tion for sequences, (A1) becomes

$$\epsilon^2 = \frac{1}{T}\int_0^T d\tau \int_{-1/2}^{1/2} dw S_{\hat{d}}(w)|C_\tau(w)|^2, \quad \text{(A2a)}$$

where

$$C_\tau(w) = \sum_{n=-\infty}^{\infty} c_\tau(n)e^{-i2\pi wn}, \qquad -\infty < w < \infty, \quad \text{(A2b)}$$

is the spectrum of $c_\tau(\cdot)$. Now $c_\tau(n)$ is the sample of $g_1(\tau - t)$ at $t = nT$. Since $g_1(\tau - t)$ $(-\infty < t < \infty)$ is band-limited to $|f| \leqslant 1/2T$ and has Fourier transform $G^*(f)e^{-i2\pi f\tau}$, we conclude from (3) that

$$C_\tau(w) = \frac{1}{T}G_1^*\left(\frac{w}{T}\right)\exp\left\{-\frac{i2\pi w\tau}{T}\right\}, \qquad |w| \leqslant \frac{1}{2}.$$

Thus (A2) is

$$\epsilon^2 = \frac{1}{T}\int_{-1/2}^{1/2} dw\, S_{\hat{d}}(w)\left|\frac{G_1\left(\frac{w}{T}\right)}{T}\right|^2 \int_0^T d\tau$$

$$= \int_{-1/2}^{1/2} S_{\hat{d}}(w)\left|\frac{G_1\left(\frac{w}{T}\right)}{T}\right|^2 dw,$$

which is the theorem.

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A Comparison of the Delsarte and Lovász Bounds

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***Abstract*—Delsarte's linear programming bound (an upper bound on the cardinality of cliques in association schemes) is compared with Lovász's θ-function bound (an upper bound on the Shannon capacity of a graph). The two bounds can be treated in a uniform fashion. Delsarte's linear programming bound can be generalized to a bound $\theta'(G)$ on the independence number $\alpha(G)$ of an arbitrary graph G, such that $\theta'(G) \leqslant \theta(G)$. On the other hand, if the edge set of G is a union of classes of a symmetric association scheme, $\theta(G)$ may be calculated by linear programming. For such graphs the product $\theta(G)\cdot\theta(\bar{G})$ is equal to the number of vertices of G.**

I. Introduction

The PURPOSE of this note is to compare two upper bound functions, both applying to numbers which are more-or-less motivated by information-theoretic problems: Delsarte's linear programming bound on the cardinality of cliques in association schemes and Lovász's θ-function bound on the Shannon capacity of a graph. The first bound may be conceived of as a bound for the independence number $\alpha(G)$ of a certain graph G, whereas Lovász's bound limits $\alpha(G^k)$, the independence number of the normal product of k copies of G.

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We first briefly state these bounds and their background. (A *graph* is an undirected graph without loops or multiple edges.)

Association Schemes and Delsarte's Linear Programming Bound [2], [5]

A pair $(X, \mathfrak{R})$, where $\mathfrak{R} = (R_0, \cdots, R_n)$ is a partition of $X \times X$, is called a (*symmetric*) *association scheme* with *intersection numbers* p_{ij}^k $(i,j,k = 0,\cdots,n)$ if

$$R_0 = \{(x,x)|x \in X\}; \quad (1)$$

$$R_k^{-1} = \{(y,x)|(x,y) \in R_k\} = R_k, \qquad \text{for } k = 0,\cdots,n; \quad (2)$$

for all $i,j,k = 0,\cdots,n$, and $(x,y) \in R_k$:

$$|\{z|(x,z) \in R_i \text{ and } (z,y) \in R_j\}| = p_{ij}^k. \quad (3)$$

So $p_{ij}^k = p_{ji}^k$. We may consider the pair (X, R_i) as a graph, for $i = 1,\cdots,n$. (X,R_i) is regular of valency $v_i = p_{ii}^0$ $(v_0 = 1)$. Therefore, $p_{ij}^0 = \delta_{ij}v_i$. Let D_i be the adjacency matrix of (X, R_i); D_0 is the identity matrix. Since, by (3), the symmetric matrices $D_0,\cdots,D_n$ commute, there exists a matrix $P = (P_k^u)_{k,u=0}^n$ such that $P_k^0,\cdots,P_k^n$ are the eigenvalues of D_k $(k = 0,\cdots,n)$, and the eigenvalues $P_0^u,\cdots,P_n^u$ of $D_0,\cdots,D_n$, respectively, have a common eigenvector ($u =$

