

Algorithm Development in CFD : Multidimensional Upwinding and Multiple Semi-Coarsening Multigrid

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Abstract. The paper describes the results of a Community research effort aimed at improving accuracy and efficiency of flow solvers for aerodynamical applications. Two computational technologies have been considered which have the potential to lead to a significant step forward in the development of future flow solvers. The first, multidimensional upwinding, improves on the state-of-the-art space discretization methods by using multidimensional physics as the basis for the upwinding, instead of the dimension-by-dimension application of one-dimensional physics (1D Riemann solver). The second technology is concerned with the convergence acceleration on structured grids. The idea of semi-coarsening to improve multigrid efficiency for anisotropic flows like boundary layers has been generalized in a consistent and theoretically founded way by using all possible nested semi-coarsenings in three directions. During the project, the theoretical framework has been developed, and the feasibility of the two approaches has been demonstrated by testing on standard airfoil and wing testcases in the subsonic and transonic regime.

1 MULTIDIMENSIONAL UPWINDING

1.1 Introduction

The development of inherently multidimensional upwind discretizations for the convective terms is the first concern of this research. It follows from the observation that the present-day methods used in two and three-dimensional high resolution upwind solvers are merely an extension of one-dimensional theory on a dimension-by-dimension basis.

The basis of these methods is a fundamental contribution made by the Russian mathematician Godunov [6], with as key ingredient the solution of the Riemann problem, this is the one-dimensional flow which results from bringing into contact two fluids at different but constant states, and for which an exact solution is known. Godunov applied this theory to construct discrete solutions in 2D and 3D: the flow domain

is subdivided into a discrete number of finite volumes, each having a constant state, and the one-dimensional theory is applied along each of the cell faces between two adjacent finite volumes, thereby neglecting interactions tangential to the coordinate direction (the "corner effects"). Subsequently, approximate Riemann solvers have been developed, e.g. by Roe, van Leer, Osher and others. They replace the exact Riemann problem solver by an approximate Riemann solver, which is computationally less expensive.

Second-order accurate schemes have been constructed along the same principles based on a piecewise linear reconstruction of the two states used in the Riemann problem, combined with limiting which reduces the scheme to the first-order method if oscillations are detected. In this way monotonic profiles are preserved over steep gradients like shocks and shears. These are the so-called upwind-TVD solvers.

The new developments considered here are no longer based on a dimension-by-dimension extension of the one-dimensional theory. Instead, they aim at a truly multidimensional generalization of the different concepts used in the one-dimensional theory. Typically, the new upwind schemes not only consider whether the propagation of signals is arriving from the right or from the left at a given meshpoint, but they also take into account the precise angle of orientation (e.g. the streamline for entropy).

A more numerical concern apart from the lack of multidimensional physics in the system schemes for the Euler equations is the following : suppose one applies the Riemann solver technology to a scalar convection equation ($u_t + au_x + bu_y = 0$), the most simple case of a hyperbolic equation. Also here, the standard scheme reduces to a dimension-by-dimension application of the first-order upwind scheme for each of the operators au_x and bu_y . It turns out that this is a very bad first-order scheme in terms of numerical diffusion, with as only merits that it is monotone and simple, although it is the best possible monotone scheme for a pure one-dimensional equation ($u_t + au_x = 0$).

To regain optimality in the sense of lowest diffusion while maintaining monotonicity, truly multidimensional upwinding in the direction of the convection speed \vec{a} (with Cartesian

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components a and b) is needed, as shown in the optimal linear monotonic first-order convection schemes on a structured Cartesian grid, discovered by Rice and Schnipke [20] in 2D and by Roe and Sidilkover in 3D, and studied in detail in this project by Hirsch and Van Ransbeeck [18].

