1$ if $r > 0$ and $0 \leqq \varepsilon \leqq p - 1$ if $r = 0$. Note also that $\tau_{n/i}(t) = c_{p^r 0}(i)$ is impossible because then $\iota_{n/i}(t) = -1$ which contradicts $i \geqq k + 1$. The number $(p^{r+1} - 1)^{-1}(e_K - d_{p^r})$ is an integer because $(F, K_\infty/K)$ is nice, $p^i \equiv 0 \bmod p^{b+\alpha_{n/i}(t)}$ and $(p^{r+1} - 1)^{-1}(p - 1)(e_1 - m_0 - 1) > 0$. Now

$$(e_1 - m_0)p^{\alpha_{n/i}(t)} \geqq (e_1 - m_0) \geqq \frac{p-1}{p^{r+1}-1}(e_1 - m_0 - 1) + \frac{p\varepsilon}{p^{r+1}-1},$$

because of (8. 1. 7) it follows that case 3) can only occur if

$$\tau_{n/i}(t) = \frac{e_K - d_{p^r}}{p^{r+1}-1} p^i - p^{\alpha_{n/i}(t)}(e_1 - m_0). \tag{8. 1. 9}$$

Suppose first that $i = k + 1$. This gives us

$$\tau_{n/i-1}(t) = p^{r+i} \cdot \frac{e_K - d_{p^r}}{p^{r+1}-1} - p^r p^{\alpha_{n/i}(t)}(e_1 - m_0) + d_{p^r} p^{i-1}.$$

Further $d_{n/i}(t) = \tau_{n/i}(t) + p^{\alpha_{n/i}(t)}(e_1 - m_0) \geqq c_{p^r 0}(i)$ and $c_{p^r 0}(i)$ is an actual change point so that

$$\sigma_{i/i-1}(d_{n/i}(t)) = \sigma^0_{i/i-1}(d_{n/i}(t)) = \left[p^{-1}\left((m_i + 1)(p-1) + p^i \cdot \frac{e_K - d_{p^r}}{p^{r+1}-1}\right)\right]$$
$$\geqq \frac{(p^i - 1)e_K + (p-1)m_0}{p} + p^{i-1} \cdot \frac{e_K - d_{p^r}}{p^{r+1}-1}.$$

The difference $\sigma_{i/i-1}(d_{n/i}(t)) - \tau_{n/i+1}(t)$ is larger than or equal to

$$p^r \cdot p^{\alpha_{n/i}(t)}(e_1 - m_0) - \frac{p-1}{p}(e_1 - m_0)$$

because

$$p^{i-1}e_K + p^{i-1}\frac{e_K - d_{p^r}}{p^{r+1}-1} - p^{r+i}\cdot\frac{e_K - d_{p^r}}{p^{r+1}-1} - d_{p^r}p^{i-1} = 0.$$

Further

$$p^r \cdot p^{\alpha_{n/i}(t)}(e_1 - m_0) - (e_1 - m_0) \geqq (p^{\alpha_{n/i-1}} - 1)(e_1 - m_0) + 1$$

because $\alpha_{n/i-1}(t) = \alpha_{n/i}(t) + r$ and $(e_1 - m_0) \geqq p$. And according to (8. 1. 6) we have that

$$\mathrm{Norm}_{k/b}(F^{t''}(K_k)) < F^{\tau_{n/b}(t)+1}(K_b)$$

if $t'' \geqq \tau_{n/k}(t) + (p^{\alpha_{n/k}(t)} - 1)(e_1 - m_0) + 1$, because $\iota_{k/k}(t'') = -1$. This proves (ii) for $k = i + 1$.

Finally if $i > k + 1$ then case 2) cannot occur. For suppose that

$$(8.\,1.\,10) \qquad \iota_{n/i}(t) = r,\ \tau_{n/i}(t) = p^i\frac{e_K - d_{p^r}}{p^{r+1}-1} - p^{\alpha_{n/i}(t)}(e_1 - m_0).$$

All (j_1, j_2)-level i-change points are $\equiv 0 \bmod p^{i-1}$ if $j_1, j_2 \geqq 1$. There are therefore because $c_{p^r0}(i) < p^i\frac{e_K - d_{p^r}}{p^{r+1}-1}$ no change points of type $c_{j_1,j_2}(i), j_1, j_2 \geqq 1$ between $\tau_{n/i}(t)$ and $c_{p^r0}(i)$. It follows that (cf. 7. 4)

$$(8.\,1.\,11) \qquad c_{0,j}(i) \leqq c_{0,p^r}(i) \quad \text{if} \quad p^r > j > 0.$$

Using Lemma 7. 5 we find

$$(8.\,1.\,12) \qquad 1 + c_{0,p^r}(i-1) \geqq c_{0j}(i-1), \qquad 0 < j < p^r.$$

Now suppose we can show that

$$(8.\,1.\,13) \qquad \sigma_{n/i-1}(t) \geqq c_{0,p^r}(i-1) + 1.$$

We know that $\iota_{n/i}(t) = r$, therefore $\iota_{n/i-1}(t) \leqq r$, and (8. 1. 12), (8. 1. 13) then imply $\iota_{n/i-1} = -1$, (cf. Lemma's 6. 1 and 7. 3). This is a contradiction because $i > k + 1$ and k is the largest index such that $\iota_{n/k}(t) = -1$.

It remains to prove (8. 1. 13), this calculation is done below in 8. 5. This proves (ii).

(iii) follows from (ii) because $\tau_{n/b}(t)$ is of the form $\chi_{b/0}(s')$, $\iota_{n/i}(t) = -1$ for $i \leqq b$ and $d_{n/i}(t) = \tau_{n/i}(t) + 1$ for $i \leqq b$.

To prove (iv) we distinguish two cases A) $\alpha_{n/k}(t) > 0$, B) $\alpha_{n/k}(t) = 0$ where $k = k_n(t)$ as before. First suppose that $\alpha_{n/k}(t) > 0$. We shall show that if $\iota_{n/s}(t) = r$ then

$$(8.\,1.\,14) \qquad a_i N^i_{s/s-1}(\pi_s^{\sigma_{n/s}(t)} A_s) < \pi_{s-1}^{d_{n/s-1}(t)} A_{s-1}, \qquad 1 \leqq s \leqq n,\ i > p^r,$$

$$(8.\,1.\,15) \qquad \mathrm{Tr}_{s/s-1}(\pi_s^{2\sigma_{n/s}(t)} A_s) < \pi_{s-1}^{d_{n/s-1}(t)} A_{s-1}, \qquad 1 \leqq s \leqq n.$$

Let $j = 0$ if $r = -1$, $j = p^r$ if $r \geqq 0$. First suppose that $s \geqq k$, then either $\alpha_{n/s}(t) > 0$ or $\iota_{n/s}(t) > 0$ (or both), otherwise we would have $\alpha_{n/k}(t) = 0$ by (5. 1. 8) and Lemma 6. 1. Now $\tau_{n/s}(t) \equiv (e_1 - m_0)p^{\alpha_{n/s}(t)} \bmod p^{\alpha_{n/s}(t)+b}$, $2 \leqq e_1 - m_0 < p^{b-1}$, $\alpha_{n/s}(t) + b - 1 < s$ (cf. (6. 3. 5)). We can therefore apply Lemma 7. 6 (with $g = b - 1$, $f = \alpha_{n/s}(t)$) to conclude

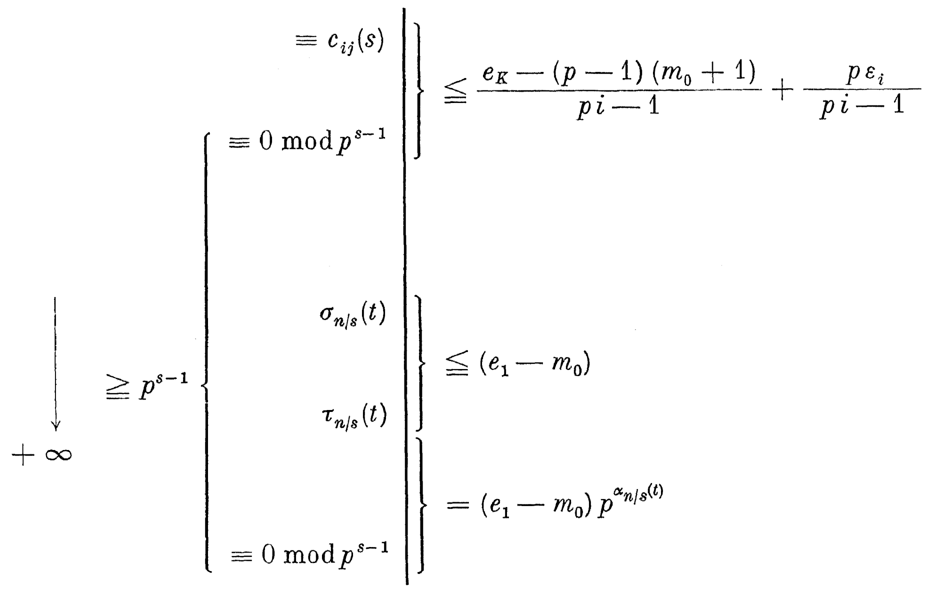
$$(8.\,1.\,16) \qquad \tau_{n/s}(t) = \sigma_{n/s}(t) > c_{ij}(s), \qquad s \geqq k,\ i > j.$$

