SUPEREXTENSIONS WHICH ARE HILBERT CUBES

by

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Abstract

It is shown that each separable metric, not totally disconnected, topological space admits a superextension homeomorphic to the Hilbert cube. Moreover, for simple spaces, such as the closed unit interval or the n-spheres $S_n$, we give easily described subbases for which the corresponding superextension is homeomorphic to the Hilbert cube.

1. Introduction

In [6], DE GROOT defined a space $X$ to be supercompact provided that it possesses a binary closed subbase, i.e., a closed subbase $\mathcal{S}$ with the property that if $\mathcal{S}' \subset \mathcal{S}$ and $\bigcap \mathcal{S}' = \emptyset$ then there exist $S_0, S_1 \in \mathcal{S}'$ such that $S_0 \cap S_1 = \emptyset$. Clearly, according to the lemma of ALEXANDER, every supercompact space is compact. The class of supercompact spaces contains the compact orderable spaces, compact tree-like spaces (BROUWER & SCHRIJVER [4], VAN MILL [10]) and compact metric spaces (STROK & SZYMAŃSKI [14]). Moreover, there are compact Hausdorff spaces which are not supercompact (BELL [2], VAN MILL [12]). There is a connection between supercompact spaces and graphs (see e.g., DE GROOT [7], BRUIJNING [5], SCHRIJVER [13]); moreover, supercompact spaces can be characterized by means of so-called interval structures (BROUWER & SCHRIJVER [4]).

Let $X$ be a $T_1$-space and $\mathcal{S}$ a closed $T_1$-subbase for $X$ (a closed subbase $\mathcal{S}$ for $X$ is called $T_1$ if for all $S \in \mathcal{S}$ and $x \in X$ with $x \notin S$, there exists an $S_0 \in \mathcal{S}$ with $x \in S_0$ and $S_0 \cap S = \emptyset$). The superextension $\lambda_q(X)$ of $X$ relative the subbase $\mathcal{S}$ is the set of all maximal linked systems $\mathcal{S}' \subset \mathcal{S}$ (a subsystem of $\mathcal{S}$ is called linked if every two of its members meet; a maximal linked system or mls is a linked system not properly contained in another linked system) topologized by taking $\{\{S \in \lambda_q(X) \mid S \notin \mathcal{S}'\} \mid S \notin \mathcal{S}\}$ as a closed subbase. Clearly, this subbase is binary, hence $\lambda_q(X)$ is supercompact, while moreover $X$ can be embedded in $\lambda_q(X)$ by the natural embedding $i : X \to \lambda_q(X)$ defined by $i(x) = \{S \in \mathcal{S} \mid x \notin S\}$. VERBEEK’s monograph [15] is a good place to find the basic theorems about superextensions. In this paper we will show that for many spaces there are superextensions homeomorphic to the Hilbert cube $Q$; moreover for simple spaces such as the unit interval or the n-spheres $S_n$ we will present easily described subbases for which the corresponding super-


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extension is homeomorphic to $Q$. Here, a classical theorem of Keller [8], which says that each infinite-dimensional compact convex subset of the separable Hilbert space is homeomorphic to $Q$ (for a more up-to-date proof of this fact, see also Bessaga & Pełczyński [3]), is of great help.

2. Some examples

In this section we will give some examples. If $X$ is an ordered space, then the Dedekind completion of $X$ will be denoted by $\overline{X}$. Roughly speaking, $\overline{X}$ can be obtained from $X$ by filling up every gap. We define $\overline{X}$ to be that ordered space which can be obtained from $X$ by filling up every gap with two points, except for possible endgaps, which we supply with one point. The compact space $\overline{X}$ thus obtained, clearly contains $X$ as a dense subspace. Define

$$\mathcal{Q}_1 = \{ A \subset X \mid \exists x \in X : A = (x, -x) \text{ or } A = [\overline{x}, \overline{x}] \}$$

and

$$\mathcal{F}_1 = \{ A \subset X \mid A \text{ is a closed half-interval} \}$$

(as usual, a half-interval is a subset $A \subset X$ such that either for all $a, b \in X$:

if $b \leq a \in A$ then $b \in A$, or for all $a, b \in X$: if $b \in A$ then $b \in A$)

and

$$\mathcal{F}_2 = \{ A \subset X \mid \exists A_0, A_1 \in \mathcal{F}_1 : A = A_0 \cup A_1 \text{ or } A = A_0 \cap A_1 \},$$

respectively.

