

ON CUTTING PLANES*

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We give a geometrical description of Chvátal's version of Gomory's cutting plane method. Restricting ourselves to rational spaces, we prove that the derived geometrical objects are polyhedra again, and that the method also works for unbounded polyhedra.

1. Introduction

For any polyhedron P let P_I denote the convex hull of the lattice points contained in P .¹ If H is the half-space $\{x \mid wx \leq d\}$, where w is a vector whose components are relatively prime integers and d is a rational number, then one easily sees that $H_I = \{x \mid wx \leq \lfloor d \rfloor\}$. Geometrically, H_I arises by shifting H until its bounding hyperplane contains lattice points. So for half-spaces H there is an easy way to determine H_I . Since for each half-space H , the inclusion $P \subset H$ implies that $P_I \subset H_I$, we know that

$$P_I \subset \bigcap_H H_I \tag{1}$$

where the intersection ranges over all half-spaces H with $P \subset H$. We denote this intersection by P' . Below we show that P' is a polyhedron again, and that $P^{(t)} = P_I$ for some natural number t . (As usual, $P^{(0)} = P$, and $P^{(t+1)} = P^{(t)'}.$)

This is the essence of Chvátal's [1] formalization of Gomory's [4, 5, 6] *cutting plane method* for solving integer linear programming problems (cf. Rosenberg [11]). Chvátal's original method applies to *bounded* polyhedra in *real* space. However, the fact that the method works for these polyhedra follows from its effectiveness for rational polyhedra (see (i) of Section 4 below).

Clearly, we may restrict the range of the intersection (1) to *supporting* half-spaces, i.e., to half-spaces whose bounding hyperplane supports P . We shall see

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¹ P is a *polyhedron* if $P = \{x \mid Ax \leq b\}$ for some matrix A and some vector b . When using expressions like $Ax \leq b$ we implicitly assume compatibility of sizes of matrices and vectors. *We work within rational spaces, rather than real ones.* So any matrix and any vector is supposed to be rational. — A *lattice point* is an integral vector, i.e., a vector with integer components. $w \cdot x$ denotes the standard inner product of vectors w and x . $\lfloor d \rfloor$ denotes the lower integer part of a rational number d . For the theory of polyhedra and cones we refer to Grünbaum [7], Rockafellar [10] and Stoer and Witzgall [13].

below that we may restrict the intersection to finitely many supporting half-spaces, namely to those half-spaces corresponding to a so-called *totally dual integral* system of linear inequalities determining P (cf. (ii) of Section 2 below). Therefore, the use of this method to determine P_I depends on the capability to find these half-spaces.

2. Two preliminaries

(i) For each polyhedron P the set P_I is a polyhedron again. This not surprising fact can be derived from Motzkin's [9] theorem that each polyhedron P can be decomposed as $P = Q + C$, where Q is a bounded polyhedron and C is a polyhedral convex cone. Motzkin's theorem implies also that there are half-spaces $H_1, \dots, H_s, L_1, \dots, L_s$ such that $P_I = L_1 \cap \dots \cap L_s$, $P \subset H_1 \cap \dots \cap H_s$, and $L_i \subset H_i$ for $i = 1, \dots, s$ (so the bounding hyperplanes of L_i and H_i are parallel).

(ii) A system of linear inequalities $Ax \leq b$ is called *totally dual integral* if the linear programming minimum

$$\min\{yb \mid y \geq 0, yA = w\} \quad (2)$$

is attained by an integral vector y , for each integral vector w for which the minimum exists. Hoffman [8] and Edmonds and Giles [2] showed that if $Ax \leq b$ is totally dual integral and b is integral, then each face of the polyhedron $P = \{x \mid Ax \leq b\}$ contains lattice points (i.e., $P = P_I$). That is, if b is integral and the right-hand side of the linear programming duality equation

$$\max\{wx \mid Ax \leq b\} = \min\{yb \mid y \geq 0, yA = w\} \quad (3)$$

is achieved by an integral vector y for each integral vector w for which the minimum exists, then also the left-hand side is achieved by an integral vector x , for each such vector w .

It follows from the results of Giles and Pulleyblank [3] that for each polyhedron P there exists a totally dual integral system $Ax \leq b$ such that A is integral and $P = \{x \mid Ax \leq b\}$. (In [12] it is shown that there exists a unique minimal such system, provided that P has nonempty interior, that is, provided that P has full dimension.) Chvátal's method to determine P_I as described in the present paper calls for algorithmic methods to determine such a totally dual integral system.

