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THE DEPENDENCE OF SOME LOGICAL AXIOMS ON DISJOINT TRANSVERSALS AND LINKED SYSTEMS

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1. Introduction and definitions. Let $\mathscr{B} = (B, \wedge, \vee, ^-, 0, 1)$ be a Boolean algebra. A subset L of B is called a *linked system* if $a \wedge b \neq 0$ for all a and b in L. A linked system L is called *maximal* if L is not contained in another linked system. Consider the following axioms on the existence of these linked systems:

LA (WEAK LINKING AXIOM). *Each Boolean algebra has a maximal linked system.*

LA′ (STRONG LINKING AXIOM). *Each linked system in a Boolean algebra is contained in some maximal linked system.*

It is easy to see that these two axioms follow from Zorn's lemma; J. van Mill proved that these axioms follow from the Boolean prime ideal theorem. We shall use the Boolean prime ideal theorem in the following two forms, both clearly equivalent to the usual Boolean prime ideal theorem [1].

FA (WEAK FILTER AXIOM). *Each Boolean algebra has an ultrafilter.*

FA′ (STRONG FILTER AXIOM). *Each filter in a Boolean algebra is contained in some ultrafilter.*

The equivalence of FA and FA′ follows easily if we make the quotient algebra of the Boolean algebra modulo the filter.

Here we prove that also LA and LA′ are equivalent. Furthermore, we prove that FA is independent of LA (does not follow from LA) and that LA is independent of the ZF-axioms by showing that LA follows from the order extension principle and that C_2 follows from LA.

OEP (ORDER EXTENSION PRINCIPLE). *Each partial order on a set can be extended to a total order.*

C_n (AXIOM OF CHOICE FOR n-SETS). *Each family of n-sets has a choice-function.*

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[1] See T. J. Jech, *The axiom of choice*, Amsterdam-London 1973.

Recall that an *n-set* is a set with n elements. Note that OEP follows from FA and that C_2 follows from OEP, but that FA is independent of OEP and that OEP is independent of C_2 (op. cit.). That is,

$$\mathrm{FA} \to \mathrm{OEP} \to \mathrm{C}_2,$$

while none of these arrows can be reversed. We prove

$$\mathrm{OEP} \to \mathrm{LA} \leftrightarrow \mathrm{LA}' \to \mathrm{C}_2.$$

Another approach to these axioms is by means of so-called *disjoint transversals*. Let $\mathscr{U}$ and $\mathscr{V}$ be subsets of $\mathscr{P}(X)$, the power set of a set X. A subset Y of X is called a *$\mathscr{U}$-transversal* if $U \cap Y \neq \emptyset$ for all $U \in \mathscr{U}$. The proposition "there exist a $\mathscr{U}$-transversal and a $\mathscr{V}$-transversal which are mutually disjoint" is denoted by $\mathrm{dt}(\mathscr{U}, \mathscr{V})$.

Clearly, $\mathrm{dt}(\mathscr{U}, \mathscr{V})$ if and only if $\mathrm{dt}(\mathscr{V}, \mathscr{U})$. A set Y is called *$\mathscr{U}$-independent* if no $U \in \mathscr{U}$ is contained in Y. Then we have $\mathrm{dt}(\mathscr{U}, \mathscr{V})$ if and only if there exists a $\mathscr{U}$-independent $\mathscr{V}$-transversal.

Let $\mathscr{P}_{\mathrm{finite}}(X)$ be the collection of all finite subsets of a set X and let $\mathscr{P}_n(X)$ be the collection of all subsets X' of X with $|X'| \leqslant n$.

We consider the following axioms.