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$0,\cdots,n$). We may assume that $P_k^0 = v_k$ for all k. Set

$$Q_k^u = \frac{\mu_u}{v_k} P_k^u \tag{4}$$

where μ_u is the dimension of the common eigenspace of $D_0,\cdots,D_n$ belonging to $P_0^u,\cdots,P_n^u$, respectively ($u=0,\cdots,n$). It can be shown that

$$\sum_{u=0}^{n} P_k^u Q_l^u = m\cdot\delta_{kl} \quad\text{and}\quad \sum_{k=0}^{n} P_k^u Q_k^v = m\cdot\delta_{uv}, \tag{5}$$

where $m=|X|$. So P and $m^{-1}\cdot Q^T$ are inverse matrices.

Coding theorists are interested in two families of association schemes: the so-called Hamming and Johnson schemes. Let n and q be natural numbers, and let X be the set of vectors of length n, with entries in $\{0,\cdots,q-1\}$. For $k=0,\cdots,n$ let

$$R_k = \{(x,y)\in X\times X \mid d_H(x,y)=k\} \tag{6}$$

where $d_H(x,y)$ denotes the *Hamming distance* between the vectors x and y, i.e., the number of coordinate places in which x and y differ. Let $\mathfrak{R}=(R_0,\cdots,R_n)$. As can be easily checked $(X,\mathfrak{R})$ is a symmetric association scheme; schemes obtained in this way are called *Hamming schemes*. For Hamming schemes the values of v_k, μ_u and P_k^u are given by

$$v_k = \binom{n}{k}\cdot(q-1)^k \qquad \mu_u=\binom{n}{u}\cdot(q-1)^u$$

$$P_k^u = K_k(u) = \sum_{j=0}^{k}(-1)^j(q-1)^{k-j}\binom{u}{j}\binom{n-u}{k-j}$$

$$= \sum_{j=0}^{k}(-q)^j(q-1)^{k-j}\binom{n-j}{k-j}\binom{u}{j}, \tag{7}$$

for $k,u=0,\cdots,n$ ($K_k(u)$ is a *Krawtchouk polynomial* of degree k in the variable u).

The second family is obtained as follows. Let v and n be natural numbers and let X be the set of 0,1-vectors of length v with exactly n ones ($n\leqslant \frac{1}{2}v$). For $k=0,\cdots,n$, let

$$R_k = \{(x,y)\in X\times X \mid d_J(x,y)=k\} \tag{8}$$

where $d_J(x,y)=\frac{1}{2}d_H(x,y)$ is the *Johnson distance* between x and y. Let $\mathfrak{R}=(R_0,\cdots,R_n)$. Then $(X,\mathfrak{R})$ is a symmetric association scheme; schemes constructed in this manner are called *Johnson schemes*. Their parameters are

$$v_k = \binom{n}{k}\binom{v-n}{n-k}$$

$$\mu_u = \binom{v}{u}-\binom{v}{u-1} = \frac{v-2u+1}{v-u+1}\cdot\binom{v}{u}$$

$$P_k^u = E_k(u)$$

$$= \sum_{j=0}^{k}(-1)^{k-j}\binom{n-j}{k-j}\binom{n-u}{j}\binom{v-n+j-u}{j}$$

$$= \sum_{j=0}^{k}(-1)^j\binom{u}{j}\binom{n-u}{k-j}\binom{v-n-u}{k-j}, \tag{9}$$

for $k,u=0,\cdots,n$ ($E_k(u)$ is an *Eberlein polynomial* of degree $2k$ in the variable u).

(A third family of symmetric association schemes is given by *strongly regular graphs*. These are exactly those graphs (X,R_1) such that $(X,\mathfrak{R})$ is a symmetric association scheme, where $\mathfrak{R}=(R_0,R_1,R_2)$, $R_2=(X\times X)\setminus(R_0\cup R_1)$. It follows that the complementary graph of a strongly regular graph is also strongly regular.)