Notice that $\mathcal{Q}_1$ equals $\mathcal{F}_1$ in case $X$ is compact or connected. It is easy to see that $\lambda_{\mathcal{Q}_1}(X) \cong \overline{X}$ and that $\lambda_{\mathcal{F}_1}(X) \cong \overline{X}$.

What about $\lambda_{\mathcal{F}_1}(X)$?

**Example (i).** If $X = I$, then $\lambda_{\mathcal{Q}_1}(X) = \lambda_{\mathcal{F}_1}(X) \cong I$. On the other hand $\lambda_{\mathcal{F}_1}(X)$ is homeomorphic to the Hilbert cube $Q$ (see Section 4).

**Example (ii).** If $X = Q$, then $\lambda_{\mathcal{Q}_1}(X) \cong I$ and $\lambda_{\mathcal{F}_1}(X)$ is a non-metrizable separable compact ordered space, which has much in common with the well-known Alexandroff double of the closed unit interval. In this case, $\lambda_{\mathcal{F}_1}(X)$ is a compact totally disconnected perfect space of weight $2^n$. (The total disconnectedness of $\lambda_{\mathcal{F}_1}(X)$ follows from the following observation: for every $T_0, T_1 \in \mathcal{F}_2$ with $T_0 \cap T_1 = \emptyset$ there exists a $T_0 \in \mathcal{F}_2$ such that $T_0 \subset T_0$ and $T_0 \cap T_1 = \emptyset$ and $X \setminus T_0 \subset \mathcal{F}_2$. For every finite linked system $\{X \setminus T_1, T_i \in \mathcal{F}_2, i \in \{1, 2, \ldots, n\}\}$ it is easy to construct two distinct mls's $\mathcal{Q}_0$ and $\mathcal{F}_1$ belonging to $\bigcap_{i=1}^n \{ \exists T \in \lambda_{\mathcal{F}_1}(X) \mid T \notin T_i \}$ showing that $\lambda_{\mathcal{F}_1}(X)$ is perfect. Finally $\lambda_{\mathcal{F}_1}$ can be embedded in $\lambda_{\mathcal{F}_1}(X)$; hence weight $\lambda_{\mathcal{F}_1}(X) = 2^n$.}
Example (iii). If $X = R \setminus Q$, then $\lambda_{q_1}(X) \simeq I$, while $\lambda_{q_2}(X) \simeq \lambda_{q_1}(X) \simeq C$, the Cantor discontinuum, for it is easy to see that $\lambda_{q_1}(X)$ and $\lambda_{q_2}(X)$ both are totally disconnected compact metric perfect spaces.

Finally define

$$G_2 = \{ A \subset X | \exists A_0, A_1 \in G_1: A = A_0 \cup A_1 \text{ or } A = A_0 \cap A_1 \}.$$ 

Notice that $G_2$ equals $T_2$ in case $X$ is compact or connected.

Example (i). If $X = I$, then $\lambda_{q_1}(X) \simeq Q$ (Section 4).

Example (ii). If $X = Q$, then $\lambda_{q_1}(X) \simeq Q$.

Example (iii). If $X = R \setminus Q$, then $\lambda_{q_1}(X) \simeq Q$.

The fact that $\lambda_{q_1}(Q) \simeq \lambda_{q_2}(R \setminus Q) \simeq Q$ can be derived from the result $\lambda_{q_1}(I) \simeq Q$. To see this, define

$$G_2 = \{ A \subset I | A \in G_2 \text{ and } A \text{ has rational endpoints} \}$$

and

$$G_2' = \{ A \subset I | A \in G_2 \text{ and } A \text{ has irrational endpoints} \}.$$

By Theorem 5 and Theorem 7 of [11] (cf. Theorem 3.1 below), it follows that

$$\lambda_{q_1}(I) \simeq \lambda_{q_1}(I) \simeq \lambda_{q_1}(Q)$$

and

$$\lambda_{q_1}(I) \simeq \lambda_{q_1}(I) \simeq \lambda_{q_1}(R \setminus Q).$$