3. Theorems

Theorem 1. *For any polyhedron P the set P' is a polyhedron again.*

Proof. Let $P = \{x \mid Ax \leq b\}$. We may suppose that A is an integral matrix, and, by (ii) of Section 2, that the system $Ax \leq b$ is totally dual integral. We show that $P' = \{x \mid Ax \leq \lfloor b \rfloor\}$ (where $\lfloor b \rfloor$ arises from b by taking componentwise lower integer parts), which yields that P' is a polyhedron.

First, $P' \subset \{x \mid Ax \leq \lfloor b \rfloor\}$, as each linear inequality in the system $Ax \leq b$ gives a half-space H , while the corresponding inequality in $Ax \leq \lfloor b \rfloor$ contains H_r .

Conversely, suppose $H = \{x \mid wx \leq d\}$ is a half-space containing P as a subset. We may suppose that the components of w are relatively prime integers, so $H_r = \{x \mid wx \leq \lfloor d \rfloor\}$. Now

$$d \geq \max \{wx \mid Ax \leq b\} = \min \{yb \mid y \geq 0, yA = w\}. \tag{4}$$

As the system $Ax \leq b$ is totally dual integral we know that the minimum in (4) is achieved by some integral vector y_0 . Therefore $\lfloor d \rfloor \geq \lfloor y_0 b \rfloor \geq y_0 \lfloor b \rfloor$, which yields that $wx \leq \lfloor d \rfloor$ for all x with $Ax \leq \lfloor b \rfloor$. This implies $\{x \mid Ax \leq \lfloor b \rfloor\} \subset H_r$. So $\{x \mid Ax \leq \lfloor b \rfloor\} \subset P'$.

Theorem 2. *For any polyhedron P there exists a number t such that $P^{(t)} = P_r$.*

Proof. We prove the theorem by induction on the dimension of the space, and on the (affine) dimension of (the affine hull of) P . If both are zero the theorem is easy.

If the dimension of P is less than the dimension of the space, then P is contained in some hyperplane K . If K contains no lattice points, then it is easy to see that $P_r = P' = \emptyset$. If K contains lattice points, then there exists an affine transformation of the space which brings K to the subspace K_0 of vectors with last component zero, and which brings the set of lattice points onto the set of lattice points. Moreover, the image of P is again a polyhedron, say P_0 . By induction we know that in the space K_0 we have: $(P_0)^{(t)} = (P_0)_r$, for some natural number t . Since each half-space H_0 of K_0 can be extended to a half-space H of the original space such that $H \cap K_0 = H_0$ and $H_r \cap K_0 = (H_0)_r$, it follows that also in the original space $(P_0)^{(t)} = (P_0)_r$. Since the collection of half-spaces and the set of lattice points are invariant under the affine transformation we know that also $P^{(t)} = P_r$.

Now suppose the dimension of P is equal to the dimension of the space. Let $H = \{x \mid wx \leq d\}$ be a half-space containing P_r as a subset, such that P is contained in some half-space $\{x \mid wx \leq d'\}$. We prove that there is a number s such that $P^{(s)} \subset H$. As P_r is the intersection of a finite number of such half-spaces H (cf. (i) of Section 2) the theorem follows.

Suppose, to obtain a contradiction, that for no s we have that $P^{(s)}$ is contained in H . Since $P' \subset \{x \mid wx \leq \lfloor d' \rfloor\}$, there exists an integer $d'' > d$ such that $P^{(s)} \subset \{x \mid wx \leq d''\}$ for all $s \geq r$, for some number r , but for no s we have $P^{(s)} \subset \{x \mid wx \leq d'' - 1\}$. Let K be the hyperplane $\{x \mid wx = d''\}$. We prove below that if Q is a polyhedron contained in $\{x \mid wx \leq d''\}$, then $Q' \cap K \subset (Q \cap K)'$. By induction on u this implies that $Q^{(u)} \cap K \subset (Q \cap K)^{(u)}$ for all u . Since the dimension of $P^{(r)} \cap K$ is less than the dimension of P , and since $P^{(r)} \cap K$ does not contain lattice points, we know by the induction hypothesis that $(P^{(r)} \cap K)^{(u)} = \emptyset$ for some u .

Therefore, $P^{(r+u)} \cap K = \emptyset$, which implies that $P^{(r+u+1)} \subset \{x \mid wx \leq d'' - 1\}$, contradicting our assumption.

We prove that for any polyhedron Q with $Q \subset \{x \mid wx \leq d''\}$ one has $Q' \cap K \subset (Q \cap K)'$ (and equality follows). It is sufficient to show that for each half-space H with $Q \cap K \subset H$ there is a half-space G such that $Q \subset G$ and $G_I \cap K \subset H_I$.