The main problem in combinatorial coding theory is to estimate the maximum size of any subset or "code" C of a Hamming or Johnson scheme such that no two elements in C have (Hamming or Johnson) distance less than a given value d. To translate this problem into the language of association schemes, we need the notion of an M-clique: given $0\in M\subset\{0,\cdots,n\}$ a subset Y of X is an *M-clique* if $(x,y)\in\cup_{k\in M}R_k$ for all $x,y\in Y$. So the coding problem is to determine the maximum cardinality of $\{0,d,d+1,\cdots,n\}$-cliques in Hamming and Johnson schemes.

To obtain an upper bound on the size of a clique in a symmetric association scheme $(X,\mathfrak{R})$ define, for $Y\subset X$, the *inner distribution* $(a_0,\cdots,a_n)$ of Y by

$$a_k = \frac{|R_k\cap(Y\times Y)|}{|Y|}, \tag{10}$$

for $k=0,\cdots,n$; so $a_0=1$ and $\Sigma_{k=0}^n a_k=|Y|$. Moreover, if Y is an M-clique, then $a_k=0$ if $k\notin M$. Delsarte showed that the inner distribution of any subset Y of X satisfies

$$\sum_{k=0}^{n} a_k Q_k^u \geqslant 0, \tag{11}$$

for $u=0,\cdots,n$. Therefore, for any M-clique Y,

$$|Y| \leqslant \max\left\{\sum_{k=0}^{n} a_k \,\middle|\, a_0,\cdots,a_n\geqslant 0;\ a_0=1;\ a_k=0 \text{ for } k\notin M;\ \sum_{k=0}^{n} a_k Q_k^u \geqslant 0\right\}$$

$$= \min\left\{\sum_{u=0}^{n} b_u \,\middle|\, b_0,\cdots,b_n\geqslant 0;\ b_0=1;\ \sum_{u=0}^{n} b_u P_k^u \leqslant 0 \text{ for } k\in M\setminus\{0\}\right\}. \tag{12}$$

The equality in (12) follows from the duality theorem of linear programming. This bound on the size of a clique is called *Delsarte's linear programming bound*. One may apply linear programming techniques to calculate its value —see [1] for applications in coding theory.

The following result of Delsarte shows that the linear programming bound is a sharpening of the Hamming bound in coding theory. Let $(X,\mathfrak{R})$ be a symmetric association scheme, with $\mathfrak{R}=(R_0,\cdots,R_n)$, and let $0\in M\subset\{0,\cdots,n\}$ and $\overline{M}=\{0\}\cup(\{0,\cdots,n\}\setminus M)$. Then

> the product of the linear programming bound for M-cliques and the linear programming bound for $\overline{M}$-cliques is at most $|X|$. (13)

Hence $|Y|\cdot|Z|\leqslant|X|$ for M-cliques Y and $\overline{M}$-cliques Z. By taking $M=\{0,d,d+1,\cdots,n\}$ in a Hamming scheme, we obtain the Hamming bound.

The Shannon capacity and Lovász's bound

For any graph G, Lovász [4] defined a number $\theta(G)$, which is an upper bound on the "Shannon capacity" $\Theta(G)$. Let $\alpha(G)$ be the maximum number of independent (i.e., pairwise nonadjacent) points in a graph G, and let $G\cdot H$ denote the (normal) product of the graphs G and H, i.e., the point set of $G\cdot H$ is the Cartesian product of the point sets of G and H, and two distinct points of $G\cdot H$ are adjacent iff in both coordinate places the elements are adjacent or equal. G^k denotes the product of k copies of G.

Shannon [9] introduced the number

$$\Theta(G)=\sup_k \sqrt[k]{\alpha(G^k)} = \lim_{k\to\infty} \sqrt[k]{\alpha(G^k)} \qquad (14)$$

now called the *Shannon capacity* of G. If one considers the points of G as letters in an alphabet, two points being adjacent iff they are "confoundable," then $\alpha(G^k)$ may be interpreted as the maximum number of k-letter messages such that any two of them are inconfoundable in at least one coordinate place.