1.2 Scalar convection schemes

The basis of the research in this project are the optimal monotone first-order schemes for scalar convection mentioned before [18], and their generalization for unstructured grids composed of triangles or tetrahedra developed by Roe, Struijs and Deconinck (the N-scheme) [5, 21, 15]. These optimal scalar first-order schemes show dramatically improved accuracy compared to the first-order dimensionally split finite volume scheme, in the unstructured grid case formulated on the dual mesh formed by the medians of the triangles (or tetrahedra). This is achieved without any addition in computational cost or loss of robustness; in fact the stencil used is in general smaller than for the first-order upwind finite volume scheme.

Starting from these optimal first-order schemes, nonlinear monotonic *second-order* versions have been developed during this project [21, 15, 18]. This was obtained through a limiting procedure, which uses multidimensional gradients. As a result, the stencil of the second-order scheme remains limited to the nearest neighbours, as opposed to the dimension-by-dimension second-order schemes which need a widening of the stencil. Basing the limiting on an optimal first-order scheme also reduces the nonlinearity of the second-order scheme, thus improving robustness as well as accuracy.

1.3 The system of Euler equations

Once satisfactory scalar multidimensional schemes were developed, the key remaining and most difficult issue was their extension to a non-commuting system of hyperbolic equations. On the other hand, this extension is almost trivial for the case of the standard first-order upwind scheme thanks to the dimension-by-dimension approach leading to a one-dimensional Riemann problem in each coordinate direction. Such a system generalization has been searched in two different directions, which can be used in combination :

1. A first direction tries to avoid (or at least to minimize) the problem by converting the system in a series of scalar wave equations (with eventual remaining coupling terms being considered as source terms). Different mechanisms such as the use of simple wave solutions [3, 21] or approximate diagonalization [19, 14] of the system have been investigated, with good success, at least in 2D. The most recent results of the work carried out in this context (hyperbolic-elliptic splitting, Paillère [16] and Mesaros [10]) happen to be very close to the work on local preconditioning of the Euler system (e.g. by Merkle, Turkel and van Leer) [23], initially intended to reduce disparities in wavespeeds. Indeed, also in this context an optimal decoupling of the

equations is sought to allow a separate timestep restriction for the different characteristic contributions. For example, for 2D steady supersonic flow the equations in characteristic variables reduce to convection of entropy and total enthalpy along the streamline, and two acoustic variables along the Machlines, i.e. a set of four scalar convection equations. In subsonic flow however, (and even for supersonic flow in 3D) coupling terms cannot be avoided and strict control of monotonicity is lost when a monotonic scalar convection scheme is applied. Nevertheless, excellent results were obtained in 2D by Paillère [14] and Mesaros [10], both for subsonic and transonic flow.

2. More recently in this project, an algebraic generalization of the optimal first-order scheme was achieved by van der Weide (matrix N-scheme) [22], applicable to general non-commuting hyperbolic systems, and reducing to the optimal scalar scheme applied to each of the decoupled equations, in case the system is diagonalizable. This scheme has been shown by Barth to be energy stable for a linear hyperbolic system when combined with Euler explicit time marching, and application to the Euler equations in 2D by van der Weide and 3D by Bonfiglioli [1], compares very favorably in accuracy with the standard system first-order finite volume scheme (e.g. using Roe's Riemann solver) on the same mesh, without losing robustness and without increase of the stencil. Nonlinear limited second-order extensions have been developed and good results in terms of accuracy and monotonicity have been obtained [22]. Applications in turbomachinery cascades have been studied in [2]. However, satisfactory robustness and convergence still fail and are the subject of current research.

Due to the compactness of their stencil in space (at most the Galerkin FE stencil), the multidimensional upwind schemes are well suited for multigrid acceleration as developed in this project by Napolitano and coworkers [3]. Also implicit time integration, application of Newton's method for steady state computations and parallelization [8] strongly profit from the compactness of the stencil. Application to the *incompressible* Euler equations was studied in this project by Michelsen [11].

1.4 Numerical results

Two example calculations for flow over a NACA0012 airfoil are given.