If $b < s < k$ we can apply Lemma 7.7 (with $\sigma_{n/s+1}(t)$ for t) to conclude

$$(8.1.17) \qquad \tau_{n/s}(t) \geqq \sigma_{n/s}(t) > c_{0i}(s), \qquad i > 0,\, b < s < k.$$

It follows that for $b < s \leqq n$

$$(8.1.18) \qquad \begin{matrix} \tau_{n/s}(t) - c_{ji}(s) \geqq (e_1 - m_0)(p^h + 2)\, p^{\alpha_{n/s}(t)}, \\ \sigma_{n/s}(t) - c_{ji}(s) \geqq (e_1 - m_0)(p^h + 1)\, p^{\alpha_{n/s}(t)}, \end{matrix} \qquad j < i \leqq p^{h-1};\, b < s \leqq n.$$



$$\left(\varepsilon_i < 1 \text{ if } i > 1;\ \varepsilon_i \leqq \frac{p-1}{p} \text{ if } i = 1;\ \text{cf. } (7.1)\right)$$

This is most easily seen by looking at the picture drawn above. Use $s \geqq b + \alpha_{n/s}(t)$, $\alpha_{n/s}(t) \geqq 0$ and condition (8.1.3) of the proposition. That the relative position of $\tau_{n/s}(t)$, $\sigma_{n/s}(t)$ and $c_{ij}(s)$ with respect to points $\equiv 0 \bmod p^{s-1}$ is as indicated, follows from the following facts:

$$c_{ij}(s) < \sigma_{n/s}(t) \leqq \tau_{n/s}(t) \ (8.1.16),\ (8.1.17) \quad c_{ij}(s) \equiv 0 \bmod p^{s-1} \text{ if } i > j > 0$$

$$c_{i0}(s) \equiv -\frac{e_K - (p-1)(m_0+1)}{pi - 1} - \frac{\varepsilon p}{pi - 1} \bmod p^{s-1}$$

where $0 \leqq \varepsilon < 1$, if $0 < i \leqq p^{h-1}$ (cf. 7.1) and $e_K - (p-1)(m_0+1) > 0$.

Finally for $b < s \leqq n$ we have

$$(e_1 - m_0)\, p^{\alpha_{n/s}} \geqq (e_1 - m_0) \geqq \frac{e_K - (p-1)(m_0+1)}{pi - 1} + \frac{p\varepsilon_i}{pi - 1}$$

which in view of (8.1.16) shows that the picture represents things correctly and thus establishes (8.1.18). It is now a matter of some straightforward calculations to prove (8.1.14) for $b < s \leqq n$. This is done in 8.6 below.

Now let $s \leqq b$; let r_0 be the smallest natural number such that $p^{r_0} \geqq (e_1 - m_0)$. Note that $r_0 < b$ because of (8.1.4). Because $\alpha_{n/k_n(t)} > 0$ we know that $b < k = k_n(t)$. It follows that

$$(8.1.19) \qquad \sigma_{n/s}(t) = \sigma^0_{s+1/s}(\sigma_{n/s+1}(t)) \geqq \sigma^0_{s+1/s}(1) \geqq e_K p^s - (e_1 - m_0).$$

Using (8. 1. 19) one checks directly that (8. 1. 14) holds for $b \geqq s > b - r_0$. Cf. 8. 8.

Now suppose that $1 \leqq s \leqq b - r_0$. Because $\tau_{n/b}(t)$ is of the form $\chi_{b/0}(b')$ for some b' (cf. the proof of (i) above), we have that

(8. 1. 20) $$\lambda_{b/s}(\tau_{n/s}(t) - \alpha) = \lambda_{b/s}(\tau_{n/s}(t)) \quad \text{if} \quad \alpha < p^{s-b}.$$

It follows from this that (because $\sigma_{n/s}(t) \geqq \tau_{n/s}(t) - (e_1 - m_0)$ and $p^{r_0} > (e_1 - m_0)$)

(8. 1. 21) $$\tau_{n/s}(t) = \sigma_{n/s}(t) \quad \text{if} \quad s \leqq b - r_0.$$

To prove (8. 1. 14) for $s \leqq b - r_0$ it therefore suffices to show that

(8. 1. 22) $$\sigma^i_{s/s-1}(\sigma_{n/s}(t)) > \sigma^0_{s/s-1}(\sigma_{n/s}(t)), \qquad s \leqq b - r_0, \ i \geqq 1,$$

and this follows from 7. 8 because $s \geqq 1$, $e_K \geqq (e_1 - m_0)$, $(e_1 - m_0) \geqq p$.

We have now proved (8. 1. 14) for all $1 \leqq s \leqq n$, in the case $\alpha_{n/k_n(t)} > 0$. As to (8. 1. 15) we remark that

(8. 1. 23) $$\sigma^0_{s/s-1}(2\sigma_{n/s}(t)) - \sigma^0_{s/s-1}(\sigma_{n/s}(t)) \geqq p^{-1}\sigma_{n/s}(t) - 1.$$

Using this it is not difficult to prove (8. 1. 15) for $s \geqq b$. Cf. 8. 9 below. To prove (8. 1. 15) for $b > s > b - r_0$, where r_0 is again the smallest natural number such that $p^{r_0} \geqq (e_1 - m_0)$, we use the fact that

(8. 1. 24) $$\sigma_{n/s}(t) \geqq e_K p^s - (e_1 - m_0)$$

which follows from $\iota_{b/b}(\sigma_{n/b}(t)) = -1$. Cf. (8. 1. 19). To check that (8. 1. 15) holds for $n > s > b - r_0$ is now straightforward. Cf. 8. 10 below.

Finally for $s \leqq b - r_0$ we have that $\tau_{n/s}(t) = \sigma_{n/s}(t)$ (cf. (8. 1. 21) above); moreover $\tau_{n/s}(t)$ is of the form $\chi_{n/s}(t')$ for these s. So that

(8. 1. 25) $$\lambda_{s/s-1}(\tau_{n/s}(t) + 1) = \lambda_{s/s-1}(\tau_{n/s}(t)) + 1 = \tau_{n/s-1}(t) + 1$$

and this proves (8. 1. 15) for $s \leqq b - r_0$. We have now proved (8. 1. 15) for all $1 \leqq s \leqq n$. Note that the hypothesis $\alpha_{n/k_n(t)} > 0$ has not been used in the proof of (8. 1. 15).

We are now in a position to prove statement (iv) of the proposition in case A; i. e. in case $\alpha_{n/k_n(t)} > 0$.

Let π be a uniformizing element of K; and let π_s, $s = 1, 2, \ldots, n$ be uniformizing elements of K_s, $s = 1, \ldots, n$, chosen such that $N_{s/s-1}(\pi_s) = \pi_{s-1}$, $N_{1/0}(\pi_1) = \pi$. Let $x \in A_K$; it follows from (8. 1. 14) and (8. 1. 15) that for $s \geqq k_n(t) = k$

(8. 1. 26) $$\mathrm{Norm}_{n/s}(x\pi_n^t) \equiv b_{u_s}x^{u_s} + b_{u_s-1}x^{u_s-1} + \cdots + b_1 x + b_0 \bmod \pi_s^{d_{n/s}(t)}$$

where

(8. 1. 27) $$\begin{aligned} & b_{u_s} = z_s \pi_s^{t(s)},\ t(s) = p^{\alpha_{n/s}(t)}t,\ z_s \in A_K,\ v_s(b_{u_s}) = \tau_{n/s}(t) = \sigma_{n/s}(t), \\ & v_s(b_i) \geqq \sigma_{n/s}(t),\ i = 0, 1, \ldots, u_s - 1,\ u_s = p^{r_n+1} + \cdots + p^{r_{s+1}+1} \\ & \qquad\qquad\qquad\qquad \text{if} \quad \iota_{n/j} = r_j,\ j = n, \ldots, s+1. \end{aligned}$$

This can e. g. be seen as follows. One uses induction. Suppose (8. 1. 26), (8. 1. 27) have been proved for $s + 1$. Because $\tau_{n/s+1}(t) = \sigma_{n/s+1}(t) \geqq p^{\alpha_{n/s+1}(t)}(e_1 - m_0)$ we also have

$$\mathrm{Norm}_{n/s+1}(x\pi_n^t) \equiv b_{u_{s+1}}x^{u_{s+1}} \underset{F}{+} \cdots \underset{F}{+} b_1 x \underset{F}{+} b_0 \bmod \pi_{s+1}^{d_{n/s+1}(t)}.$$

It follows that

$$\mathrm{Norm}_{n/s}(x\pi_n^t) \equiv \mathrm{Norm}_{s+1/s}(b_{u_{s+1}}x^{u_{s+1}}) \underset{F}{+} \cdots \underset{F}{+} \mathrm{Norm}_{s+1/s}(b_0) \bmod \pi_s^{d_{n/s}(t)}$$

and hence

$$\mathrm{Norm}_{n/s}(x\pi_n^t) \equiv \mathrm{Norm}_{s+1/s}(b_{u_{s+1}}x^{u_{s+1}}) + \cdots + \mathrm{Norm}_{s+1/s}(b_0) \bmod \pi_s^{d_{n/s}(t)}.$$

Now let $\iota_{n/s+1}(t) = r_{s+1}$, then we have using (8. 1. 14), (8. 1. 15)

$$\mathrm{Norm}_{s+1/s}(b_{u_{\nu+1}}x^{u_{s+1}}) \equiv b_{u_s}x^{u_s} + \cdots + b_1'x + b_0' \bmod \pi_s^{d_{n/s}(t)}$$

with $v_s(b_j') \geqq \sigma_{n/s}(t)$, and for $i = 0, 1, \ldots, u_{s+1} - 1$

$$\mathrm{Norm}_{s+1/s}(b_ix^i) \equiv b_{i'}''x^{i'} + \cdots + b_0''x + b_0'' \bmod \pi_s^{d_{n/s}(t)}$$

where $i' \leqq p^{r_{s+1}+1} + i < u_s$ and $v_s(b_j'') \geqq \sigma_{n/s}(t)$. This proves the induction step.