Since $\alpha(G)^k \leqslant \alpha(G^k)$, it follows that $\alpha(G)\leqslant\Theta(G)$. Equality does not hold in general; e.g., $\alpha(C_5)=2$, whereas $\alpha(C_5^2)=5\leqslant\Theta(C_5)^2$. Lovász showed that $\Theta(C_5)=\sqrt{5}$. In fact, Lovász gave a general upper bound for $\Theta(G)$ as follows.

Let $G=(V,E)$ be a graph, with vertex set $V=\{1,\cdots,n\}$, and define

$$\theta(G)=\min\ \{\text{lev } A \mid A=(a_{ij}) \text{ is a symmetric } n\times n\text{-matrix such that } a_{ij}=1 \text{ if } \{i,j\}\notin E\}, \qquad (15)$$

where lev A denotes the largest eigenvalue of A. Now if $\alpha(G)=k$, each matrix A satisfying the conditions mentioned in (15) has a $k\times k$ all-one principal submatrix (with largest eigenvalue k), hence lev $A\geqslant k$. Therefore, $\alpha(G)\leqslant\theta(G)$. Since, as Lovász proved, $\theta(G\cdot H)=\theta(G)\cdot\theta(H)$ for all graphs G and H, it follows that $\alpha(G^k)\leqslant\theta(G)^k$, which yields the stronger inequality $\Theta(G)\leqslant\theta(G)$ (Haemers [3] showed the existence of graphs G with $\Theta(G)<\theta(G)$). Moreover, Lovász showed that

$$\theta(G)=\max\ \{\textstyle\sum_{i,j}b_{ij} \mid B=(b_{ij}) \text{ is an } n\times n \text{ positive semi-definite matrix, with Tr } B=1, \text{ and } b_{ij}=0 \text{ whenever } \{i,j\}\in E\}. \qquad (16)$$

So $\theta(G)$ may be considered as both a maximum and a minimum, which makes the function θ easier to handle. Lovász found, *inter alia*, that for graphs G with n points

$$\theta(G)\cdot\theta(\overline{G})\geqslant n \text{ (where } \overline{G} \text{ is the complementary graph), with equality if } G \text{ is vertex-transitive,} \qquad (17)$$

and

$$\theta(G)\leqslant\frac{-n\lambda_n}{\lambda_1-\lambda_n} \text{ if } G \text{ is regular } (\lambda_1 \text{ and } \lambda_n \text{ being the largest and smallest eigenvalues of the adjacency matrix of } G\text{), with equality if } G \text{ is edge-transitive.} \qquad (18)$$

A consequence of (18) is, let $v\geqslant 2n$ and let $K(v,n)$ be the graph whose vertices are the n-subsets of some fixed v-set, two vertices being adjacent iff they are disjoint; such graphs are called Kneser graphs. Then

$$\theta(K(v,n))=\binom{v-1}{n-1} \qquad (19)$$

(by (18) it is sufficient to calculate the eigenvalues of $K(v,n)$), generalizing the Erdös–Ko–Rado theorem, which says that

$$\alpha(K(v,n))=\binom{v-1}{n-1}.$$

The theories of Delsarte and Lovász appear to have certain common characteristics, such as bounding cliques or independent sets in graphs, using eigenvalue techniques on matrices determined by graphs, relating a graph and its complement, and being applicable to allied structures such as "constant weight codes" and Kneser graphs. The purpose of this note is to go further into this relationship.

Clearly Delsarte's linear programming bound may be conceived of as an upper bound on $\alpha(G)$ for graphs G whose edge set is a union of classes R_i of a symmetric association scheme $(X,\mathfrak{R})$. We show that Delsarte's bound can be extended to a bound $\theta'(G)$ on $\alpha(G)$ for arbitrary graphs G; the description of $\theta'(G)$ has many features in common with Lovász's $\theta(G)$. It will follow that $\theta'(G)\leqslant\theta(G)$ (in general $\theta'(G)\neq\theta(G)$). On the other hand, if the edge set of G is a union of classes of a symmetric association scheme $(X,\mathfrak{R})$, the number $\theta(G)$ may be calculated by means of a linear program obtained from (12) by dropping the nonnegativity constraints for $a_0,\cdots,a_n$. It follows that for these graphs G, one also has $\theta(G)\cdot\theta(\overline{G})=|X|$ (cf. (13) and (17)).