1.4.1 Subcritical NACA0012 airfoil

The subcritical flow ($M_\infty = 0.63$, 2° angle of attack) over a NACA0012 airfoil has been computed on a very coarse mesh, with only 489 nodes, and 60 nodes on the body, Figure 1(a). The solution (Figure 1) is computed with the 'hyperbolic/elliptic' splitting, applying the compact nonlinear second-order N-scheme on the hyperbolic part and the system SUPG-scheme on the elliptic part. The solution has very low false entropy production and the aerodynamic coefficients $c_L = 0.323$, $c_D = 0.004$ compare well with fine-mesh results found in the literature, despite the coarseness of the mesh used.

1.4.2 Transonic NACA0012, $M_\infty = 0.85$, $\alpha = 1^\circ$

This computation has been made with the new monotone system distribution schemes, the linear system N-scheme and the second-order compact limited system N-scheme. The unstructured mesh (Figure 2) has 2355 nodes of which 140 on the airfoil. First, the convergence history (Figure 3) is shown based on a Newton iterative solver [8]. The first-order system N-scheme shows full Newton convergence in 25 iterations. The computation with the second-order matrix N-scheme is started from the first-order solution, and the residual stagnates after about 6 orders of reduction (40 nonlinear iterations). Mach number isolines for the system limited N-scheme, applied to the preconditioned Euler equations are given in Figure 4(a). In Figure 4(b), the entropy distribution on the body for the limited N-scheme applied to the full Euler equations is compared with the entropy distribution for the same scheme, applied to the preconditioned equations (hyperbolic/elliptic splitting, HE). It is clear that the entropy production in the nose region is lower for the hyperbolic/elliptic splitting. Note that the entropy for both solutions remains perfectly monotonic in the shock. The most significant difference between both solutions however is found in the total enthalpy distribution: for the solution with hyperbolic-elliptic splitting total enthalpy remains perfectly constant, also in the shock profile and at the leading edge, as shown in Figure 4(c).

2 MULTIPLE SEMI-COARSENING MULTIGRID

2.1 Introduction

A difficulty of standard multigrid methods in solving 3D problems, as compared to solving 2D problems, is that the requirements imposed on the smoother are more severe. On a structured grid in 3D, standard coarsening implies restriction from $2 \times 2 \times 2$ cells to a single cell only. Because a $2 \times 2 \times 2$ -set of cells can support more high-frequency errors than a 2×2 -set, 3D standard multigrid imposes stronger requirements on the smoother than 2D standard multigrid. Therefore, standard multigrid generally works fine for 2D problems, i.e. gives (nearly) grid-independent convergence rates, but for 3D problems this may not be the case. This is illustrated in Figure 5, showing the convergence results for the ONERA-M6 wing at transonic flow conditions, as obtained with the standard multigrid method. The convergence is clearly grid dependent. A fix might be found in deriving a more powerful smoother, keeping all other components of the numerical method the same, but the type of smoother then depends on the dimension of the problem. A more natural approach followed in this project is to apply multiple semi-coarsening (Figure 6b) instead of standard (i.e. full) coarsening (Figure 6a). Though multigrid with semi-coarsening is expected to be most fruitful for 3D problems, (as far as we know) applications of semi-coarsening only existed in 2D, before the start of this project. The pioneering work in 2D has been done by Mulder [12],

who introduced semi-coarsening as a fix for the poor convergence results observed in computing nearly grid-aligned flows governed by the steady, 2D Euler equations. In [17], Radespiel and Swanson embroider on Mulder's approach for the steady, 2D Euler equations, paying particular attention to the prolongation operators.

In a first phase of this project, investigation of semi-coarsened multigrid for second-order elliptic (Poisson-type) equations has been carried out by De Zeeuw [4], based on the 2D work of Naik and Van Rosendale [13]. Just as in [17], much attention was paid to specific prolongation operators for semi-coarsening.