DT (DISJOINT TRANSVERSAL AXIOM). *If X is a set, $\mathscr{U}$ and $\mathscr{V}$ are subsets of $\mathscr{P}_{\mathrm{finite}}(X)$ and each two finite subcollections $\mathscr{U}_0$ of $\mathscr{U}$ and $\mathscr{V}_0$ of $\mathscr{V}$ have $\mathrm{dt}(\mathscr{U}_0, \mathscr{V}_0)$, then $\mathrm{dt}(\mathscr{U}, \mathscr{V})$.*

$\mathrm{DT}_{m,n}$. *If X is a set, $\mathscr{U}$ is a subset of $\mathscr{P}_m(X)$, $\mathscr{V}$ is a subset of $\mathscr{P}_n(X)$ and each two finite subcollections $\mathscr{U}_0$ of $\mathscr{U}$ and $\mathscr{V}_0$ of $\mathscr{V}$ have $\mathrm{dt}(\mathscr{U}_0, \mathscr{V}_0)$, then $\mathrm{dt}(\mathscr{U}, \mathscr{V})$.*

In this paper we prove that DT and $\mathrm{DT}_{2,3}$ both are equivalent to the Boolean prime ideal theorem (or to FA). $\mathrm{DT}_{2,2}$ is too weak to imply DT; we prove that $\mathrm{DT}_{2,2}$ is equivalent to LA.

We may consider a subset of $\mathscr{P}_2(X)$ as the edge set of a graph with vertex set X. As a side result we give a characterization of pairs of graphs $G_1 = (X, \mathscr{U})$ and $G_2 = (X, \mathscr{V})$ such that $\mathrm{dt}(\mathscr{U}, \mathscr{V})$, i.e. such that there are disjoint subsets X_1 and X_2 of X with the following properties: X_1 meets every edge of G_1, and X_2 meets every edge of G_2. A corollary is a characterization of classes of graphs

$$\{G_i \mid i \in I\} = \{(X, \mathscr{U}_i) \mid i \in I\} \quad (I \text{ is an index set})$$

such that there is a collection $\{X_i \mid i \in I\}$ of pairwise disjoint subsets of X with the property that X_i meets each edge of G_i for all $i \in I$.

2. Equivalence of DT, $\mathrm{DT}_{2,3}$, and FA. In this section we prove the equivalence of FA, FA$'$, DT, and all $\mathrm{DT}_{m,n}$ in case $m \geqslant 2$, $n \geqslant 2$ and not $m = n = 2$. Since, clearly, (i) FA implies FA$'$, (ii) DT implies each $\mathrm{DT}_{m,n}$, and (iii) $\mathrm{DT}_{m,n}$ implies $\mathrm{DT}_{2,3}$ in case $m \geqslant 2$, $n \geqslant 2$ and not $m = n = 2$, it is enough to prove (iv) FA′ implies DT and (v) $\mathrm{DT}_{2,3}$ implies FA.

PROPOSITION 2.1. FA′ *implies* DT.

Proof. Let X be a set and let $\mathscr{U}$ and $\mathscr{V}$ be subsets of $\mathscr{P}_{\text{finite}}(X)$ such that $\mathrm{dt}(\mathscr{U}_0, \mathscr{V}_0)$ for all finite $\mathscr{U}_0 \subset \mathscr{U}$ and $\mathscr{V}_0 \subset \mathscr{V}$. Assume that $\mathscr{B} = (B, \wedge, \vee, ^{-}, 0, 1)$ is the Boolean algebra freely generated by X. Let $G \subset B$ be the collection

$$G = \{ \bigvee_{x \in U} x \mid U \in \mathscr{U}\} \cup \{ \bigvee_{x \in V} \bar{x} \mid V \in \mathscr{V}\}.$$

We prove that if $g_1, g_2, \ldots, g_m \in G$, then $g_1 \wedge g_2 \wedge \ldots \wedge g_m \neq 0$. For this let $\mathscr{U}_0 \subset \mathscr{U}$ and $\mathscr{V}_0 \subset \mathscr{V}$ be finite. We have to prove

$$a = \Big(\bigwedge_{U \in \mathscr{U}_0} \bigvee_{x \in U} x \Big) \wedge \Big(\bigwedge_{V \in \mathscr{V}_0} \bigvee_{x \in V} \bar{x} \Big) \neq 0.$$

