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VERTEX-CRITICAL SUBGRAPHS OF KNESER GRAPHS

A. SCHRIJVER

ABSTRACT

We show that if the stable (independent) n-subsets of a circuit with 2n+k vertices are split into k+1 classes, one of the classes contains two disjoint n-subsets; this yields a (k+2)-vertex-critical subgraph of Lovász's Kneser-graph $KG_{n,k}$.

1. INTRODUCTION

Let n and k be natural numbers and let X be a set with 2n+k elements. Call a collection of subsets of X a *clan* if it does not contain two disjoint sets. The following question arises naturally: *What is the minimal number of clans into which the collection of all* n-*subsets of* X *can be split?*
[An n-*subset* is a subset with n elements.]

In 1955 KNESER [5] raised the conjecture that the following splitting has the minimal number of clans, where we lose no generality by assuming that X = {1,...,2n+k}. For i = 1,...,2n+k, let K_i contain all n-subsets of X whose smallest element is i. Then

$$K_1,\dots,K_{k+1},K_{k+2}\cup\dots\cup K_{k+n+1}$$

divides the n-subsets of X into k+2 clans. So Kneser conjectured that no splitting of the n-subsets into k+1 clans is possible.

In 1977 LOVÁSZ [6] was able to prove this conjecture. His interesting proof uses homotopy theory and the following theorem of BORSUK [2] (cf. DUGUNDJI [3]) from 1933, where S^k ($\subset \mathbb{R}^{k+1}$) denotes the kdimensional sphere:

BORSUK'S THEOREM (closed form): *If* $S^k = F_1 \cup \ldots \cup F_{k+1}$, *where* $F_1,\ldots,F_{k+1}$ *are closed subsets of* S^k, *then one of the sets* F_i *contains two antipodal points.*

In 1977 as well, BÁRÁNY [1] demonstrated that the truth of Kneser's conjecture immediately follows from the following form of Borsuk's theorem:

BORSUK'S THEOREM (open form): *If* $S^k = U_1 \cup \ldots \cup U_{k+1}$, *where* $U_1,\ldots,U_{k+1}$ *are open subsets of* S^k, *then one of the sets* U_i *contains two antipodal points*

(by using simple topological arguments the two forms of Borsuk's theorem can be deduced from each other), together with a theorem of GALE [4] from 1956:

GALE'S THEOREM: *One can select* $2n+k$ *points on* S^k *such that each open hemisphere of* S^k *contains at least* n *of these points.*

(In the present paper we give explicitly a possible choice of these $2n+k$ points, found earlier by PETTY [7]). Bárány's method runs as follows. Suppose we could divide all n-subsets of X into k+1 clans, say $C_1,\ldots,C_{k+1}$. In this case we may suppose, without loss of generality, that the $2n+k$ elements of X are situated on S^k in a way as formulated in Gale's theorem. For $i = 1,\ldots,k+1$, let U_i be the (open) set of the centers of those open hemispheres which enclose an element of C_i (this element being a subset of X and hence of S^k). Since, by Gale's theorem, each open hemisphere includes at least one n-subset of X, we know that $S^k = U_1 \cup \ldots \cup U_{k+1}$. Hence Borsuk's theorem assures the existence of two antipodal points in, say, U_i. But antipodal points are the centers of disjoint open hemispheres, and these hemispheres include necessarily disjoint n-subsets in C_i, contradicting the fact that C_i is a clan.

One may translate Kneser's conjecture in the language of graphs, by defining the *Kneser-graph* $KG_{n,k}$ as follows. The vertices of $KG_{n,k}$ are the n-subsets of X, two vertices being adjacent iff they are, as n-subsets, disjoint. So clans induce independent sets of vertices of $KG_{n,k}$. $KG_{2,1}$ is the well-known Petersen-graph.

Now dividing n-subsets of X into ℓ clans coincides with colouring the vertices of $KG_{n,k}$ with ℓ colours such that adjacent vertices have different colours; the vertices coloured with some fixed colour together form a clan. So Kneser's conjecture, i.e. Lovasz's result, can be formulated as: the colouring number of $KG_{n,k}$ equals k+2.