In a second step, semi-coarsened multigrid for the steady, 3D Euler equations was developed [9], with particular attention for the prolongation operators as well.

2.2 Semi-coarsened multigrid method

As the smoothing technique for the first-order discretized Euler equations, simple collective symmetric point Gauss-Seidel relaxation is applied, combined with the nonlinear multigrid (FAS) scheme. Let $\Omega_{l,m,n}$, $l = 0, 1, \dots, l_{max}$, $m = 0, 1, \dots, m_{max}$, $n = 0, 1, \dots, n_{max}$ be the set of semi-coarsened, nested grids, with $\Omega_{0,0,0}$ the coarsest and $\Omega_{l_{max},m_{max},n_{max}}$ the finest grid.

Then, nested iteration (FMG) is applied to obtain a good initial solution on the finest grid. The nested iteration starts with a user-defined initial estimate on the coarsest grid, $\Omega_{0,0,0}$, which is improved by relaxation. The improved solution $q_{0,0,0}$ is prolonged (level-by-level) to all grids up to and including level 3, with a specifically developed 3D prolongation [7]. Next, the solution $q_{1,1,1}$ is improved by a single nonlinear multigrid cycle and prolonged to all grids up to and including level 6. For simplicity, we assume that $l_{max} = m_{max} = n_{max}$. Then, the above process can be repeated in a straightforward manner up to and including level $3 \times l_{max}$.

A single nonlinear multigrid cycle from level $l + m + n$ is recurrently defined by the following steps:

1. Compute on all grids at the next coarser level, $(l + m + n) - 1$ the same right-hand sides as in standard multigrid, using a specific restriction operator developed during the project.
2. Approximate the solutions on the coarser level $(l + m + n) - 1$ by the application of a single nonlinear multigrid cycle.
3. Correct the current solutions on level $l + m + n$ by one of two alternative correction prolongations also developed during this project.
4. Improve the solutions on level $l + m + n$ by the application of n_{post} post-relaxations.

So, as nonlinear multigrid cycles, here we also use saw-tooth cycles. Whereas in the standard multigrid method one may also apply pre-relaxations, in the present semi-coarsened multigrid technique this is not possible in fact. Pre-relaxation leads to incoherent right-hand side representations. For an explanation of this we refer to [4].

2.3 Numerical results

As test case we consider the ONERA-M6 wing at the transonic conditions $M_\infty = 0.84$, $\alpha = 3.06^\circ$. The grids used are of C-O-type (Figures 7). The wing as well as the grids are symmetric with respect to the plane through the wing's leading and trailing edges.

In the convergence results given in Figure 8, the finest grid considered is the $64 \times 16 \times 16$ -grid. In all four graphs, the convergence measure is the density residual on the finest mesh, non-dimensionalized by the value at the start of the computation. The improvement of both semi-coarsened multigrid methods [9] with respect to the standard multigrid method is significant. Of both semi-coarsened methods, the one with the fixed prolongation weights (Figure 8d) performs best.

2.4 Outlook : sparse grid multigrid

A disadvantage of full-grid-of-grid semi-coarsening is that very many points are needed. With N the total number of points on the finest grid, asymptotically standard multigrid uses $\frac{9}{8}N$ grid points versus $8N$ points for the full-grid-of-grid approaches. An efficiency improvement is obtained by thinning out the grid of grids. Most ambitious in this respect is the sparse-grid-of-grids approach (see [7] and the further references in there). The efficiency gain by the reduction of the numbers of grid points is enormous.

Future research will be directed to modifying the full-grid-of-grids technique into the direction of a sparse-grid-of-grids technique, maintaining as far as possible the solution accuracy on the finest grid. Theoretically, the sparse-grid-of-grids approach has the best ratio of discrete accuracy over number of grid points used. In the ideal case the full grid-of-grids will be completely replaced by a sparse grid-of-grids.