II. A Comparison of the Delsarte and Lovász Bounds

First recall the following strong form of the duality theorem of linear programming. Let C and D be closed convex cones in $\boldsymbol{R}^k$ and $\boldsymbol{R}^m$, respectively, with dual cones C^* and D^* (that is, C^* consists of all vectors in $\boldsymbol{R}^k$ having a nonnegative inner product with each element of C). Let M be a real-valued $m\times k$-matrix, and let $c\in\boldsymbol{R}^k$ and $d\in\boldsymbol{R}^m$. Then

$$\max\ \{cx \mid x\in C;\ d-Mx\in D\} = \min\ \{yd \mid y\in D^*;\ yM-c\in C^*\}, \qquad (20)$$

provided that these sets are nonempty and closed. Furthermore, notice that the closed convex cone of all real-valued symmetric positive semi-definite $n\times n$-matrices, conceived of as n^2-vectors, has as dual cone the set of real-valued $n\times n$-matrices U such that $y^TUy\geqslant 0$ for all real n-vectors y. (So symmetric matrices in the dual cone are positive semi-definite.) For convenience we use the following inner product notation for $n\times n$-matrices $A=(a_{ij})$ and $B=(b_{ij})$:

$$A*B=\sum_{i,j=1}^{n} a_{ij}\cdot b_{ij}, \tag{21}$$

that is, $A*B=\mathrm{Tr}(A^TB)$. So $A*I=\mathrm{Tr}\,A$ and $A*J=\sum_{i,j=1}^{n}a_{ij}$.

Let G be a graph, with point set $\{1,\cdots,n\}$. Lovász defined

$$\theta(G)=\max\ \{\textstyle\sum_{i,j}b_{ij}|B=(b_{ij}) \text{ is a symmetric positive semi-definite } n\times n\text{-matrix with } \mathrm{Tr}\ B=1, \text{ and } b_{ij}=0 \text{ if } \{i,j\}\in E\}=\min\ \{\text{lev } A|A=(a_{ij}) \text{ is a symmetric } n\times n\text{-matrix with } a_{ij}=1 \text{ if } \{i,j\}\notin E\}. \tag{22}$$

Now define $\theta'(G)$ as follows:

$$\theta'(G)=\max\ \{\textstyle\sum_{i,j}b_{ij}|B=(b_{ij}) \text{ is a nonnegative symmetric positive semi-definite } n\times n\text{-matrix with } \mathrm{Tr}\ B=1, \text{ and } b_{ij}=0 \text{ if } \{i,j\}\in E\}, \tag{23}$$

which differs from (22) in that only nonnegative matrices B are considered.

Theorem 1: $\alpha(G)\leqslant\theta'(G)\leqslant\theta(G)$.

Proof: Clearly $\theta'(G)\leqslant\theta(G)$. Suppose $Y\subset\{1,\cdots,n\}$ is an independent set with $\alpha(G)=k$ elements. Define $b_{ij}=1/k$, if $i,j\in Y$, and $b_{ij}=0$, otherwise. Then $B=(b_{ij})$ is nonnegative and positive semi-definite with trace 1, and $b_{ij}=0$ if $\{i,j\}\in E$. Furthermore $\sum_{i,j}b_{ij}=k$. Hence $\alpha(G)=k\leqslant\theta'(G)$. □

Theorem 2: $\theta'(G)=\min$ {lev $A|A=(a_{ij})$ is a symmetric $n\times n$-matrix with $a_{ij}\geqslant 1$ if $\{i,j\}\notin E$}.