For $s < k = k_n(t)$ one sees from (8. 1. 14) and (8. 1. 15) that

(8. 1. 28) $$\mathrm{Norm}_{n/s}(x\pi_n^t) \equiv \mathrm{Tr}_{k/s}(\mathrm{Norm}_{n/k}(x\pi_n^t)) \bmod \pi_s^{d_{n/s}(t)}.$$

We now use the trace lemma of Section 4 above to keep track of what happens to the "leading coefficient" of (8. 1. 26). We have

(8. 1. 29) $$\mathrm{Tr}_{s/s-1}(z_s\pi_s^{t(s)}) \equiv z_s p \pi_{s-1}^{t(s-1)} \bmod \pi_{s-1}^{d_{n/s-1}(t)}, \qquad b < s \leqq k = k_n(t)$$

if $v_s(z_s\pi_s^{t(s)}) = \tau_{n/s}$, $t(s) = p^{\alpha_{n/s}(t)}t$, $t(s-1) = p^{-1}t(s) = p^{\alpha_{n/s-1}(t)}$, $z_s \in A_K$.
This follows directly from proposition 4. 1 and condition (8. 1. 3). Cf. 8. 11 below.

From (8. 1. 26), (8. 1. 27), (8. 1. 28), (8. 1. 29) and (5. 1. 5) we obtain for $b \leqq s < k$

(8. 1. 30) $$\mathrm{Norm}_{n/s}(x\pi_n^t) \equiv b_ux^u + b_{u-1}x^{u-1} + \cdots + b_1x + b_0 \bmod \pi_s^{d_{n/s}(t)}$$

where $u = u_k = p^{r_n+1} + \cdots + p^{r_{k+1}+1}$, $b_u = z_s\pi_s^{t(s)}$, $t(s) = p^{\alpha_{n/s}(t)}t$, $v_s(b_u) = \tau_{n/s}(t)$, $v_s(b_i) \geqq \sigma_{n/s}(t)$, $i = 0, 1, \ldots, u-1$, $b \leqq s < k$. In particular we have that

(8. 1. 31) $$\mathrm{Norm}_{n/b}(x\pi_n^t) \equiv c_ux^u + c_{u-1}x^{u-1} + \cdots + c_1x + c_0 \bmod \pi_b^{d_{n/b}(t)}$$

$u = u_k$, $c_u = z_b\pi_b^{t(b)}$, $t(b) = t$, $v_b(c_u) = \tau_{n/b}(t)$, $v_b(c_i) \geqq \sigma_{n/b}(t)$, $i = 0, 1, \ldots, u-1$.

Now $\tau_{n/b}(t)$ is of the form $\chi_{b/0}(t')$ for some t'; further

$$v_b(c_i) \geqq \sigma_{n/b}(t) \geqq \tau_{n/b}(t) - (e_1 - m_0) > \tau_{n/b}(t) - p^b,\ i = 0, 1, \ldots, u-1.$$

It follows that

(8. 1. 32) $$\begin{aligned} &v_0(\mathrm{Tr}_{b/0}(c_u)) = \lambda_{b/0}(\tau_{n/b}(t)) = \tau_{n/0}(t) = \sigma_{n/0}(t), \\ &v_0(\mathrm{Tr}_{b/0}(c_i)) \geqq \lambda_{b/0}(\tau_{n/b}(t)) = \tau_{n/0}(t) = \sigma_{n/0}(t),\ i = 0, 1, \ldots, u-1. \end{aligned}$$

Putting (8. 1. 31), (8. 1. 28) and (8. 1. 32) together yields

(8. 1. 33) $$\begin{aligned} &\mathrm{Norm}_{n/0}(x\pi_n^t) \equiv c_u'x^u + \cdots + c_1'x + c_0' \bmod \pi^{d_{n/0}(t)}, \\ &v_0(c_u') = \sigma_{n/0}(t),\ v_0(c_i') \geqq \sigma_{n/0}(t),\ i = 0, 1, \ldots, u-1 \end{aligned}$$

and this proves statement (iv) of the proposition in the case that $\alpha_{n/k_n(t)} > 0$.

Now suppose that we are in case B; i. e. $\alpha_{n/k}(t) = 0$ where $k = k_n(t)$. Then $\iota_{n/s}(t) = 0$ for $s = n, \ldots, k+1$; $\iota_{n/s}(t) = -1$ for $s = k, k-1, \ldots, 2, 1$; $b = k$. It is not difficult to check that the proof given for (8. 1. 14) in case A for $n \geqq s \geqq k$ also works in case B provided $s > k$. (The hypothesis $\alpha_{n/k}(t) > 0$ is used twice: to establish (8. 1. 16) in case $s = k$ and to establish (8. 1. 19) in case $s = b$.) The arguments used to prove (8. 1. 14) for $s \leqq b$ in case A also remain valid, except in the case $s = b$ (cf. 8. 1. 19). In the case $s = b = k$ we have instead of (8. 1. 16) (cf. Lemma 7. 6 exceptional case)

(8. 1. 34) $\quad \tau_{n/b}(t) = \sigma_{n/b}(t) > c_{i0}(b),\ i > 1;\ \tau_{n/b}(t) = \sigma_{n/b}(t) \geqq c_{10}(b).$

There are therefore two possibilities in case B.

B_1. $\tau_{n/b}(t) > c_{10}(b)$ in which case (8. 1. 14) also holds for $s = k = b$ (cf. (8. 12)).

B_2. $\tau_{n/b}(t) = c_{10}(b)$ in which case (8. 1. 14) fails to hold for $s = k = b$.

Formula (8. 1. 15) has been proved above without any hypothesis on $\alpha_{n/k}(t)$. We are now in a position to prove (iv) also in case B. Exactly as above one shows that

$$\text{(8. 1. 35)} \qquad \begin{aligned} &\mathrm{Norm}_{n/k}(x\pi_n^t) \equiv b_u x^u + \cdots + b_1 x + b_0 \bmod \pi_k^{d_{n/k}(t)} \\ &v_k(b_u) = \tau_{n/k}(t) = \sigma_{n/k}(t);\ v(b_i) \geqq \sigma_{n/k}(t), \\ &\qquad\qquad i = 0, 1, \ldots, u-1, \quad u = (n-k)p. \end{aligned}$$

Applying $\mathrm{Norm}_{k/k-1}$ to this we find

$$\text{(8. 1. 36)} \qquad \mathrm{Norm}_{n/k-1}(x\pi_n^t) \equiv c_{u'} x^{u'} + \cdots + c_1 x + c_0 \bmod \pi_{k-1}^{d_{n/k-1}(t)}$$

where $u' = up$ in case B_2, and then $v_{k-1}(c_{u'}) = \tau_{n/k-1}(t) = \sigma_{n/k-1}(t)$ because $\alpha_{n/k}(t) = 0$ and $\tau_{n/k}(t) = c_{10}(k)$; or $u' = u$ in case B_1 and then $v_{k-1}(c_{u'}) = \tau_{n/k-1}(t) = \sigma_{n/k-1}$ because $\tau_{n/k}(t)$ is of the form $\chi_{k/0}(t')$ for some t'. Also in both cases $v_{k-1}(c_i) \geqq \sigma_{n/k-1}(t)$. Using (8. 1. 36) instead of (8. 1. 31) one obtains in the same way as in case A that

$$\text{(8. 1. 37)} \qquad \begin{aligned} &\mathrm{Norm}_{n/0}(x\pi_n^t) \equiv c'_{u'} x^{u'} + \cdots + c'_1 x + c'_0 \bmod \pi^{d_{n/0}(t)}, \\ &v_0(c'_{u'}) = \tau_{n/0}(t) = \sigma_{n/0}(t);\quad v_0(c'_i) \geqq \tau_{n/0}(t) = \sigma_{n/0}(t),\quad i = 0, 1, \ldots, u'-1 \end{aligned}$$

which proves statement (iv) of the proposition in case B.

8. 2. Lemma. *Let* $(F, K_\infty/K)$ *be a nice pair, let* $\alpha_{n/i}(t) = i - b,\ i > b$ *and* $\iota_{n/i}(t) = -1$, *Then*

$$\chi_{i/b}(\tau_{n/b}(t)) = (p^{i-b} - 1)(e_1 - m_0) + \tau_{n/i}(t).$$

(Note that $\chi_{i/b}(\tau_{n/b}(t))$ is defined, because $\lambda_{i/b}(\tau_{n/i}(t)) \geqq \tau_{n/b}(t)$, from which we also see that $\chi_{i/b}(\tau_{n/b}(t)) \geqq \tau_{n/i}(t)$).