3 CONCLUSION

It is anticipated that multiple semi-coarsening multigrid and multidimensional upwinding discretizations could form the heart of a new generation of superior solvers for high Reynolds number compressible flows. Although more effort will be needed before this goal will be reached, especially in 3D, the results of this project show the potential.

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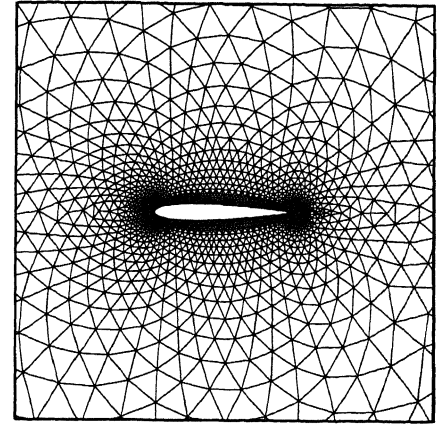


Figure 2. Mesh for transonic NACA0012 airfoil (2355 meshpoints, 140 on airfoil).

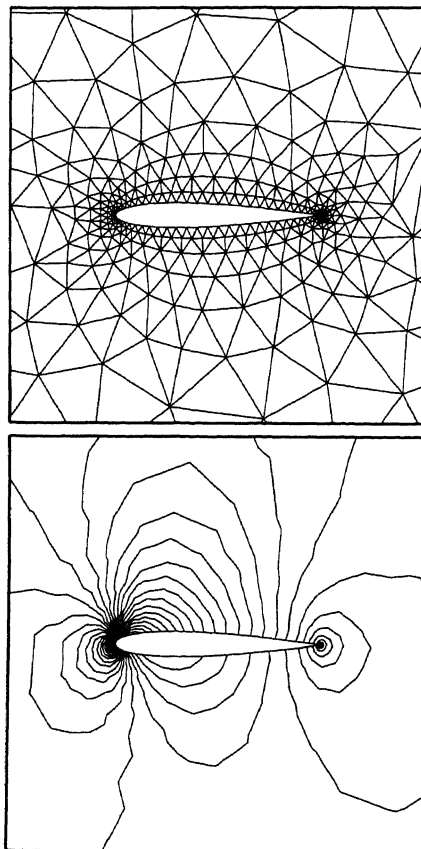


Figure 1. Subcritical NACA0012 airfoil: unstructured mesh with 489 nodes (of which 60 on the body), and Mach number isolines obtained with the hyperbolic/elliptic splitting. PSI scheme on hyperbolic equations and SUPG scheme on elliptic equations.

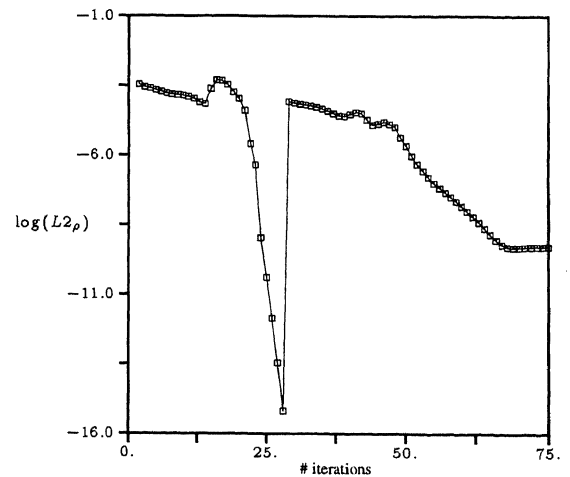


Figure 3. Convergence history for the HE splitting for the transonic NACA0012 airfoil ($M_\infty = 0.85$, $\alpha = 1^\circ$) for the system N- (first part) and for the system limited N-scheme (second part).