3. Superextensions which are Hilbert cubes

In this section we will show that for each separable metric, not totally disconnected topological space $X$, there exists a normal closed $T_1$-subbase $\mathcal{S}$ such that $\lambda_{q}(X)$ is homeomorphic to the Hilbert cube $Q$. First we will give some preliminary definitions and recapitulate some well-known results from the literature, which are needed in the remainder of this section. A closed subset $B$ of $Q$ is called a Z-set ([1]) if for any non-empty homotopically trivial open subset $O$ of $Q$, the set $O \setminus B$ is again non-empty and homotopically trivial. Examples of Z-sets are compact subsets of $(0, 1)^\omega$ and closed subsets of $Q$ which project onto a point in infinitely many coordinates. In fact, Z-sets can be characterized by the property that for every Z-set $B$ there exists an autohomeomorphism $\Phi$ of $Q$ which maps $B$ onto a set which projects onto a point in infinitely many coordinates ([1]). Obviously the property of being a Z-set is a topological invariant. Moreover, it is easy to show that a closed countable union of Z-sets is again a Z-set (cf. Kroonenberg [9]). The importance of Z-sets is illustrated by the following theorem due to Anderson [1].
Theorem. Any homeomorphism between two $Z$-sets in $Q$ can be extended to an autohomeomorphism of $Q$.

We will apply this theorem to show that every separable metric, not totally disconnected topological space $X$ can be embedded in $Q$ in such a way that $Q$ has the structure of a superextension of $X$, i.e., every point of $Q$ represents an $m_s$ in a suitable closed subbase for $X$. The canonical binary subbase for $Q$ is

$$\mathcal{S} = \{ A \subset Q | A = \Pi_{n=1}^\infty [0, x] \text{ or } A = \Pi_{n=1}^\infty [x, 1], \text{ with } n \in \mathbb{N} \text{ and } x \in I \}$$

and consequently, if we embed $X$ in $Q$ in such a way that for every two elements $T_0, T_1 \in \mathcal{S}$ with $T_0 \cap T_1 \neq \emptyset$ we have that $T_0 \cap T_1 \cap X = \emptyset$, then $Q$ is a superextension of $X$; this is a consequence of the following theorem ([11] Theorem 5).

Theorem 3.1. Let $X$ be a subspace of the topological $T_1$-space $Y$. Then $1$ is homeomorphic to a superextension of $X$ if and only if $Y$ possesses a binary, closed subbase $\mathcal{S}$ such that for all $T_0, T_1 \in \mathcal{S}$ with $T_0 \cap T_1 \neq \emptyset$ we have that $T_0 \cap T_1 \cap X \neq \emptyset$.

In particular, in Theorem 3.1 $Y \cong \lambda_{x_0} \mathcal{S}(X)$, where $\mathcal{S} \cap X = \{ T \cap X | T \in \mathcal{S} \}$

Theorem 3.2. For every separable metric, not totally disconnected topological space $X$ there exists a normal closed $T_1$-subbase $\mathcal{S}$ such that $\lambda_{\mathcal{S}}(X)$ is homeomorphic to the Hilbert cube $Q$.

Proof. Assume that $X$ is embedded in $Q(= I^n)$ and let $C$ be a non-trivial component of $X$. Choose a convergent sequence $B$ in $C$. Furthermore, define a sequence $\{ y_n \}_{n=0} \in Q$ by

$$ (y_n)_i = \begin{cases} 1 & \text{if } i \neq n \\ 0 & \text{if } i = n, \end{cases} $$

for $i = 1, 2, \ldots$.