Proof. According to Lemma 8. 3 below we have

$$\chi_{i/b}(\tau_{n/b}(t)) = (\tau_{n/b}(t) - p^b e_K(i-b) - m_0)\,p^{i-b} + m_0 + e_K \frac{p^{i-b}-1}{p-1}$$

and because $\iota_{n/i}(t) = -1$ and $\alpha_{n/i}(t) = i - b$ we have that

$$\tau_{n/i}(t) = p^{i-b}\tau_{n/b}(t) - (i-b)p^i e_K$$

(by the definition of the functions $\tau_{n/k}$). q. e. d.

8. 3. Lemma. *Let $(F, K_\infty/K)$ be a nice pair. If the function $\chi_{i/b}$, $i > b$ is defined for a certain s, then*

$$\chi_{i/b}(s) = (s - p^b e_K(i-b) - m_0)p^{i-b} + m_0 + e_K \frac{p^{i-b}-1}{p-1}$$

(and $\chi_{i/b}(s)$ is defined for all s for which this expression is positive).

Proof. For calculations like this the following fact is useful. If $a_1, a_2, \ldots, a_r$ is a series of integers, $t \in \mathbb{R}$ and we define

$$t_1 = \left[\frac{a_1+t}{p}\right], \ldots, \quad t_r = \left[\frac{a_r + t_{r-1}}{p}\right], \quad t_1' = \frac{a_1+t}{p}, \ldots, \quad t_r' = \frac{a_r+t}{p},$$

then $t_r = [t_r']$. Using this we find

$$\lambda_{i/b}(t) = \left[p^b e_K(i-b) + m_0 - \frac{m_0}{p^{i-b}} + 1 - \frac{1}{p^{i-b}} - e_K p^{-(i-b)} \frac{p^{i-b}-1}{p+1} + \frac{t}{p^{i-b}}\right].$$

The lemma follows from this.

8. 4. Proof of (8. 1. 6). Because $v_i(x) \geqq (p^{(i-b)}-1)(e_1 - m_0) + 1$ and Lemma 8. 2 we have that $v_i(x) \geqq \chi_{i/b}(\tau_{n/b}(t)) + 1$ which implies $\lambda_{i/b}(v_i(x)) \geqq \tau_{n/b}(t) + 1 = d_{n/b}(t)$ which implies (8. 1. 6) because $\iota_{n/i}(t) = -1$, $\iota_{n/i-1}(t) = -1, \ldots, \iota_{n/b+1}(t) = -1$.

8. 5. Proof of (8. 1. 13). We have

$$\iota_{n/i}(t) = r > 0, \quad \tau_{n/i}(t) = p^i \frac{e_K - d_{pr}}{p^{r+1}-1} - p^{\alpha_{n/i}(t)}(e_1 - m_0)$$

and we must show that $\sigma_{n/i-1}(t) \geqq c_{pr0}(i-1) + 1$. First, suppose that $0 \leqq r < h-1$, then $d_{pr} \geqq 1$ and we find

$$\begin{aligned}\sigma_{n/i-1}(t) - c_{pr0}(i-1) - 1 &= \tau_{n/i-1}(t) - c_{pr0}(i-1) - 1 \\ &\geqq p^{i+r}\frac{e_K - d_{pr}}{p^{r+1}-1} - p^r p^{\alpha_{n/i}(t)}(e_1 - m_0) \\ &\quad + d_{pr}p^{i-1} - p^{i-1}\frac{e_K - d_{pr}}{p^{r+1}-1} + \frac{e_K - (p-1)(m_0+1)}{p^{r+1}-1} - 1 \geqq 0\end{aligned}$$

because $p^{i-1} \geqq p^{r+\alpha_{n/i}(t)}(e_1 - m_0)$ as $r + \alpha_{n/i}(t) = \alpha_{n/i-1}(t) \leqq i-1-b$ and $p^b \geqq e_1 - m_0$ and

$$p^{r+i}\frac{e_K - d_{pr}}{p^{r+1}-1} - p^{i-1}\frac{e_K - d_{pr}}{p^{r+1}-1} - 1 \geqq 0$$

(because $(p^{r+1}-1)^{-1}(e_K - d_{pr})$ is an integer > 0).

Now let $r = h-1$, then $d_{pr} = 0$ and we find

$$\begin{aligned}\sigma_{n/i-1}(t) - c_{pr0}(i-1) - 1 &\geqq p^{i+h-1}\frac{e_K}{p^h-1} - p^{h-1}p^{\alpha_{n/i}(t)} \\ &\quad - p^{i-1}\frac{e_K}{p^h-1} + \frac{e_K - (p-1)(m_0+1)}{p^h-1} - 1 \\ &\geqq p(p^h-1)p^{i-1}\frac{e_K}{p^h-1} - p^{\alpha_{n/i-1}(t)}(e_1 - m_0 + 1) \geqq 0\end{aligned}$$

because $p^b \geqq e_1 - m_0$ and $\alpha_{n/i-1}(t) \leqq i-1-b$.

8. 6. Proof of (8. 1. 14) in the case $\alpha_{n/k_n(t)} > 0$, $n \geqq s > l_n(t) = b$. Let $n \geqq s > l_n(t) = b$. First suppose that $r = \iota_{n/s}(t) \geqq 0$, let $j = p^r$, and $i > j$ then

$$\sigma^i_{s/s-1}(\sigma_{n/s}(t)) - \sigma^i_{s/s-1}(c_{ji}(s)) = i(\sigma_{n/s}(t) - c_{ji}(s)),$$
$$\sigma^j_{s/s-1}(\sigma_{n/s}(t)) - \sigma^j_{s/s-1}(c_{ji}(s)) = p^r(\sigma_{n/s}(t) - c_{ji}(s)),$$
$$\text{and} \quad \sigma^j_{s/s-1}(\sigma_{n/s}(t)) = \sigma_{n/s-1}(t),\ \sigma^j_{s/s-1}(c_{ji}(s)) = \sigma^i_{s/s-1}(c_{ji}(s)).$$

It follows that

$$\sigma^i_{s/s-1}(\sigma_{n/s}(t)) - \sigma_{n/s-1}(t) = (i - p^r)\,(\sigma_{n/s}(t) - c_{ji}(s)).$$

Using (8. 1. 18) and (6. 2. 5) we see that for $i > j$

$$\begin{aligned}\sigma^i_{s/s-1}(\sigma_{n/s}(t)) - \tau_{n/s-1}(t)) &\geqq (i - p^r)(e_1 - m_0)(p^h + 1)p^{\alpha_{n/s}(t)} - (e_1 - m_0)\\ &\geqq (p^h + 1)p^{\alpha_{n/s}(t)}(e_1 - m_0) - (e_1 - m_0)\\ &\geqq p^{h+\alpha_{n/s}(t)}(e_1 - m_0) \geqq p^{\alpha_{n/s-1}(t)}(e_1 - m_0)\end{aligned}$$

which proves (8. 1. 14) for the case $r \geqq 0$, $p^r < i \leqq p^{h-1}$, $b < s \leqq n$.

Now let $r = \iota_{n/s}(t) = -1$, then for $i \geqq 1$

$$\sigma^i_{s/s-1}(\sigma_{n/s}(t)) - \sigma^i_{s/s-1}(c_{i0}(s)) = i(\sigma_{n/s}(t) - c_{i0}(s))$$

and

$$\begin{aligned}\sigma^0_{s/s-1}(\sigma_{n/s}(t)) - \sigma^0_{s/s-1}(c_{i0}(s)) &= \left[\frac{(m_s+1)(p-1)+\sigma_{n/s}(t)}{p}\right] - \left[\frac{(m_s+1)(p-1)+c_{i0}(s)}{p}\right]\\ &\leqq \frac{(m_s+1)(p-1)+\sigma_{n/s}(t)}{p} - \frac{(m_s+1)(p-1)+c_{i0}(s)}{p} + 1\\ &= p^{-1}(\sigma_{n/s}(t) - c_{i0}(s)) + 1.\end{aligned}$$

It follows that $\sigma^i_{s/s-1}(\sigma_{n/s}(t)) - \sigma_{n/s-1}(t) \geqq (i - p^{-1})\,(\sigma_{n/s}(t) - c_{i0}(s)) - 1$ and hence

$$\begin{aligned}\sigma^i_{s/s-1}(\sigma_{n/s}(t)) - \tau_{n/s-1}(t) &\geqq (1 - p^{-1})\,(p^h + 1)\,(e_1 - m_0)p^{\alpha_{n/s}(t)} - 1 - (e_1 - m_0)\\ &= (p - 1)\,(p^h + 1)\,(e_1 - m_0)p^{\alpha_{n/s-1}(t)} - 1 - (e_1 - m_0)\\ &= ((p - 1)\,(p^h + 1)p^{\alpha_{n/s-1}(t)} - 1)(e_1 - m_0) - 1\\ &\geqq p^h p^{\alpha_{n/s-1}(t)}(e_1 - m_0) - 1 \geqq p^{\alpha_{n/s-1}(t)}(e_1 - m_0)\end{aligned}$$

which proves (8. 1. 14) for the case $r = -1$, $1 \leqq i \leqq p^{h-1}$, $b < s \leqq n$.