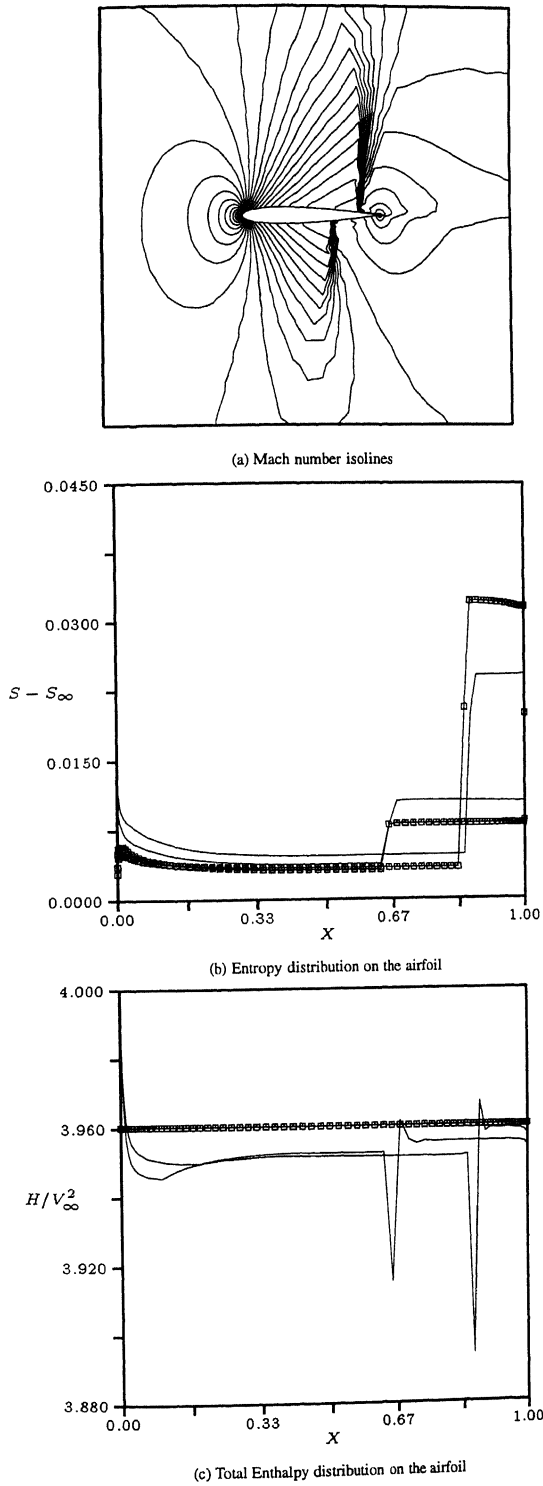


Figure 4. Transonic NACA0012 airfoil, $M_\infty = 0.85$, $\alpha = 1^\circ$. System limited N-scheme. (a) : Mach number isolines for the HE splitting. (b),(c) : Entropy and Total enthalpy distribution on the airfoil for the preconditioned (symbols) and full (solid line) Euler equations.

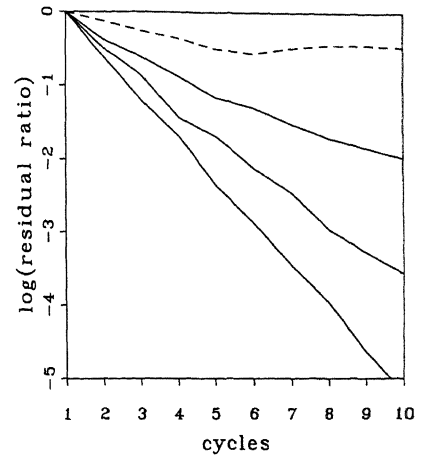


Figure 5. Convergence behavior of standard multigrid method for ONERA-M6 wing at transonic conditions, $M_\infty = 0.84$, $\alpha = 3.06^\circ$ (dashed: single grid $16 \times 4 \times 4$, solid: multigrid, with as finest grid, from below to above: $16 \times 4 \times 4$, $32 \times 8 \times 8$, $64 \times 16 \times 16$)