Let $H = \{x \mid vx \leq e\}$ be such a half-space, where the vector v consists of relatively prime integers. As

$$H \supset Q \cap K = Q \cap \{x \mid -wx \leq -d''\}, \tag{5}$$

there exist, by Farkas' lemma, $\lambda \geq 0$, and v', e' such that $v = v' - \lambda w$, $e \geq e' - \lambda d''$, and $Q \subset \{x \mid v'x \leq e'\}$. We may suppose that λ is an integer, since replacing v' by $v' + \mu w$, λ by $\lambda + \mu$, and e' by $e' + \mu d''$, for some nonnegative number μ , does not violate the required properties of v' . Let $G = \{x \mid v'x \leq e'\}$. As $v' = v + \lambda w$ is integral we have that

$$G_I \cap K \subset \{x \mid v'x \leq \lfloor e' \rfloor, wx = d''\} \subset \{x \mid vx \leq \lfloor e \rfloor\} = H_I, \tag{6}$$

which finishes the proof.

4. Some remarks

(i) Chvátal restricted himself to bounded polyhedra, but he considered real spaces. However, the analogue of Theorem 2 for bounded polyhedra in real space may be derived from Theorem 2 as follows (note that the analogue of Theorem 2 is, in general, not true for unbounded polyhedra in real space, as is shown by the polyhedron $\{(x, x\sqrt{2}) \mid x \geq 0\}$). If P is a compact convex region in real euclidean space, define $P' = \bigcap_H H_I$ where the intersection ranges over all rational half-spaces H with $P \subset H$ (a polyhedron is *rational* if it is determined by rational linear inequalities, i.e., if it is the convex hull of the rational vectors contained in it). Now one easily proves that each compact convex region P is contained in a bounded rational polyhedron Q such that $P_I = Q_I$. It follows from Theorem 2 that $Q^{(t)} = Q_I$ for some t . Since $P_I \subset P^{(t)} \subset Q^{(t)}$, it follows that also $P^{(t)} = P_I$.

Chvátal worked with collections of linear inequalities rather than convex geometrical objects. If \mathcal{L} is a (possibly infinite) collection of linear inequalities in the vector variable x , then \mathcal{L}' is defined to be the collection of linear inequalities occurring in \mathcal{L} , together with all linear inequalities $wx \leq d$, where w is an integral vector and d is an integer, such that $wx \leq d'$ is a nonnegative linear combination of a finite number of linear inequalities in \mathcal{L} , for some $d' < d + 1$.

Now if P is the set of vectors x satisfying all inequalities in \mathcal{L} , then P' is the set of vectors x satisfying all inequalities in \mathcal{L}' , provided that P is bounded. This follows from the fact that if $P \subset H$, where H is a rational half-space, then there exists a rational half-space K such that $P \subset K$, $H_I = K_I$ and the linear inequality defining K is a nonnegative linear combination of a finite number of inequalities

occurring in \mathcal{L} . Hence, for some t , P_t is the set of all vectors satisfying all inequalities in $\mathcal{L}^{(t)}$. One easily checks that this implies that P_t is determined by finitely many inequalities in $\mathcal{L}^{(t+1)}$.

We do not know whether the analogue of Theorem 1 is true in real spaces. We were able to show only that if P is a bounded polyhedron in real space, and P' has empty intersection with the boundary of P , then P' is a (rational) polyhedron.

(ii) In [12] we proved that for each polyhedron P with full dimension there exists a unique minimal totally dual integral system $Ax \leq b$, with A integral, such that $P = \{x \mid Ax \leq b\}$. Now the proof of Theorem 1 above gives: $P' = \{x \mid Ax \leq \lfloor b \rfloor\}$. However, not every linear inequality in $Ax \leq \lfloor b \rfloor$ is necessary for defining P' . Otherwise, each face of P of codimension 1 would have a parallel face in P' , which is obviously not true.

(iii) As corollaries of Theorem 2 we have:

(a) each face of a polyhedron P contains lattice points, if and only if each supporting hyperplane of P contains lattice points (this is equivalent to: $P = P_t$ if and only if $P = P'$; one may derive from this that any totally dual integral system of linear inequalities with integral right-hand sides yields a polyhedron each face of which contains lattice points—cf. (ii) of Section 2);

(b) any affine subspace containing no lattice points is contained in a hyperplane containing no lattice points (this is equivalent to: a system $Ax = b$ of linear equations has no integral solution, if and only if there exists a vector y such that yA is integral and yb is not an integer—cf. Van der Waerden [14, Section 108]; for real spaces this theorem may be proved easily by induction on the codimension of the subspace, using cardinality arguments).

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