Proof: By definition

$$\theta'(G)=\max\ \{B*J|B=(b_{ij}) \text{ is a symmetric positive semi-definite } n\times n\text{-matrix such that: } B*I=1,\ B*F_{ij}=0 \text{ for } \{i,j\}\in E, \text{ and } B*F_{ij}\geqslant 0 \text{ for } \{i,j\}\notin E\} \tag{24}$$

where F_{ij} is the $n\times n-(0,1)$-matrix with only ones in the positions (i,j) and (j,i). From the above-mentioned form of the duality theorem it follows that this maximum equals

$$\min\ \{\lambda\in\boldsymbol{R}|M=(m_{ij}) \text{ is a symmetric } n\times n\text{-matrix}; m_{ij}\leqslant 0 \text{ if } \{i,j\}\notin E;\ \lambda I+M-J \text{ is positive semi-definite}\}. \tag{25}$$

Putting $A=J-M$ and using the fact that for symmetric A, the largest eigenvalue of A is equal to the minimum value of λ such that $\lambda I-A$ is positive semi-definite, one obtains

$$\theta'(G)=\min\ \{\text{lev } A|A=(a_{ij}) \text{ is a symmetric } n\times n\text{-matrix such that } a_{ij}\geqslant 1 \text{ if } \{i,j\}\notin E\}. \quad \square \tag{26}$$

Since the largest eigenvalue of a matrix is not increased by decreasing diagonal elements, we may suppose that the minimum is attained by some A with ones on the diagonal.

We next prove that for graphs derived from symmetric association schemes $\theta'(G)$ coincides with Delsarte's linear programming bound.

Let $(X,\mathfrak{R})$ be a symmetric association scheme, with $\mathfrak{R}=(R_0,\cdots,R_n)$, and let $0\in M\subset\{0,\cdots,n\}$. Let $G=(X,E)$ be the graph with $E=\cup_{i\notin M}R_i$. Clearly M-cliques in the association scheme coincide with independent sets in G.

Theorem 3: $\theta'(G)$ is equal to the linear programming bound for M-cliques in $(X,\mathfrak{R})$.

Proof: The linear programming bound is, by definition (cf. (12))

$$\max\ \{\textstyle\sum_{k=0}^{n}a_k|a_0,\cdots,a_n\geqslant 0;\ a_0=1;\ a_k=0 \text{ for } k\notin M;\ \sum_{k=0}^{n}a_kQ_k^u\geqslant 0 \text{ for } u=0,\cdots,n\}. \tag{27}$$

Let $a_0,\cdots,a_n$ attain this maximum, and put

$$(b_{ij})=B=\sum_{k=0}^{n}\frac{a_k}{m\cdot v_k}D_k, \tag{28}$$

where m, v_k, and D_k are as in Section I. Then B satisfies the conditions mentioned in (23); B is positive semi-definite since, by the commutativity of $D_0,\cdots,D_n$, the matrix B has eigenvalues

$$\sum_{k=0}^{n}\frac{a_k}{m\cdot v_k}P_k^u=\sum_{k=0}^{n}\frac{a_k}{m\cdot\mu_u}Q_k^u, \tag{29}$$

for $u=0,\cdots,n$. Since $D_k*J=v_k\cdot m$, it follows that $B*J=\sum_{i,j}b_{ij}=\sum_k a_k$. Therefore the linear programming bound is at most $\theta'(G)$. To prove the converse, let $b_0,\cdots,b_n$ attain the minimum in (12), and let $\lambda=\sum_u b_u$. Define

$$\begin{aligned}A&=\lambda I-\sum_{k,u=0}^{n}\frac{b_u}{\mu_u}Q_k^u\cdot D_k+J\\&=\lambda I-\sum_{k=0}^{n}\left(\sum_{u=0}^{n}\frac{b_u}{\mu_u}Q_k^u-1\right)\cdot D_k.\end{aligned} \tag{30}$$

Since $\lambda I-A$ has eigenvalues

$$\sum_{k=0}^{n}\left(\sum_{u=0}^{n}\frac{b_u}{\mu_u}Q_k^u-1\right)\cdot P_k^v=\sum_{u=0}^{n}\frac{b_u}{\mu_u}\cdot m\cdot\delta_{uv}-m\delta_{v0}\geqslant 0 \tag{31}$$

$(v=0,\cdots,n)$, the matrix A has largest eigenvalue at most λ. Furthermore, by (4) and (12), $a_{ij}\geqslant 1$ if $\{i,j\}\notin E$. Therefore, the minimum in (12) is at least the minimum of

Theorem 2, or the linear programming bound is at least $\theta'(G)$. □

If $(X,\mathfrak{R})$ is a Johnson scheme with n classes (cf. Delsarte [2]) and $M=\{0,\cdots,n-1\}$, then $G=K(v,n)$. As Lovász showed that

$$\theta(K(v,n))=\binom{v-1}{n-1},$$

Delsarte's linear programming bound also yields the Erdös–Ko–Rado theorem.