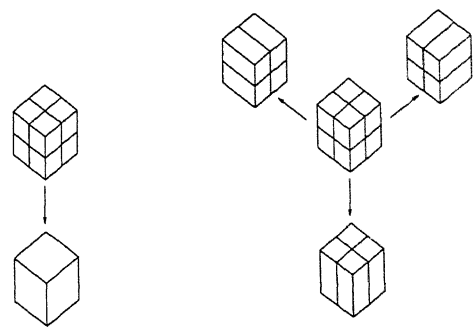
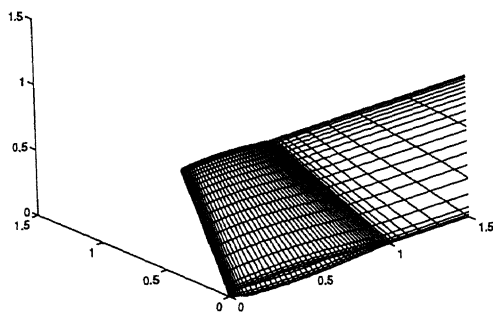
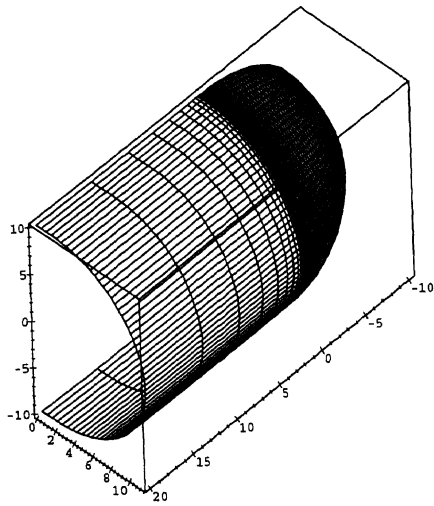


Figure 6. (a) full coarsening (b) semicoarsening

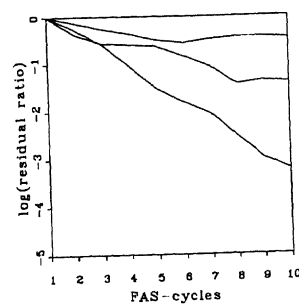


a. Upper side wing.

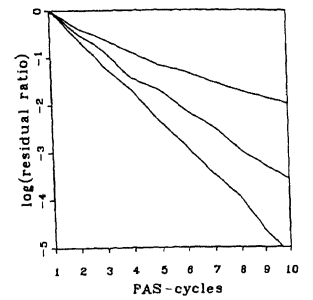


b. Far-field boundary.

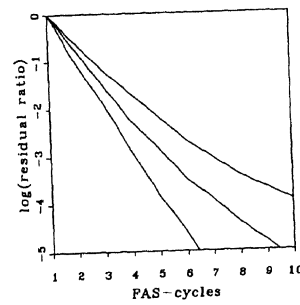
Figure 7. Views of $128 \times 32 \times 32$ C-O-type grid for ONERA-M6 wing



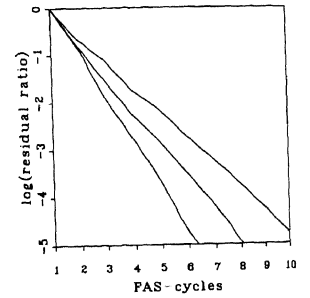
a. Single-grid.



b. Standard multigrid.



c. Semi-coarsened MG (1).



d. Semi-coarsened MG (2).

Figure 8. Convergence behavior of different solution methods for ONERA-M6 wing at transonic conditions, $M_\infty = 0.84$, $\alpha = 3.06^\circ$. Semi-coarsened multigrid (MG) results using two different prolongation weights, (1) defect dependent, (2) fixed (finest grid in all four graphs, from below to above: $16 \times 4 \times 4$, $32 \times 8 \times 8$, $64 \times 16 \times 16$).