It is clear that

$$ \lim_{n \to \infty} y_n = y_0. $$

Moreover define $z \in Q$ by $z_i = 0$ ($i = 1, 2, \ldots$). Then

$$ E = \{ y_n | n \in \mathbb{N} \} \cup \{ z \} \cup \{ y_0 \} $$

is a convergent sequence and therefore is homeomorphic to $B$. Since $B$ and both are closed countable unions of $Z$-sets in $Q$, they themselves are $Z$-sets. Choose a homeomorphism $\Phi : B \to E$ and extend this homeomorphism to autohomeomorphism of $Q$. This procedure shows that we may assume t
$X$ is embedded in $Q$ in such a way that $E \subseteq C$. Let $T_0, T_1 \in \mathcal{F}$ such that $T_0 \cap T_1 \neq \emptyset$, where $\mathcal{F}$ is the canonical binary closed subbase for $Q$. We need only consider the following 4 cases:

Case 1: $T_0 = \Pi_{n_0}^{-1} [0, x]$; $T_1 = \Pi_{n_1}^{-1} [y, 1]$ ($x \geq y$). Since $z \in T_0$ and $y \in T_1$ and $C$ is connected, it follows that $\emptyset \neq T_0 \cap T_1 \cap C \subset T_0 \cap T_1 \cap X$.

Case 2: $T_0 = \Pi_{n_0}^{-1} [0, x]$; $T_1 = \Pi_{n_1}^{-1} [y, 1]$ ($n_0 \neq n_1$). Then $y \in T_0 \cap T_1 \cap X$.

Case 3: $T_0 = \Pi_{n_0}^{-1} [0, x]$; $T_1 = \Pi_{n_1}^{-1} [0, y]$. Then $y \in T_0 \cap T_1 \cap X$.

Case 4: $T_0 = \Pi_{n_0}^{-1} [x, 1]$; $T_1 = \Pi_{n_1}^{-1} [y, 1]$. Then $y \in T_0 \cap T_1 \cap X$.

This completes the proof of the theorem.

4. A superextension of the closed unit interval

In the present section we will prove that $\lambda_{\mathcal{F}}(I)$ is homeomorphic to the Hilbert cube, where $\mathcal{F} = \{[x, y] \mid x, y \in I\} \cup \{[0, x] \cup [y, 1] \mid x, y \in I\}$. For this purpose we introduce

$$\mathcal{F} = \{f : I \to I \mid f(0) = 0 \text{ and if } x, y \in I \text{ and } x \leq y \text{ then } 0 \leq f(y) - f(x) \leq y - x\}.$$ 

Hence each $f \in \mathcal{F}$ is continuous and monotone non-decreasing. On $\mathcal{F}$ we define a topology by considering $\mathcal{F}$ as a subspace of $C[I, I]$ with the point-open topology. We obtain the same topology on $\mathcal{F}$ by ordering $\mathcal{F}$ partially as follows:

$$f \leq g \text{ iff for each } x \in I : f(x) \leq g(x), \ (f, g \in \mathcal{F}),$$

and then taking as a closed subbase for $\mathcal{F}$ the collection of all subsets of the form $\{f \in \mathcal{F} \mid f \leq f_0\}$ or $\{f \in \mathcal{F} \mid f \geq f_0\}$, where $f_0$ runs through $\mathcal{F}$. We first prove that $\mathcal{F} \cong Q$ and next that $\lambda_{\mathcal{F}}(I) \cong \mathcal{F}$; we conclude that $\lambda_{\mathcal{F}}(I) \cong Q$.

Notice that by Keller’s theorem each compact metrizable convex infinite-dimensional subspace $X$ of $I^1$ is homeomorphic to the Hilbert cube $Q$, since, by the fact that $X$ is metrizable, $X$ can be embedded as a convex subspace of $I^\infty$; finally $I^\infty$ can be affinely embedded in $l_\infty$. This observation will be used in the proof of Theorem 4.1 and Theorem 5.1.

Theorem 4.1. $\mathcal{F} \cong Q$.

Proof. We show that $\mathcal{F}$ is a compact, infinite-dimensional, convex subspace of $I^1$, with countable base; hence, by Keller’s theorem, $\mathcal{F}$ is homeomorphic to the Hilbert cube $Q$. 

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\( \mathcal{F} \) is clearly a convex subspace of \( I^1 \); it is also clear that \((\mathcal{F}, \leq)\), as defined above, is a complete lattice, whence \( \mathcal{F} \) is compact. \( \mathcal{F} \) has a countable subbase, since the collection of all subsets of the forms \( \{ f \in \mathcal{F} : f(x) \leq y \} \) and \( \{ f \in \mathcal{F} : f(x) \leq y \} \) where \( x, y \in Q \cap I \), forms a countable closed subbase for \( \mathcal{F} \).