Using techniques similar to those used in the proof of Theorem 3, we can prove the following result for symmetric association schemes $(X,\mathfrak{R})$ and graphs G that are related as mentioned before Theorem 3.

Theorem 4: $\theta(G)=\max\ \{\sum_{k=0}^{n}a_k \mid a_0=1;\ a_k=0 \text{ for } k\notin M;\ \sum_{k=0}^{n}a_k Q_k^u \geqslant 0 \text{ for } u=0,\cdots,n\}=\min\ \{\sum_{u=0}^{n}b_u \mid b_0,\cdots,b_n\geqslant 0;\ b_0=1;\ \sum_{u=0}^{n}b_u P_k^u=0 \text{ for } k\in M\setminus\{0\}\}$.

So for graphs derived from symmetric association schemes there is an easier way to calculate the θ-value. As a generalization of Delsarte's result (13) we have the following.

Theorem 5: Let the edge set E of the graph $G=(V,E)$ be the union of some classes of a symmetric association scheme. Then $\theta(G)\cdot\theta(\overline{G})=|X|$.

Proof: Lovász proved that for all graphs G: $\theta(G)\cdot\theta(\overline{G})\geqslant|X|$. Now suppose E is the union of some classes of an association scheme, as described before Theorem 3. Then by Theorem 4, $\theta(G)=\sum_k a_k$, for some $a_0,\cdots,a_n$, where $a_0=1$, $a_k=0$ for $k\notin M$ and $\sum_k a_k Q_k^u\geqslant 0$ for $u=0,\cdots,n$. Set

$$b_u=\frac{\sum_{k=0}^{n}a_k Q_k^u}{\theta(G)}. \tag{32}$$

Then $b_0,\cdots,b_n\geqslant 0$ and $b_0=1$; furthermore, for $k\notin M$ (cf. (5)):

$$\sum_{u=0}^{n}b_u P_k^u=\frac{1}{\theta(G)}\cdot\sum_{u,l}a_l Q_l^u P_k^u$$
$$=\frac{1}{\theta(G)}\cdot\sum_{l}a_l\cdot m\cdot\delta_{kl}=\frac{a_k\cdot m}{\theta(G)}=0, \tag{33}$$

so $b_0,\cdots,b_n$ satisfy the conditions mentioned in the last line of Theorem 4, with $\overline{G}$ instead of G. Also

$$\sum_{u=0}^{n}b_u=\frac{1}{\theta(G)}\cdot\sum_{k,u}a_k Q_k^u$$
$$=\frac{1}{\theta(G)}\cdot\sum_k a_k\cdot\sum_u Q_k^u$$
$$=\frac{1}{\theta(G)}\cdot\sum_k a_k\cdot m\cdot\delta_{k0}=\frac{|X|}{\theta(G)}. \tag{34}$$

Since, by Theorem 4, $\sum_u b_u\geqslant\theta(\overline{G})$, we have shown that $\theta(G)\cdot\theta(\overline{G})\leqslant|X|$. □

Because there are many strongly regular graphs that are not vertex-transitive (cf. Seidel [8]), Theorem 5 is not included in (17). M. R. Best found the following example of a graph G with $\theta'(G)<\theta(G)$. The points of G are all vectors in $\{0,1\}^6$, two vectors being adjacent if and only if their Hamming distance is at most 3 (so the edge set is the union of some classes of a Hamming scheme). Then $\theta'(G)=4$, whereas $\theta(G)=16/3$.

After completing this research we learned that partially similar results have been independently obtained by McEliece, Rodemich, and Rumsey [7] (cf. [6]). Their functions $\alpha_L(G)$ and $\theta_L(G)$ are equal to $\theta'(G)$ and $\theta(G)$, respectively.

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