It remains to prove (8. 1. 14) for $b < s \leqq n$, $i > p^{h-1}$. We have (cf. 8. 7 below)

(8. 6. 1) $$\sigma_{n/s}(t) \geqq p^{\alpha_{n/s}(t)}t.$$

For $i > p^{h-1}$ we have

$$\begin{aligned}\sigma^i_{s/s-1}(\sigma_{n/s}(t)) - \tau_{n/s-1}(t) &\geqq \sigma^i_{s/s-1}(\sigma_{n/s}(t)) - \sigma_{n/s-1}(t) - (e_1 - m_0)\\ &\geqq \sigma^i_{s/s-1}(\sigma_{n/s}(t)) - \sigma^{p^{h-1}}_{s/s-1}(\sigma_{n/s}(t)) - (e_1 - m_0)\\ &\geqq (i - p^{h-1})\sigma_{n/s}(t) - (e_1 - m_0) \geqq \sigma_{n/s}(t) - (e_1 - m_0)\\ &\geqq p^{\alpha_{n/s}(t)}t - (e_1 - m_0) \geqq p^{\alpha_{n/s-1}(t)}(e_1 - m_0)\end{aligned}$$

because $t \geqq (p^h + 1)(e_1 - m_0)$, according to condition (8. 1. 2). This concludes the proof of (8. 1. 14) in case $\alpha_{n/k_n(t)} > 0$, $n \geqq s > l_n(t) = b$.

8. 7. Lemma. *If* $b \leqq s \leqq n$, $b = l_n(t) > 0$, *then* $\sigma_{n/s}(t) \geqq p^{\alpha_{n/s}(t)} t$.

Proof. This is obvious if $s \geqq k_n(t)$. Using $\sigma^0_{s/s-1}(t') \geqq \sigma^0_{s/s-1}(t)$ if $t' \geqq t$, it therefore remains to show that $\sigma^0_{s/s-1}(pt') \geqq t'$ which follows from the definition of $\sigma^0_{s/s-1}$.

8. 8. Proof of (8. 1. 14) in the case $\alpha_{n/k_n(t)} > 0$, $b \geqq s > b - r_0$. If i is not a multiple of p^{h-1} we have

$$\begin{aligned}\sigma^i_{s/s-1}(\sigma_{n/s}(t)) - \tau_{n/s-1}(t) &\geqq \sigma^i_{s/s-1}(\sigma_{n/s}(t)) - \sigma_{n/s-1}(t) - (e_1 - m_0)\\ &\geqq \sigma^i_{s/s-1}(\sigma_{n/s}(t)) - \sigma^0_{s/s-1}(\sigma_{n/s}(t)) - (e_1 - m_0).\end{aligned}$$

And because $\sigma_{n/s}(t) \geqq e_K p^s - (e_1 - m_0)$ this is larger than or equal to (cf. (8. 8. 1) below)

$$\begin{aligned}&\sigma^i_{s/s-1}(e_K p^s - (e_1 - m_0)) - \sigma^0_{s/s-1}(e_K p^s - (e_1 - m_0)) - (e_1 - m_0)\\ &\quad\geqq e_K p^s - (e_1 - m_0) + p^{s-1} - p^{-1}(e_K p^s - (e_1 - m_0) + (p-1)(m_s + 1)) - (e_1 - m_0)\\ &\quad\geqq e_K p^s - (e_1 - m_0) + p^{s-1}\\ &\qquad - p^{-1}(e_K p^s - (e_1 - m_0) + p^s e_K - e_K + (p-1)m_0 + (p-1)) - (e_1 - m_0)\\ &\quad\geqq p^{s-1} - (e_1 - m_0) + p^{-1}(e_1 - m_0) + p^{-1}e_K - p^{-1}(p-1)m_0 - \frac{p-1}{p} - (e_1 - m_0)\\ &\quad= p^{s-1} - (e_1 - m_0) - \frac{p-1}{p} \geqq p^{b-r_0} - (e_1 - m_0 + 1) \geqq 1 = d_{n/s-1}(t) - \tau_{n/s-1}(t)\end{aligned}$$

because of conditions (8. 1. 4) and $(e_1 - m_0) \geqq p$.

If i is a multiple of p^{h-1} we have as above

$$\begin{aligned}\sigma^i_{s/s-1}(\sigma_{n/s}(t)) - \tau_{n/s-1}(t) &\geqq \sigma^i_{s/s-1}(e_K p^s - (e_1 - m_0))\\ &\quad - \sigma^0_{s/s-1}(e_K p^s - (e_1 - m_0)) - (e_1 - m_0)\end{aligned}$$

which is larger than or equal to

$$\begin{aligned}&p^{h-1}(e_K p^s - (e_1 - m_0)) - 2e_K p^{s-1} - (e_1 - m_0)\\ &\quad\geqq p^{h-1}(e_K p^s - (e_1 - m_0)) - e_K p^s - (e_1 - m_0)\\ &\quad= (p^{h-1} - 1)(e_K p^s - (e_1 - m_0)) - 2(e_1 - m_0)\\ &\quad\geqq e_K p^s - 3(e_1 - m_0) \geqq p^{b-r_0+1} - 3(e_1 - m_0) \geqq 1\end{aligned}$$

because of (8. 1. 4). Note that we have used

$$\text{(8. 8. 1)} \qquad i > j,\ t' > t,\ t', t \in \mathbb{N} \Rightarrow \sigma^i_{s/s-1}(t') - \sigma^j_{s/s-1}(t') \geqq \sigma^i_{s/s-1}(t) - \sigma^j_{s/s-1}(t).$$

This is obvious from the definitions of $\sigma^i_{s/s-1}$, $\sigma^j_{s/s-1}$ if $j \geqq 1$. If $j = 0$ to prove (8. 8. 1) one uses

$$\begin{aligned}\sigma^0_{s/s-1}(t') &\leqq \sigma^0_{s/s-1}(t) + 1 + \frac{t' - t - 1}{p},\\ \sigma^i_{s/s-1}(t') &= \sigma^i_{s/s-1}(t) + i(t' - t) \qquad (i > 0).\end{aligned}$$

8. 9. Proof of (8. 1. 15) for $n \geqq s \geqq b = l_n(t)$. If $s \geqq b$ then (cf. Lemma (8. 7))

$$\text{(8. 9. 1)} \qquad \sigma_{n/s}(t) \geqq p^{\alpha_{n/s}(t)} t.$$

Using (8. 1. 23) and (8. 1. 2) we see from this that

$$\sigma^0_{s/s-1}(2\sigma_{n/s}(t)) - \tau_{n/s-1}(t) \geqq \sigma^0_{s/s-1}(2\sigma_{n/s}(t)) - \sigma_{s/s-1}(\sigma_{n/s}(t)) - (e_1 - m_0)$$
$$\geqq \sigma^0_{s/s-1}(2\sigma_{n/s}(t)) - \sigma^0_{s/s-1}(\sigma_{n/s}(t)) - (e_1 - m_0)$$
$$\geqq p^{-1}p^{\alpha_{n/s}(t)}t - (e_1 - m_0) - 1$$
$$\geqq p^{-1}p^{\alpha_{n/s}(t)}p^{h+1}(e_1 - m_0) + p^{-1}p^{\alpha_{n/s}(t)}p(e_1 - m_0 + 2) - (e_1 - m_0) - 1$$
$$\geqq p \cdot p^{\alpha_{n/s-1}(t)}(e_1 - m_0) \geqq p^{\alpha_{n/s}(t)}(e_1 - m_0).$$

8. 10. Proof of (8. 1. 15) for $b > s > b - r_0$. Using (8. 1. 23), (8. 1. 24), (8. 1. 4) we see that

$$\sigma^0_{s/s-1}(2\sigma_{n/s}(t)) - \tau_{n/s-1}(t) \geqq \sigma^0_{s/s-1}(2\sigma_{n/s}(t)) - \sigma^0_{s/s-1}(\sigma_{n/s}(t)) - (e_1 - m_0)$$
$$\geqq p^{-1}\sigma_{n/s}(t) - 1 - (e_1 - m_0) \geqq p^{-1}(e_K p^s - (e_1 - m_0)) - 1 - (e_1 - m_0)$$
$$\geqq e_K p^{s-1} - \frac{e_1 - m_0}{p} - 1 - (e_1 - m_0) \geqq e_K p^{b-r_0} - 2(e_1 - m_0) - 1 \geqq 1$$
$$= d_{n/s-1}(t) - \tau_{n/s-1}(t).$$

8. 11. Proof of (8. 1. 29). Because $b < s \leqq k = k_n(t)$ we have $\alpha_{n/s}(t) > 0$ so that $t(s)$ is a multiple of p. We can therefore apply proposition (4. 1) to obtain (using that $z_s \in A_k$)

$$\mathrm{Tr}_{s/s-1}(z_s \pi_s^{t(s)}) \equiv p z_s \pi_{s-1}^{p^{-1}t(s)} \bmod z_s \pi_{s-1}^{2r_s + p^{-1}t(s)+1}$$

where $r_s = p^{-1}[(m_s + 1)(p-1) + 1]$. It therefore only remains to show that

$$v_{s-1}(z_s) + 2r_s + p^{-1}t(s) + 1 \geqq d_{n/s-1}(t).$$

Now $v_s(z_s) + t(s) = \tau_{n/s}(t)$, so that $v_{s-1}(z_s) + p^{-1}t(s) = \tau_{n/s-1}(t) - p^{s-1}e_K$ which means that we must show

$$2r_s + 1 - p^{s-1}e_K \geqq p^{\alpha_{n/s-1}(t)}(e_1 - m_0).$$

Because $\alpha_{n/s-1}(t) = s - 1 - b$ we have

$$2r_s + 1 - p^{s-1}e_K \geqq 2p^{s-1}e_K - \frac{2(p-1)}{p}(e_1 - m_0) + \frac{2}{p} + 1 - p^{s-1}e_K$$
$$\geqq p^{s-1}e_K - 2(e_1 - m_0) \geqq p^{\alpha_{n/s-1}(t)}(p^b - 2(e_1 - m_0))$$
$$\geqq p^{\alpha_{n/s-1}(t)}(e_1 - m_0)$$

because $p^b \geqq 3(e_1 - m_0)$ according to condition (8. 1. 3).