Finally, \( \mathcal{F} \) is infinite-dimensional, because \( Q \) can be embedded in \( \mathcal{F} \).

For, let \( a = (a_1, a_2, a_3, \ldots) \in I^\mathbb{N} \). Let \( G(a) \) be the smallest function \( f \) in \( \mathcal{F} \) (in the ordering \( \leq \) of \( \mathcal{F} \)) such that for each \( i = 1, 2, 3, \ldots \) the following holds:

\[
\left( \frac{3}{2^{i+1}} \right) \geq \frac{1}{2^{i+1}} + \frac{1}{2^{i+1}} a_i.
\]

It can be seen easily that \( G \) defines a topological embedding of \( Q \) in \( \mathcal{F} \).

**Theorem 4.2.** \( \lambda_Q(I) \cong \mathcal{F} \).

**Proof.** Define a function \( K: \lambda_Q(I) \to I \) by:

\[
K(\mathcal{M}) = \inf \{ x \in I : [0, x] \in \mathcal{M} \}, \quad (\mathcal{M} \in \lambda_Q(I)),
\]

and a function \( H: \lambda_Q(I) \to \mathcal{F} \) by:

\[
H(\mathcal{M}) (i) = \inf \{ x \in I : [0, x] \cup [y, 1] \in \mathcal{M}, \quad x + y = K(\mathcal{M}) + i \},
\]

\[
(i \in I, \mathcal{M} \in \lambda_Q(I)).
\]

We prove that \( H \) is an homeomorphism between \( \lambda_Q(I) \) and \( \mathcal{F} \).

First we observe that:

- \( K(\mathcal{M}) \leq x \) iff \([0, x] \in \mathcal{M}\);
- \( K(\mathcal{M}) \geq x \) iff \([x, 1] \in \mathcal{M}\);
- \( K(\mathcal{M}) = x \) iff \([0, x] \in \mathcal{M} \) and \([x, 1] \in \mathcal{M}\);
- \( H(\mathcal{M}) (i) \leq x \) iff \([0, x] \cup [K(\mathcal{M}) + i] \in \mathcal{M}\);
- \( H(\mathcal{M}) (i) \geq x \) iff \([x, K(\mathcal{M}) + i] \in \mathcal{M}\);
- \( H(\mathcal{M}) (i) = x \) iff \([0, x] \cup [K(\mathcal{M}) + i] \in \mathcal{M}\) and \([x, K(\mathcal{M}) + i] \in \mathcal{M}\);

these facts follows easily from the fact that \( \mathcal{M} \) is a maximal linked system in \( \mathcal{G}_2 \). Also we have \( K(\mathcal{M}) = H(\mathcal{M})(1) \).

Next we show that \( H(\mathcal{M}) \in \mathcal{F} \), for each maximal linked system \( \mathcal{M} \). I n fact (i) \( H(\mathcal{M})(0) = 0 \), for \([0, 0] \cup [K(\mathcal{M}), 1] \in \mathcal{M} \) and \([0, K(\mathcal{M})] \in \mathcal{M} \); (ii) \( i \leq j \), \( H(\mathcal{M})(i) = x \), \( H(\mathcal{M})(j) = y \), then \( x \leq y \), for \([x, K(\mathcal{M}) + j] \in \mathcal{M} \)

\( \subset [x, K(\mathcal{M}) + i \leq x] \in \mathcal{M} \), hence \([x, K(\mathcal{M}) + j] \in \mathcal{M} \) and \( y = H(\mathcal{M})(j) \geq \infty \).
also $y - x \leq j - i$, for $[y - j + i, K(\mathfrak{M}) + i - (y - j + i)] \supseteq [y, K(\mathfrak{M}) + j - y] \subseteq \mathfrak{M}$, hence $x = H(\mathfrak{M})(i) \geq y - j + i$.