Do we always need the graph $KG_{n,k}$ completely to conclude that this graph is not (k+1)-colourable? Evidently not, since if k = 1, the existence of an odd circuit in $KG_{n,1}$ is already enough for knowing that $KG_{n,1}$ is not 2-colourable. Therefore one may ask for minimal not-(k+1)-colourable induced subgraphs of $KG_{n,k}$, i.e. for induced subgraphs of $KG_{n,k}$ which are not-(k+1)-colourable, but if we delete any vertex of these subgraphs they will be (k+1)-colourable. Otherwise stated, find collections, consisting of n-subsets of X, which cannot be split into k+1 clans, but whose proper subcollections all are partitionable into k+1 (or less) clans.

A graph whose proper induced subgraphs all have a lower colouring number than the colouring number c of the graph itself is called c-*vertex-critical*. So we are looking for (k+2)-vertex-critical subgraphs of $KG_{n,k}$; in this note we present such subgraphs.

To this end, define an n-subset X' of X = $\{1,\dots,2n+k\}$ to be *stable* if for no i = $1,\dots,2n+k-1$ both $i \in X'$ and $i+1 \in X'$, nor both $2n+k \in X'$ and $1 \in X'$; i.e. a subset is stable if it does not contain two neighbours in the cyclic ordering of $\{1,\dots,2n+k\}$.

By giving an explicit embedding of 2n+k points on S^k satisfying the claim of Gale's theorem we prove the non-(k+1)-colourability of the subgraph of $KG_{n,k}$ induced by those vertices of $KG_{n,k}$ representing stable n-subsets of X = $\{1,\dots,2n+k\}$. That is, the collection of stable n-subsets of X cannot be divided into k+1 clans. We also show that this last indeed is possible for each of its proper subcollections.

Note that in case k = 1 the stable n-subsets of $\{1,\dots,2n+1\}$ induce an odd circuit in $KG_{n,1}$.

2. VERTEX-CRITICAL SUBGRAPHS

We first give an explicit embedding of $2n+k$ points on the k-dimensional sphere S^k such that each open hemisphere contains at least n of these points (cf. PETTY [7]). For this we need the following observation about the values of polynomials.

<u>OBSERVATION</u>. *Let* $p(x)$ *be a non-zero polynomial of degree at most* k, *with real coefficients. Then there is a stable* n*-subset* X' *of* $\{1,\ldots,2n+k\}$ *such that* $(-1)^i p(i) > 0$ *whenever* $i \in X'$.

<u>PROOF</u>. Let $p(x)$ be such a polynomial. Define inductively the sequence $i_0, i_1, i_2, i_3 \ldots$ of integers by

(1) i_0 is the largest nonpositive integer such that $(-1)^{i_0} p(i_0) > 0$;

(2) i_1 is the smallest positive integer such that $(-1)^{i_1} p(i_1) > 0$;

(3) i_ℓ is the smallest integer such that $i_\ell \geq i_{\ell-1} + 2$ and $(-1)^{i_\ell} p(i_\ell) > 0$, for $\ell = 2,3,4\ldots$.

Clearly, this sequence is infinite. Now take $X' = \{i_1,\ldots,i_n\}$. We are ready once we have proved that $i_n \leq 2n+k$ and $i_n - i_1 \leq 2n+k-2$.

To this end let, for real numbers r and s, $Z(r,s)$ be the number of zeros of $p(x)$ contained in the open interval (r,s), counting f-fold zeros f times ($f \in \mathbb{N}$). We prove that

(4) $$i_\ell - i_{\ell-1} \leq 2 + Z(i_{\ell-1}, i_\ell), \qquad \text{for } \ell = 1,2,3,\ldots \; .$$

Appropriately adding some of these inequalities yields the inequalities we need, since $Z(-\infty, i_n) \leq k$.

First remark that if all integers in the open interval (c,d) are zeros of $p(x)$, where c and d are integers, then $Z(c,d) \geq d-c-1$. If furthermore $(-1)^c p(c)(-1)^d p(d) > 0$ then $Z(c,d) \equiv d-c \pmod 2$, whence

$Z(c,d) \geq d-c$.