8. 12. Proof of (8. 1. 14) for $s = k_n(t) = b$ in case B_1. Because (8. 1. 16) holds in case B_1 we have (8. 1. 18)

$$\tau_{n/s}(t) \geqq c_{0i}(s) + (p^h + 1)(e_1 - m_0).$$

Further because we are in case B, $\sigma_{n/s}(t) = \tau_{n/s}(t)$ for all s. As in 8. 6 we now see that

$$\sigma^i_{s/s-1}(\sigma_{n/s}(t)) - \sigma_{n/s-1}(t) \geqq (i - p^{-1})(\sigma_{n/s}(t) - c_{0i}(s)) - 1$$
$$\geqq (1 - p^{-1})(p^h + 1)(e_1 - m_0) - 1 \geqq \frac{1}{2} \cdot (2 + 1) \cdot 2 - 1 > 1,$$

which proves (8. 1. 14) in this case because $d_{n/s-1}(t) = \tau_{n/s-1}(t) + 1 = \sigma_{n/s-1}(t) + 1$ (the case $i > p^{h-1}$ follows from the case $i = p^{h-1}$).

9. Calculation of $\sigma_{n/0}(t)$, $\tau_{n/0}(t)$ and $l_n(t)$

In this section we shall assume that K_∞/K is an extension such that

$$m_n = m(K_n/K_{n-1}) = (1 + p + \cdots + p^{n-1})e_K + m_0 \quad \text{for all } n.$$

9. 1. Lemma. *If* $\iota_{n/s}(t) < h - 1$, *then* $\iota_{n/s-c-1}(t) = -1$, *where* c *is the smallest integer such that* $p^c \geqq e_K$.

Proof. If $\iota_{n/s}(t) = -1$ then $\iota_{n/u}(t) = -1$ for all $u \leqq s$. (Cf. Lemma 6. 1.) Now suppose $-1 < \iota_{n/s}(t) < h - 1$ and let $j = p^{\iota_{n/s}(t)}$. Then

$$\tau_{n/s-1}(t) = \sigma_{n/s-1}(t) = \sigma^j_{s/s-1}(\sigma_{n/s}(t)) \geqq p^{s-1}$$

because $d_j > 0$ as $1 \leqq j < p^{h-1}$. Let u be the largest integer such that $\iota_{n/u}(t) = -1$. Suppose that $u < s - c - 1$, then

$$t' := \sigma_{n/u+1}(t) \geqq \sigma_{n/s-1}(t) \geqq p^{s-1}.$$

However (cf. (7. 1. 3)), for $i = 1, 2, \ldots, p^{h-1}$

$$p^{s-1} > c_{0i}(u+1) = \frac{(e_K - d_i)p^{u+1}}{pi - 1} - \frac{e_K - (p-1)(m_0+1)}{pi - 1} - \frac{p\varepsilon}{pi - 1},$$

because $s - 1 > u + c$. This shows that $\iota_{u+1/u+1}(t') = \iota_{n/u+1}(t) = -1$ contradicting our assumption that $u < s - c - 1$. Therefore $u \geqq s - c - 1$. q. e. d.

Let $j_n(t)$ be the number of indices $\leqq n$ such that $\iota_{n/s}(t) = h - 1$ (cf. (5. 1. 13)).

9. 2. Lemma. *For every integer* $b > 0$ *there is a constant* $t_0(b)$ *independent of* n *such that* $t \geqq t_0 \Rightarrow j_n(t) \leqq h^{-1}(n - b)$.

Proof. $j_n(t) > s \Leftrightarrow \iota_{n/n}(t) = h - 1 = \cdots = \iota_{n/n-s}(t) = h - 1 \Leftrightarrow \iota_{n/n-s}(t) = h - 1$. Hence if $j_n(t) > s$ then $\sigma_{n/n-s}(t) = p^{s(h-1)}t$, and $\iota_{n/n-s}(t) = h - 1$ then means

$$p^{s(h-1)}t < c_{ip^{h-1}}(n - s), \qquad i = 0, 1, 2, \ldots, p^{h-1} - 1,$$

and this implies in particular

$$p^{s(h-1)}t < c_{0p^{h-1}}(n - s) = \frac{e_K p^{n-s}}{p^h - 1} - \frac{e_K - (p-1)(m_0+1)}{p^h - 1} - \frac{p\varepsilon}{p^h - 1} \leqq e_K p^{n-s}$$

which implies $t < p^{n-sh}e_K$. Therefore if $t \geqq t_0 = e_K p^b$ then $j_n(t) \leqq h^{-1}(n - b)$.

9. 3. Corollary. $\qquad l_n(t) = n - hj_n(t) - a_{n,t}$

where the constants $a_{n,t}$ *are bounded independently of* t *and* n.

9. 4. Corollary. *For every* $b > 0$ *there is a constant* $t_0 \in \mathbb{N}$ *independent of* n *such that* $l_n(t) \geqq b$ *for all* $t \geqq t_0$.

Proof. This follows from 9. 2 and 9. 3.

9. 5. Lemma. *Let* $\bar{t}$ *be a fixed integer* > 0. *Then as* n *varies we have* $j_n(\bar{t}) = h^{-1}n + c_n$ *where the* c_n *are a bounded sequence.*

Proof. Note that

$$j_n(\bar t) > s \Leftrightarrow p^{u(h-1)}\bar t < c_{ip^{h-1}}(n-u), \qquad i = 0, 1, 2, \ldots, p^{h-1}-1;\ u = 0, 1, \ldots, s$$

$$\Leftrightarrow \begin{cases} \bar t < \dfrac{d_i p^{n-u-1}}{p^{h-1}-i}\, p^{-uh+u}, & i = 1, 2, \ldots, p^{h-1}-1;\ u = 0, 1, \ldots, s \\ \bar t < \left(\dfrac{e_K p^{n-u}}{p^h-1} - \dfrac{e_K-(p-1)(m_0+1)}{p^h-1} - \dfrac{p\varepsilon}{p^h-1}\right) p^{-uh+u}, & u = 0, 1, \ldots, s. \end{cases}$$

The lemma follows because the d_i are constants $\geqq 1$ for $i = 1, 2, \ldots, p^{h-1}-1$.

9. 6. Corollary. *Let $\bar t$ be a constant integer > 0. Then as n varies we have*

$$\tau_{n/0}(\bar t) = \frac{h-1}{h}\, n e_K + b_{n,\bar t}$$

where the $b_{n,\bar t}$ are bounded independent of n.

Proof. It follows from 9. 5 and 9. 3 that $l_n(\bar t) = d_n$ where the d_n are a bounded sequence. Let $k(n) = k_n(\bar t)$ be the largest integer k such that $\iota_{n/k}(\bar t) = -1$. Then $k(n) = n - j_n(\bar t) - d'_n$ where the d'_n are bounded by $c+1$ according to Lemma 9. 1.

Applying 9. 1 again we obtain from the definition of $\tau_{n/s}$

$$\tau_{n/k(n)}(t) = t p^{(h-1)j_n(\bar t)+c'_n} + c''_n$$

where (c'_n) and (c''_n) are bounded sequences. Therefore if $l_n(\bar t) > 0$

$$\tau_{n/l_n(\bar t)}(\bar t) = \bar t + ((h-1)j_n(\bar t) + c'_n)e_K \cdot p^{l_n(\bar t)} + p^{-(h-1)j_n(\bar t)-c'_n} \cdot c''_n,$$

$$\tau_{n/0}(\bar t) = \lambda_{l_n(\bar t)/0}(\tau_{n/l_n(\bar t)}(\bar t)) = ((h-1)j_n(\bar t) + c'_n)e_K + b''_n = \frac{(h-1)}{h} \cdot n \cdot e_K + b_{n,\bar t}$$

where the b''_n and $b_{n,\bar t}$ are bounded sequences (use the fact that $l_n(\bar t)$ is bounded and 9. 5)˙ If $l_n(\bar t) = 0$ then

$$\tau_{n/0}(\bar t) = p^{-k(n)}\tau_{n/k(n)}(\bar t) + k(n)e_K = \frac{h-1}{h} n \cdot e_K + b_{n,\bar t}$$

where $(b_{n,t})$ is a bounded sequence because $k(n) = n - j_n(\bar t) - d'_n$ where (d'_n) is bounded, and $j_n(\bar t) = h^{-1}n + d_n$ where (d_n) is bounded.

