On lower bounds for permanents

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ABSTRACT

Let Λ_n^k be the collection of $n \times n$ matrices with nonnegative integer coefficients such that each row and column sum equals k. We prove that

$$\min \{\operatorname{per} A \mid A \in \Lambda_n^k\} \leq k^{2n}/\binom{nk}{k}.$$

1. INTRODUCTION AND STATEMENT OF RESULTS

Let Λ_n^k be the collection of $n \times n$ matrices with nonnegative integer coefficients such that each row and column sum equals k. We define

(1) $\lambda_k(n) := \min \{\operatorname{per} A \mid A \in \Lambda_n^k\}$

and

$$\theta_k := \inf \{\lambda_k(n)^{1/n} \mid n \in \mathbb{N}\}.$$

If $A \in \Lambda_n^k$, $B \in \Lambda_m^k$ such that $\operatorname{per} A = \lambda_k(n)$ and $\operatorname{per} B = \lambda_k(m)$, then

$$\lambda_k(n + m) \leq \operatorname{per} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \lambda_k(n)\lambda_k(m).$$

Combining this observation with Fekete's lemma, this ensures that, actually,

$$\theta_k = \lim_{n \to \infty} (\lambda_k(n))^{1/n}.$$

Trivially, $\lambda_1(n) = 1$. It is easy to show that $\lambda_2(n) = 2$, by considering the matrices with entries 1 in the main diagonal and an adjacent diagonal. So $\theta_1 = \theta_2 = 1$. Erdös and Rényi [2] conjectured that $\theta_k > 1$ for $k \geq 3$. This conjecture was proved by M. Voorhoeve [7], T. Bang [1] and S. Friedland [3]. Voorhoeve [7] showed by elementary methods that

$$\lambda_3(n) \geq 3 \cdot \left(\frac{4}{3}\right)^{n-1},$$

so $\theta_3 \geq {}^4/_3$. Friedland [3] and Bang [1] proved by a method due to Bang that the permanent of a doubly stochastic $n \times n$ matrix is at least e^{-n}. This immediately yields

$$\lambda_k(n) \geq \left(\frac{k}{e}\right)^n; \ \theta_k \geq \frac{k}{e}.$$

Recently by a different application of Bang's method, Valiant [6] showed that $\theta_4 \geq {}^3/_2$ and $\theta_6 \geq {}^{20}/_9$. J.H. van Lint [4] gave the upper bound $\theta_3 < 1.466$. For an additional survey we refer to the book on permanents by H. Minc [5].

In this paper we give the following general upper bound for the numbers $\lambda_k(n)$.

THEOREM. The numbers $\lambda_k(n)$ defined by (1) satisfy

$$\lambda_k(n) \leq k^{2n} / \binom{nk}{k}.$$

By Stirling's formula, this yields

COROLLARY 1.

$$\theta_k \leq \frac{(k-1)^{k-1}}{k^{k-2}}$$

COROLLARY 2.

$$\theta_3 = {}^4/_3.$$

PROOF. Our upper bound and Voorhoeve's lower bound for θ_3 coincide, hence we have equality for $k = 3$. □

In the proof of our theorem the permanent is represented by the number of common systems of distinct representatives of two partitions. Our theorem is inspired by Bang's "tensor product formula" (cf. Bang [1]; Friedland [3]). A further inspection of this formula, (see Valiant [6]) and the results for θ_1, θ_2 and θ_3 lead us to conjecturing that our upper bound for the numbers θ_k is optimal.

CONJECTURE.

$$\theta_k = \frac{(k-1)^{k-1}}{k^{k-2}}.$$

2. PROOF OF THE THEOREM

Let $P_{k,n}$ be the collection of all ordered partitions of $\{1,2,\ldots,nk\}$ into n classes of size k. We have

$$p_{k,n} := |P_{k,n}| = \frac{(nk)!}{(k!)^n}.$$

A system of distinct representatives (SDR) of such a partition $\mathscr{A}=(A_1,\ldots,A_n)$ is a collection Γ of n elements such that $|\Gamma \cap A_i| = 1$ for all i. Clearly, the number of SDR's of $\mathscr{A}$ equals k^n. Now take $\mathscr{A}=(A_1,\ldots,A_n)$ and $\mathscr{B}=(B_1,\ldots,B_n)$ from $P_{k,n}$ and let $S(\mathscr{A},\mathscr{B})$ be the collection of *common* SDR's of $\mathscr{A}$ and $\mathscr{B}$. We claim that the number $s(\mathscr{A},\mathscr{B}) := |S(\mathscr{A},\mathscr{B})|$ is equal to the permanent of the matrix $\alpha = (\alpha_{i,j})$, where

$$\alpha_{i,j} = |A_i \cap B_j| \quad (i,j=1,\ldots,n).$$

Indeed, if σ is a permutation of $\{1,\ldots,n\}$, then $\prod_{i=1}^n \alpha_{i,\sigma(i)}$ is the number of common SDR's $(\gamma_1,\ldots,\gamma_n)$ of $\mathscr{A}$ and $\mathscr{B}$ such that $\gamma_i \in A_i \cap B_{\sigma(i)}$. Hence

$$\text{per } \alpha = \sum_{\sigma \in S_n} \prod_{i=1}^n \alpha_{i,\sigma(i)} = s(\mathscr{A},\mathscr{B}).$$

Moreover, since $\sum_{i=1}^n \alpha_{ij} = |B_j| = k = |A_i| = \sum_{j=1}^n \alpha_{ij}$, we have that $\alpha \in \Lambda_k^n$. Hence

(2) $$s(\mathscr{A},\mathscr{B}) \geq \lambda_k(n).$$

Let $\mathscr{A} \in P_{k,n}$ be given. A fixed SDR $\Gamma=(\gamma_1,\ldots,\gamma_n)$ of $\mathscr{A}$ is also an SDR of $n! p_{k-1,n}$ partitions $\mathscr{B}$ in $P_{k,n}$, as we can distribute $\gamma_1,\ldots,\gamma_n$ in $n!$ ways among $B_1,\ldots,B_n$, whereas the other elements of $B_1,\ldots,B_n$ can be chosen freely. Since $\mathscr{A}$ has k^n SDR's, we find

$$\sum_{\mathscr{B} \in P_{k,n}} s(\mathscr{A},\mathscr{B}) = k^n n! p_{k-1,n}.$$

Hence by (2)

$$\lambda_k(n) \leq \frac{k^n \cdot n! p_{k-1,n}}{p_{k,n}} = \frac{k^n \cdot n! (k!)^n (nk-n)!}{((k-1)!)^n (nk)!},$$

proving our theorem. $\square$

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