$H$ is a one-to-one function, for suppose $\mathfrak{M}_1, \mathfrak{M}_2 \in \lambda_q(I)$, $\mathfrak{M}_1 \supsetneq \mathfrak{M}_2$ and $H(\mathfrak{M}_2) = H(\mathfrak{M}_2)$. Let $a = K(\mathfrak{M}_1) = H(\mathfrak{M}_2)(1) = H(\mathfrak{M}_2)(1) = K(\mathfrak{M}_2)$, i.e., $[0, a] \subseteq \mathfrak{M}_1 \cap \mathfrak{M}_2$ and $[a, 1] \subseteq \mathfrak{M}_1 \cap \mathfrak{M}_2$. Since $\mathfrak{M}_1 \neq \mathfrak{M}_2$ we may suppose that there are $x', y'$ such that $[0, x'] \cup [y', 1] \subseteq \mathfrak{M}_1 \setminus \mathfrak{M}_2$. Since $[0, a] \subseteq \mathfrak{M}_2$ and $[a, 1] \subseteq \mathfrak{M}_2$, we have $x' < a < y'$. Let $i = x' + y' - a \in [x', y'] \subseteq I$. Then since $[0, x'] \cup [a + i - x', 1] = [0, x'] \cup [y', 1] \subseteq \mathfrak{M}_1 \setminus \mathfrak{M}_2$, we find that $H(\mathfrak{M}_2)(i) \leq x' < H(\mathfrak{M}_2)(i)$ and this is a contradiction. $H$ is also a surjection. Take $f \in \mathfrak{S}$ and let:

$$ \mathfrak{L} = \{[f(i), f(1) + i - f(i)] \mid i \in I\} \cup \{[0, f(i)] \cup [f(1) + i - f(i), 1] \mid i \in I\}. $$

Then by definition of $\mathfrak{S}$, it is easy to see that $\mathfrak{L}$ is a linked system in $Q^2$. $\mathfrak{L}$ is contained in some maximal linked system $\mathfrak{M}$ of $Q^2$, and for this $\mathfrak{M}$ it holds that $K(\mathfrak{M}) = f(1)$ while for each $i \in I$ $H(\mathfrak{M})(i) = f(i)$; i.e., $H(\mathfrak{M}) = f$. Finally we prove that $H$ is continuous. Let $i, x \in I$. Then

$$ \{\mathfrak{M} \in \lambda_q(I) \mid H(\mathfrak{M})(i) \leq x\} = \bigcap_{y \in I} \{\mathfrak{M} \in \lambda_q(I) \mid [0, x] \cup [y, 1] \subseteq \mathfrak{M}\} \cup \{\mathfrak{M} \in \lambda_q(I) \mid [0, x] \cup [y, 1] \supseteq [0, x] \cup [y, 1] \supseteq \mathfrak{M}\}, $$

and hence this set is closed. For, let $\mathfrak{M} \in \lambda_q(I)$ such that $H(\mathfrak{M})(i) \leq x$; this last inequality means that $[0, x] \cup [K(\mathfrak{M}) + i - x, 1] \subseteq \mathfrak{M}$. If $y \geq K(\mathfrak{M}) + i - x$, then $[0, y + x - i] \supseteq [0, K(\mathfrak{M})] \subseteq \mathfrak{M}$; if $y \leq K(\mathfrak{M}) + i - x$ then $[0, x] \cup [y, 1] \supseteq [0, x] \cup [K(\mathfrak{M}) + i - x, 1] \subseteq \mathfrak{M}$.

Conversely, suppose that

$$ [0, x] \cup [y, 1] \subseteq \mathfrak{M} \text{ or } [0, x + y - i] \subseteq \mathfrak{M}. $$

for each $y \in I$, then also $[0, x + y - i] \notin \mathfrak{M}$ for each $y < K(\mathfrak{M}) + i - x$; hence $[0, x] \cup [y, 1] \subseteq \mathfrak{M}$; we conclude that $[0, x] \cup [K(\mathfrak{M}) + i - x, 1] \subseteq \mathfrak{M}$, i.e., $H(\mathfrak{M})(i) \leq x$.

In the same way one proves:

$$ \{\mathfrak{M} \in \lambda_q(I) \mid H(\mathfrak{M})(i) \geq x\} = \bigcap_{y \in I} \{\mathfrak{M} \in \lambda_q(I) \mid [x, y] \subseteq \mathfrak{M} \text{ or } [x + y - i, 1] \subseteq \mathfrak{M}\}, $$

and hence is closed.