9. 7. Corollary (= easy half of Theorem 3. 1). *Let K_∞/K be any Γ-extension; F a one parameter formal group over A_K. Then there exists a constant c_1 such that*

$$F\text{-Norm}_{n/0}(F(K_n)) \subset F^{\alpha_n}(K)$$

where $\alpha_n = h^{-1}(h-1)e_K n - c_1$.

Proof. Take $b > 0$ large enough so that $m_n = (1 + p + \cdots + p^{n-1})e_K + m_0$ for $n \geqq b$. The Γ-extension K_∞/K_b satisfies the condition stated in the beginning of this section so that we can apply 9. 6 to find a constant c'_1 such that

$$\sigma_{n/b}(1) \geqq \tau_{n/b}(1) - (e_1 - m_0) \geqq h^{-1}(h-1)e_K p^b n - c'_1.$$

It follows from this that, for large enough n, $\sigma_{n/0}(1) = \lambda_{b/0}(\sigma_{b/0}(1))$. The corollary follows from this because, for fixed b, $\lambda_{b/0}(t) = p^{-b}t + c_t$ where (c_t) is a bounded sequence.

10. Proof of the theorem in the nice case

In this section $(F, K_\infty/K)$ is a nice pair such that $e_1 - m_0 \geqq p$ and $e_K \geqq e_1 - m_0$. To prove Theorem 3. 1 by means of proposition 8. 1 we must show that there are enough integers t to which 8. 1 applies.

10. 1. Lemma. *Let $n \in \mathbb{N}$ be fixed and let $t_0 \in \mathbb{N}$ be such that $l_n(t_0) > 0$; let $s_0 = \tau_{n/0}(t_0)$. Then for every $s \geqq s_0$ there exists an integer $t \geqq t_0$ such that*

(i) *t is of the form $\chi_{l_n(t)/0}(t')$, $t' \in \mathbb{N}$,* (ii) *$\tau_{n/0}(t) = s$.*

Proof. Let $l_0 = l_n(t_0) > 0$. Then $\lambda_{l_0/0}(\tau_{n/l_0}(t_0)) = s_0$ and hence $\chi_{l_0/0}(s_0) \geqq \tau_{n/l_0}(t_0)$. Let

(10. 1. 1) $$t_1 = t_0 + \chi_{l_0/0}(s_0) - \tau_{n/l_0}(t_0) \geqq t_0.$$

If $l_n(t_1) = l_n(t_0) = l_0$, then t_1 satisfies (i), (ii) above for $s = s_0$. If $l_n(t_1) \neq l_0$, then $l_n(t_1) > l_0$. We want to show that

(10. 1. 2) $$\tau_{n/l_0}(t_1) \leqq \tau_{n/l_0}(t_0) + t_1 - t_0.$$

This can be done as follows. Let $t_0 < t(1) < t(2) \cdots < t(r)$ be the integers between t_0 and t_1 (t_1 included) where $l_n(t)$ changes value. I. e. $t(1)$ is the smallest real number larger than t_0 such that $l_n(t(1)) > l_n(t_0)$, etc. ... One has $t_0 < t(1) < \cdots < t(r) \leqq t_1$. Let $l(i) = l_n(t(i))$, $l_1 = l_n(t_1)$, then $l_0 < l(1) < \cdots < l(r) \leqq l_1$.

One has (cf. the definition of ϱ and (6. 3. 3), (6. 3. 6))

(10. 1. 3)
$$\begin{aligned}
&\varrho_{n/l_0}(t(1)) - \varrho_{n/l_0}(t_0) = t(1) - t_0,\\
&\varrho_{n/l_0}(t(2)) - \varrho_{n/l_0}(t(1)) = (t(2) - t(1))p^{l_0 - l(1)},\\
&\varrho_{n/l_0}(t(3)) - \varrho_{n/l_0}(t(2)) = (t(3) - t(2))p^{l_0 - l(2)},\\
&\quad\vdots\\
&\varrho_{n/l_0}(t(r)) - \varrho_{n/l_0}(t(r-1)) = (t(r) - t(r-1))p^{l_0 - l(r-1)},\\
&\varrho_{n/l_0}(t_1) - \varrho_{n/l_0}(t(r)) = (t_1 - t(r))p^{l_0 - l(r)}
\end{aligned}$$

(note that $l_n(t(1) - \varepsilon) = l_n(t_0)$, thus $\varrho_{n/l_0}(t(1) - \varepsilon) - \varrho_{n/l_0}(t_0) = t(1) - \varepsilon - t_0$, taking the limit as $\varepsilon \to 0$ gives the first of the formulas above because $\varrho_{n/j}(t)$ is a continuous function of t; the other formulas of (10. 1. 3) are proved similarly). Adding the formulas (10. 1. 3) and using $l(i) \geqq l_0$, $i = 0, 1, \ldots, r$ gives

$$\varrho_{n/l_0}(t_1) - \varrho_{n/l_0}(t_0) \leqq t_1 - t_0$$

and hence (cf. (6. 3. 6), (6. 2. 5))

$$\tau_{n/l_0}(t_1) \leqq \varrho_{n/l_0}(t_1) \leqq \varrho_{n/l_0}(t_0) + t_1 - t_0 = \tau_{n/l_0}(t_0) + t_1 - t_0$$

which proves (10. 1. 2). Using (10. 1. 1) we see

$$\tau_{n/l_0}(t_1) \leqq \tau_{n/l_0}(t_0) + (t_0 + \chi_{l_0/0}(s_0) - \tau_{n/l_0}(t_0)) - t_0 = \chi_{l_0/0}(s_0).$$

Now

$$\lambda_{l_1/l_0}(\tau_{n/l_1}(t_1)) = \tau_{n/l_0}(t_1) \leqq \chi_{l_0/0}(s_0)$$

which implies $\tau_{n/l_1}(t_1) \leqq \chi_{l_1/0}(s_0)$. Now let

(10. 1. 4) $$t_2 = t_1 + \chi_{l_1/0}(s_0) - \tau_{n/l_1}(t_1).$$

If $l_n(t_2) = l_n(t_1) = l_1$ then t_2 satisfies (i) and (ii) of the lemma for $s = s_0$; if not then $l_n(t_2) = l_2 > l_1$ and we construct a t_3 from t_2 as we constructed t_2 from t_1. This process must terminate because $l_0 < l_1 < l_2 < \cdots \leqq n$. This proves the lemma for $s = s_0$.

One proceeds by induction. Let t_s satisfy (i) and (ii) for $s \geqq s_0$. Let $l = l_n(t_s)$, then $\chi_{l/0}(s+1)$ is larger than $\tau_{n/l}(t_s) = \chi_{l/0}(s)$. Let

$$t_1 = t_s + \chi_{l/0}(s+1) - \tau_{n/l}(t_s).$$

If $l_n(t_1) = l$, then t_1 satisfies (i) and (ii) for $s+1$. If not then $l_n(t_1) > l_n(t) = l \ldots$ (same argument as above) .... q. e. d.

10. 2. Proof of theorem 3. 1 in the nice case. We have that $(F, K_\infty/K)$ is nice, $e_1 - m_0 \geqq p$, $e_K \geqq e_1 - m_0$. Now choose $b \in \mathbb{N}$ such that $p^b \geqq p(e_1 - m_0)(p^h + 3)$ and $p^b \geqq 2p^{r_0}(e_1 - m_0 + 1)$ where r_0 is the smallest natural number such that $p^{r_0} \geqq (e_1 - m_0)$. According to 9. 4 there is a constant t_1 such that $l_n(t_1) \geqq b$ for all n. Let

$$t_0 = \max\{t_1, p^{h+1}(e_1 - m_0) + p^{r_0+1}(e_1 - m_0 + 2)\}, \qquad s_0 = \tau_{n/0}(t_0).$$

According to Lemma 10. 1 above there exists for every $s \geqq s_0$ a $t_s \geqq t_0$ such that $\tau_{n/0}(t_s) = s$, $\tau_{n/l_n(t_s)}(t_s)$ is of the form $\chi_{l_n(t_s)/0}(t')$ for some $t' \in \mathbb{N}$; further $l_n(t_s) \geqq b$ for all $s \geqq s_0$ because $t_s \geqq t_0$. We can therefore apply proposition 8. 1 with $t = t_s$ for all $s \geqq s_0$. This proves (cf. [3], Lemma (3. 2))

$$\operatorname{Norm}_{n/0}(F^{t_0}(K_n)) = F^{s_0}(K).$$

According to 9. 6 there exists a constant c_2 such that $\tau_{n/0}(t_0) \geqq \frac{h-1}{h} n e_K - c_2$ for all n. This proves the righthand inclusion of 3. 1. The lefthand inclusion was proved in Corollary 9. 7.