Now to show (4), for $\ell = 1,2,3,\ldots$, look at the behaviour of $p(x)$ between $i_{\ell-1}$ and i_ℓ. Let

$$i_{\ell-1} = j_0 < j_1 < \ldots < j_m = i_\ell$$

be those integers in the closed interval $[i_{\ell-1}, i_\ell]$ which are not a zero of $p(x)$, and consider the sequence of numbers

$$(-1)^{j_0}p(j_0), (-1)^{j_1}p(j_1), \ldots, (-1)^{j_m}p(j_m).$$

By (3) above these numbers are negative except the first and last one and possibly the second one. Hence at most two of the products of two consecutive terms in this sequence are negative, the remaining products being positive.

So, by our remark, we have $Z(j_{s-1}, j_s) \geq j_s - j_{s-1}$, for $s = 1,\ldots,m$ with at most two exceptions; but in all cases $Z(j_{s-1}, j_s) \geq j_s - j_{s-1} - 1$. Therefore, by adding inequalities we get

$$Z(i_{\ell-1}, i_\ell) = Z(j_0, j_m) \geq j_m - j_0 - 2 = i_\ell - i_{\ell-1} - 2,$$

thus proving (4). $\square$

Next define, for each natural number i, the vector $v_i \in \mathbb{R}^{k+1}$ by

$$v_i = (-1)^i(1, i, i^2, \ldots, i^k).$$

By projecting the vectors $v_1, \ldots, v_{2n+k}$ onto the sphere S^k we obtain $2n+k$ points

$$w_1 = \frac{v_1}{|v_1|}, \ldots, w_{2n+k} = \frac{v_{2n+k}}{|v_{2n+k}|},$$

situated on S^k and satisfying a stronger form of Gale's claim, as stated in the following theorem (cf. PETTY [7]).

<u>THEOREM 1</u>. *Each open hemisphere of* S^k *encloses an* n*-subset of* $\{w_i \mid 1 \leq i \leq 2n+k\}$ *whose indices form a stable subset of* $\{1,\ldots,2n+k\}$.

PROOF. Choose a hemisphere with center, say, $a = (a_0,\dots,a_k)$. This hemisphere contains the point w_i if and only if the inner product of a and w_i is positive, that is, if and only if

$$(-1)^i(a_0 + a_1 i + a_2 i^2 + \dots a_k i^k) > 0.$$

Since $p(x) = a_0 + a_1x + a_2x^2 + a_kx^k$ is a non-zero polynomial of degree at most k, the observation gives us that a stable n-subset X' of $\{1,\dots,2n+k\}$ exists such that $(-1)^i p(i) > 0$ whenever $i \in X'$. Hence the vectors w_i with index i in X' have the required properties. □

Now we are able to prove, in a manner analogous to Bárány's manner of proving Kneser's conjecture, the following sharpening of Kneser's conjecture.

THEOREM 2. *It is not possible to divide the stable* n*-subsets of* $\{1,\dots,2n+k\}$ *into* $k+1$ *clans.*

PROOF. Suppose $C_1,\dots,C_{k+1}$ are clans such that each stable n-subset of $\{1,\dots,2n+k\}$ is in at least one of them. Let U_i consist of all centers of those open hemispheres which enclose any n-subset $\{w_i \mid i \in X'\}$ with $X' \in C_i$ $(i=1,\dots,k+1)$. By theorem 1, $S^k = U_1 \cup \dots \cup U_{k+1}$. Since each U_i is open, the open form of Borsuk's theorem implies the existence of two antipodal points in some U_i. Since antipodal points are the centers of disjoint open hemispheres, there are disjoint n-subsets in C_i. This contradicts the fact that C_i is a clan. □

Let, by definition, the *reduced Kneser-graph* $KG'_{n,k}$ have as vertices all stable n-subsets of $\{1,\dots,2n+k\}$, two of them being adjacent if they are disjoint. So $KG'_{n,k}$ is an induced subgraph of $KG_{n,k}$. Theorem 2 in fact asserts that the colouring number of $KG'_{n,k}$ equals $k+2$.