As a consequence of these two theorems we have, as announced,

**Theorem 4.3.** $\lambda_q(I) \cong Q$. 
5. A superextension of the \( n \)-sphere

In this final section we show that the superextension of the \( n \)-sphere \( S^n \) with respect to the collection of all closed massive \( n \)-balls in \( S^n \) is homeomorphic with the Hilbert-cube. As usual, the \( n \)-sphere \( S^n \) is the space

\[
\left\{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \left| \sum_{i=0}^n x_i^2 = 1 \right. \right\}
\]

and the closed massive \( n \)-ball with centre \( x \in S^n \) and radius \( \varepsilon \geq 0 \) is the set

\[
B(x, \varepsilon) = \{ y \in S^n \mid d(x, y) \leq \varepsilon \}.
\]

Writing \( \mathcal{B} \) for the collection of all closed massive \( n \)-balls in \( S^n \), we will prove that, if \( n \geq 1 \), \( \lambda_\mathcal{B}(S^n) \cong Q \). Obviously \( \lambda_\mathcal{B}(S^n) \) is the superextension of the circle with respect to the set of closed intervals. For the definition of \( \mathcal{B} \) it does not matter whether the euclidian metric of \( \mathbb{R}^{n+1} \) or the sphere metric of \( S^n \) (in this case the distance between \( x \) and \( y \) in \( S^n \) is \( \sqrt{\sum_{i=0}^n x_i^2 y_i^2} \)), i.e., the minimum length of a curve between \( x \) and \( y \) on \( S^n \) is used. However, in the proof of the theorem we need the latter metric and we call this metric \( d \).

Furthermore we define, for each point \( x = (x_0, x_1, \ldots, x_n) \in S^n \), the antipode \( \bar{x} \) of \( x \) by \( \bar{x} = (-x_0, -x_1, \ldots, -x_n) \).

**Theorem 5.1.** If \( n \geq 1 \), \( \lambda_\mathcal{B}(S^n) \) is homeomorphic to the Hilbert-cube \( Q \).

**Proof.** In fact we show that \( \lambda_\mathcal{B}(S^n) \) is compact and infinite-dimensional and has a countable base and that \( \lambda_\mathcal{B}(S^n) \) can be embedded as a convex subspace in \( \mathbb{R}^{2n} \); hence, by Keller's theorem, \( \lambda_\mathcal{B}(S^n) \) is homeomorphic to \( Q \). Clearly, \( \lambda_\mathcal{B}(S^n) \) is compact.

To prove that \( \lambda_\mathcal{B}(S^n) \) has a countable base, let \( X \) be a countable dense subset of \( S^n \). Define \( \mathcal{B}_0 = \{ B(x, \varepsilon) \mid x \in X, \varepsilon \in \mathbb{Q}, \varepsilon > 0 \} \). It is not difficult to see that \( P : \lambda_\mathcal{B}(S^n) \to \lambda_\mathcal{B}(S^n) \) such that \( P(\mathcal{B}) = \mathcal{B} \cap \mathcal{B}_0 \) is a homeomorphism; i.e., since \( \lambda_\mathcal{B}(S^n) \) has a countable base, \( \lambda_\mathcal{B}(S^n) \) also has a countable base. Next, \( \lambda_\mathcal{B}(S^n) \) is infinite-dimensional, since \( \lambda_\mathcal{B}(I)(\cong Q) \) can be embedded in \( \lambda_\mathcal{B}(S^n) \).

For, \( Y = \{ x \in S^n \mid x = (x_0, x_1, \ldots, x_n), x_1 \geq 0, x_2 = \ldots = x_n = 0 \} \); this subspace is homeomorphic to \( I \). Let \( \mathcal{B}_2 \) be as defined in Section 3, i.e., \( \mathcal{B}_2 \) is the collection of all closed subsets \( Y' \) if \( Y \) such that \( Y' \) is connected or \( Y' \) is connected. Define \( T : \lambda_\mathcal{B}(Y) \to \lambda_\mathcal{B}(S^n) \) by \( T(\mathcal{B}) = \{ B \in \mathcal{B} \mid B \cap Y \in \mathcal{B}_2 \} \) \( (\mathcal{B} \in \lambda_\mathcal{B}(I)) \). Again it is not difficult to prove that \( T \) is a topological embedding. Hence \( \lambda_\mathcal{B}(I) \cong Q \) can be embedded in \( \lambda_\mathcal{B}(S^n) \), i.e., \( \lambda_\mathcal{B}(S^n) \) is infinite-dimensional.
Finally we embed $\lambda_\mathbb{R}(S^n)$ as a convex subspace in $\mathbb{R}^{S^n}$, by means of the function $U : \lambda_\mathbb{R}(S^n) \to \mathbb{R}^{S^n}$, determined by:

$$U(\mathcal{M})(x) = \inf \{ \varepsilon \geq 0 | B(x, \varepsilon) \in \mathcal{M} \}, \ (\mathcal{M} \in \lambda_\mathbb{R}(S^n), \ x \in S^n).$$

The mapping $U$ is continuous and one-to-one since $U(\mathcal{M})(x) \leq \varepsilon$ iff $B(x, \varepsilon) \in \mathcal{M}$, and $U(\mathcal{M})(x) \geq \varepsilon$ iff $B(x, \varepsilon - \varepsilon) \in \mathcal{M}$. And indeed, $U(\lambda_\mathbb{R}(S^n))$ is a convex subspace of $\mathbb{R}^{S^n}$. In order to show this, we need only prove: if $\mathcal{M}_1, \mathcal{M}_2 \in \lambda_\mathbb{R}(S^n)$, then there exists an $\mathcal{M} \in \lambda_\mathbb{R}(S^n)$ such that $U(\mathcal{M}) = \frac{1}{2} U(\mathcal{M}_1) + \frac{1}{2} U(\mathcal{M}_2)$ ($U(\lambda_\mathbb{R}(S^n))$ being compact and hence closed in $\mathbb{R}^{S^n}$). So take $\mathcal{M}_1, \mathcal{M}_2 \in \lambda_\mathbb{R}(S^n)$ and let $\mathcal{M}_3 = \{ B(x, \varepsilon) | x \in S^n, \ \varepsilon \geq \frac{1}{2} U(\mathcal{M}_1)(x) + \frac{1}{2} U(\mathcal{M}_2)(x) \}$. Then $\mathcal{M}_3$ is a linked system, because if $B(x, \varepsilon)$ and $B(y, \delta) \in \mathcal{M}_3$ ($x, y \in S^n$, $\varepsilon \geq \frac{1}{2} U(\mathcal{M}_1)(x) + \frac{1}{2} U(\mathcal{M}_2)(x)$, $\delta \geq \frac{1}{2} U(\mathcal{M}_1)(y) + \frac{1}{2} U(\mathcal{M}_2)(y)$), then:

$$d(x, y) \leq U(\mathcal{M}_3)(x) + U(\mathcal{M}_3)(y),$$

and

$$d(x, y) \leq U(\mathcal{M}_3)(x) + U(\mathcal{M}_3)(y);$$

hence

$$d(x, y) \leq \delta + \varepsilon,$$

i.e.,

$$B(x, \varepsilon) \cap B(y, \delta) \neq \emptyset.$$

Let $\mathcal{M}_3$ be a maximal linked system containing $\mathcal{M}_3$ (in fact $\mathcal{M}_3$ is itself a maximal linked system). Then, clearly,

$$U(\mathcal{M}_3)(x) \leq \frac{1}{2} U(\mathcal{M}_1)(x) + \frac{1}{2} U(\mathcal{M}_2)(x),$$

and

$$U(\mathcal{M}_3)(x) \leq \frac{1}{2} U(\mathcal{M}_1)(x) + \frac{1}{2} U(\mathcal{M}_2)(x) \quad \text{for each} \ x \in S^n.$$

But, since for each maximal linked system $\mathcal{M}$: $U(\mathcal{M})(x) + U(\mathcal{M})(\bar{x}) = \pi$, we have

$$U(\mathcal{M}_3)(x) = \frac{1}{2} U(\mathcal{M}_1)(x) + \frac{1}{2} U(\mathcal{M}_2)(x) \quad \text{for each} \ x \in S^n.$$

Thus

$$U(\mathcal{M}_3) = \frac{1}{2} U(\mathcal{M}_1) + \frac{1}{2} U(\mathcal{M}_2).$$
REFERENCES


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