11. Proof of Theorem 3. 1

In view of 10. 2, 9. 7 and 3. 5 it suffices to show that given a Γ-extension K_∞/K and a formal group F over A_K there exists a finite extension L/K such that

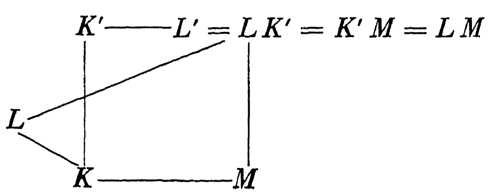
(i) $L \cdot K_\infty/L$ is a Γ-extension, (ii) $e_1(L) - m_0(L) \geqq p$,

(iii) $e_L \geqq e_1(L) - m_0(L)$, (iv) $(L \cdot K_\infty/L, F)$ is a nice pair

(where F is considered as a formal group over A_L and $e_1(L)$, $m_0(L)$ are the numbers of the Γ-extension $L \cdot K_\infty/L$ corresponding to e_1 and m_0 for K_∞/K).

11. 1. Lemma. *Let K'/K and L/K be two totally ramified galois extensions of prime degree p and p' respectively such that $m(K'/K) > m(L/K)$. Let $L' = L \cdot K'$, then L'/L and L'/K' are both totally ramified (galois) of degree p and p' respectively and*

$$m(L'/L) = m(L/K) + p\big(m(K'/K) - m(L/K)\big), \; m(L'/K') = m(L/K).$$



Proof. Let M be the maximal unramified extension of K contained in $L \cdot K'$. If $M \neq K$, then $L \cdot M = L \cdot K' = K'M$, which because M/K is unramified implies $m(L/K) = m(LM/M) = m(KM/M) = m(K'/K)$, a contradiction. Hence $M = K$ and L'/K' and L'/L are totally ramified of degree p', p respectively.

$$\begin{array}{ccc} K' & \text{———} & L' \\ | & & | \\ K & \text{———} & L \end{array}$$

The statement on $m(L'/L)$ and $m(L'/K')$ now follows from the fact that we must have

$$\Psi_{L'/L} \circ \Psi_{L/K} = \Psi_{L'/K'} \circ \Psi_{K'/K}$$

where the $\Psi_{P/Q}$ are the Herbrand Ψ-functions. Cf. [9]; if P/Q is totally ramified, galois of degree p then $\Psi_{P/Q}(t) = t$ if $t \leqq m(P/Q)$ and $\Psi_{P/Q}(t) = m(P/Q) + p(t - m(P/Q))$ if $t \geqq m(P/Q)$. q. e. d.

Let K_∞/K be a Γ-extension; L/K a finite extension. We write L_∞ for $L \cdot K_\infty$. If L_∞/L is again a totally ramified Γ-extension let $m_0(L)$ denote the natural number such that $m(L_n/L_{n-1}) = (1 + p + \cdots + p^{n-1})e_L + m_0(L)$ for n large; we write $e_1(L)$ for $(p-1)^{-1}e_L$.

11. 2. Lemma. *Let K_∞/K be a Γ-extension; $q_1, \ldots, q_r$ a finite set of integers. Then there exists a finite extension L/K such that*

(i) *L_∞/L is a totally ramified Γ-extension such that*

$$m(L_n/L_{n-1}) = (1 + p + \cdots + p^{n-1})e_L + m_0(L) \quad \textit{for all } n.$$

(ii) *q_i divides $e_{L/K} = v_L(\pi_K)$, $\quad i = 1, \ldots, r$.*

(iii) *$e_1(L) - m_0(L) \geqq p$.*

(iv) *$e_L \geqq e_1(L) - m_0(L)$.*

Proof. Let q' be the smallest common multiple of the q_i and let $q' = p^s q$ where $(p, q) = 1$. Let $b \in \mathbb{N}$ be such that

$$m(K_n/K_{n-1}) = (1 + p + \cdots + p^{n-1})e_K + m_0 \text{ for } n \geqq b.$$

Take $L^{(1)} = K_c$ where $c = \max(b, s)$. Then $L^{(1)}_\infty/L^{(1)}$ satisfies (i) and $p^s | e_{L^{(1)}/K}$. Let $L^{(2)}/L^{(1)}$ be the extension of $L^{(1)}$ obtained by adjoining a root of $X^q - \pi_{L^{(1)}}$. The extension $L^{(2)}_\infty/L^{(2)}$ then satisfies (i) and (ii). This follows from Lemma 11. 1 above. Replacing $L^{(2)}$ with $L^{(2)}_2$ if necessary we can also assume that

$$m(L^{(2)}_1/L^{(2)}) = e_{L^{(2)}} + m_0(L^{(2)}) \geqq 4.$$

Let $r = 3$ if $p = 2$ and $r = 2$ if $p > 2$. There exists a totally ramified (galois) extension $L^{(3)}/L^{(2)}$ of degree p such that $m(L^{(3)}/L^{(2)}) = r$. (E. g. a socalled Artin-Schreier extension (cf. [2] (6. 5.)) for example.) It follows from Lemma 11. 1 that $L^{(3)}_\infty/L^{(3)}$ still satisfies (i) and (ii). Further also according to Lemma 11. 1

$$\begin{aligned} m(L^{(3)}_1/L^{(3)}) = m_0(L^{(3)}) + e_{L^{(3)}} &= r + p(m(L^{(2)}_1/L^{(2)}) - r) \\ &= p m_0(L^{(2)}) + p e_{L^{(2)}} + (1-p)r \end{aligned}$$

which gives us $m_0(L^{(3)}) = p m_0(L^{(2)}) + (1-p)r$ and hence

$$e_1(L^{(3)}) - m_0(L^{(3)}) = p e_1(L^{(2)}) - p m_0(L^{(2)}) + (p-1)r \geqq p$$

i. e. $L^{(3)}_\infty/L^{(3)}$ also satisfies (iii). Finally we get $e_1(L^{(3)}_i) - m_0(L^{(3)}_i) = e_1(L^{(3)}) - m_0(L^{(3)})$

as is easily checked. To get an extension Γ such that (i)—(iv) hold we can therefore take $L = L_i^{(3)}$ for $i \in \mathbb{N}$ sufficiently large. q. e. d.

11. 3. Proof of Theorem 3. 1. Apply Lemma 11. 2 with

$$\{j - i \mid 1 \leqq i, j \leqq p^{h-1}\} \cup \{pi - 1 \mid i = 1, 2, \ldots, p^{h-1}\} \cup \{p - 1\}$$

as the set of the q_i. Now apply 10. 2, 9. 7 and 3. 5.

12. Concluding remarks

12. 1. The Proof of Theorem 3. 1 is except for some technical complications the same as the proof of the main theorem of [3] as given in [3]. This is a main reason why we proved the main theorem of [3] as we did. There does exist in fact an easier proof of [3], Theorem 6. 1. Let F be a formal group over A_K. The formal group F is isomorphic over A_K to a formal group F' with a logarithm of type $f_t(X)$; $t_1, t_2, \ldots \in A_K$, where $f_T(X) = X + \Sigma \frac{T_i}{p} f_T^{(i)}(X^{p^i}) \in \mathbb{Q}[T_1, T_2, \ldots][[X]]$ where $f_T^{(i)}$ is obtained from f_T by raising all the parameters $T_1, T_2, \ldots$ to the p^i-th power and $f_t(X)$ is obtained from $f_T(X)$ by substituting t_i for T_i; cf. [4]; $h(F') = h$ iff $v_K(t_i) \geqq 1, i = 1, \ldots, h-1, v_K(t_h) = 0$. If K is an *unramified* local field (i. e. $v_K(p) = 1$), then one shows relatively easily that if $h(F') = h$, $F'(X, Y) = f_t^{-1}(f_t(X) + f_t(Y))$ where f_t is as above, and

$$f_t(X) = \sum_{i=0}^{\infty} a_i X^{p^i}, \qquad a_0 = 1, a_i \in K$$

then

$$\begin{aligned} &v_K(a_i) \geqq 0, && i = 0, 1, \ldots, h-1, \\ (12.1.1) \qquad &v_K(a_{nh}) = -n, && n = 0, 1, 2, \ldots, \\ &v_K(a_i) \geqq -n, && nh \leqq i < (n+1)h. \end{aligned}$$

Now let K_∞/K be the cyclotomic Γ-extension and consider the commutative diagram

$$\begin{array}{ccc} F'(K_n) & \xrightarrow{\ f_t\ } & K_n^+ \\ {\scriptstyle F\text{-Norm}} \downarrow & & \downarrow {\scriptstyle \mathrm{Tr}_{n/0}} \\ F'(K) & \xrightarrow{\ f_t\ } & K^+ \end{array}$$

(K_n^+ is the additive group of K_n).

Using (12. 1. 1) and the trace lemma of [3] it is not difficult to calculate the image of $\mathrm{Tr}_{n/0} \circ f_t$. Finally f_t is an isomorphism of $F'(K)$ with the subgroup $pA_K < K^+$. This provides another proof of [3], Theorem 6. 1.

12. 2. The same method as sketched above in 12. 1 gives the same result as in Theorem 3. 1 in case K_∞/K is any Γ-extension, K a local field with perfect residue field and F a formal group defined over $A_{W(\bar{K})}$ where $W(\bar{K})$ is the maximal unramified subfield of K.

12. 3. It should be possible to use the method of 12. 1 to obtain another (and easier) proof of Theorem 3. 1. The problem is that (12. 1. 1) is not necessarily true if K is not unramified.

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