We conclude with showing that the collection of stable n-subsets is minimal (under inclusion) with these properties, in other words

THEOREM 3. *The reduced Kneser-graph* $KG'_{n,k}$ *is* $(k+2)$*-vertex-critical.*

PROOF. By theorem 2 it is enough to show that if S is a stable n-subset of $\{1,\ldots,2n+k\}$ then the collection

$$\{X' \mid X' \text{ is a stable } n\text{-subset of } \{1,\ldots,2n+k\}, \text{ and } X' \neq S\}$$

can be split into k+1 clans. So choose S.

Consider the circuit with vertices $1,\ldots,2n+k$, two vertices i and j being adjacent iff $i \equiv j + 1$ or $i \equiv j - 1 \pmod{2n+k}$. Let the set S' consist of all elements of S together with all points adjacent in C to any element of S. We may split the set S' uniquely into disjoint classes $T_1,\ldots,T_m$, such that each of them induces on C a path with both end points in $S'\setminus S$, and such that no class contains two adjacent points of $S'\setminus S$ (except in the trivial case $k = 1$). So S' has 2n+m points, and there remain k-m points which are in $\{1,\ldots,2n+k\}\setminus S'$. Each of these remaining points determines a clan consisting of all stable n-subsets containing the point; this provides us with the first k-m clans $H_1,\ldots,H_{k-m}$.

Let a and b be the two end points of the path determined by T_1; clans H_{k-m+1} and H_{k-m+2}, respectively, have as elements all stable n-subsets containing a and b, respectively.

The clans $H_1,\ldots,H_{k-m+2}$ together contain all stable n-subsets of $\{1,\ldots,2n+k\}$ except those completely contained in $S'\setminus\{a,b\}$. Now observe that

(1) each stable n-subset contained in $S'\setminus\{a,b\}$ either encloses $T_1 \cap S$ or encloses, for some $j = 2,\ldots,m$, $T_j\setminus S$,

and

(2) each stable n-subset contained in $S'\setminus\{a,b\}$ and different from S meets some $T_j\setminus S$ $(j = 2,\ldots,m)$.

(1) and (2) imply that the collections

$$H_{k-m+j+1} = \{X' \mid X' \text{ is a stable } n\text{-subset of } S'\setminus\{a,b\} \text{ such that } T_j\setminus S \subset X' \text{ or both } T_1 \cap S \subset X' \text{ and } X' \cap (T_j\setminus S) \neq \emptyset\}$$

($j = 2,\dots,m$) together contain all stable n-subsets of $S'\setminus\{a,b\}$ different from S. Since $T_1 \cap S$ and $T_j\setminus S$ are nonempty, the collections $H_{k-m+3},\dots,H_{k+1}$ are clans. Hence $H_1,\dots,H_{k+1}$ partition all stable n-subsets of $\{1,\dots,2n+k\}$ different from S into $k+1$ clans. $\square$

REFERENCES

[1] BÁRÁNY, I., *A short proof of Kneser's conjecture,* J. Combinatorial Theory (B), to appear.

[2] BORSUK, K., *Drei Sätze über die n-dimensionale euklidische Sphäre,* Fund. Math. 20 (1933) 177-190.

[3] DUGUNDJI, J., *Topology,* Allyn & Bacon, Boston, 1966.

[4] GALE, D., *Neighboring vertices on a convex polyhedron,* in: Linear inequalities and related systems (H.W. Kuhn & A.W. Tucker, eds.), Annals of Math. Studies 38, Princeton University Press, Princeton, N.J., 1956, pp. 255-263.

[5] KNESER, M., *Aufgabe* 360, Jber. Deutsch. Math.-Verein. 58 (1955/56) 2.Abt. p.27.

[6] LOVÁSZ, L., *Kneser's conjecture, chromatic number and homotopy,* J. Combinatorial Theory (B), to appear.

[7] PETTY, C.M., *Equivalent Point Arrangements in Space and on a Sphere,* Arch. d. Math. 27 (1976) 645-648.

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