Since $\mathrm{dt}(\mathscr{U}_0, \mathscr{V}_0)$ holds, there are finite subsets X_1 and X_2 of X such that

$$X_1 \cap X_2 = \emptyset,$$

$$X_1 \cap U \neq \emptyset \text{ for each } U \in \mathscr{U}_0, \quad \text{and} \quad X_2 \cap V \neq \emptyset \text{ for each } V \in \mathscr{V}_0.$$

Now let

$$z = \Big(\bigwedge_{x \in X_1} x \Big) \wedge \Big(\bigwedge_{x \in X_2} \bar{x} \Big).$$

Since X_1 and X_2 are disjoint and X is a set of free generators for $\mathscr{B}$, we have $z \neq 0$. We prove that $z \leqslant a$.

First, let $U \in \mathscr{U}_0$. Then $U \cap X_1 \neq \emptyset$; take $x_1 \in U \cap X_1$. Then

$$z \leqslant x_1 \leqslant \bigvee_{x \in U} x.$$

Second, let $V \in \mathscr{V}_0$. Then $V \cap X_2 \neq \emptyset$; take $x_2 \in V \cap X_2$. Then

$$z \leqslant \bar{x}_2 \leqslant \bigvee_{x \in V} \bar{x}.$$

Hence

$$0 < z \leqslant \Big(\bigwedge_{U \in \mathscr{U}_0} \bigvee_{x \in U} x \Big) \wedge \Big(\bigwedge_{V \in \mathscr{V}_0} \bigvee_{x \in V} \bar{x} \Big) = a.$$

So G generates a filter, and this filter is contained in an ultrafilter F (by FA′). Now let

$$X_1 = \{x \in X \mid x \in F\} \quad \text{and} \quad X_2 = \{x \in X \mid \bar{x} \in F\}.$$

Then, clearly, X_1 and X_2 are disjoint. Furthermore, X_1 is a $\mathscr{U}$-transversal. For let $U \in \mathscr{U}$. Then

$$\bigvee_{x \in U} x \in G \subset F.$$

Hence, since F is an ultrafilter, there is an $x \in U$ such that $x \in F$, i.e. such that $x \in X_1$. This means that $U \cap X_1 \neq \emptyset$. In the same way one proves that X_2 is a $\mathscr{V}$-transversal. Therefore $\mathrm{dt}(\mathscr{U}, \mathscr{V})$.

PROPOSITION 2.2. $\mathrm{DT}_{2,3}$ *implies* FA.

Proof. Let $\mathscr{B} = (B, \wedge, \vee, ^{-}, 0, 1)$ be a Boolean algebra. We prove that $\mathscr{B}$ has an ultrafilter. For this let

$$\mathscr{U} = \{\{a, \bar{a}\} \mid a \in B\}$$

and

$$\mathscr{V} = \{\{a, b, c\} \mid a, b, c \in B \text{ and } a \wedge b \wedge c = 0\}.$$

Now $\mathrm{dt}(\mathscr{U}_0, \mathscr{V}_0)$ for any finite $\mathscr{U}_0 \subset \mathscr{U}$ and $\mathscr{V}_0 \subset \mathscr{V}$, since the elements of B occurring in $\mathscr{U}_0$ and $\mathscr{V}_0$ generate a finite subalgebra $\mathscr{B}_0$ of $\mathscr{B}$. This $\mathscr{B}_0$ has an ultrafilter which is a $\mathscr{U}_0$-transversal and the complement of which in B is a $\mathscr{V}_0$-transversal.

Hence, by $\mathrm{DT}_{2,3}$, we have $\mathrm{dt}(\mathscr{U}, \mathscr{V})$. Let $F \subset B$ be such that F is a $\mathscr{U}$-transversal and $B \setminus F$ is a $\mathscr{V}$-transversal. We prove that F is an ultrafilter. F is a filter, for suppose $a, b \in F$ and $c \geqslant a \wedge b$. Then $\{a, b, \bar{c}\} \in \mathscr{V}$, whence

$$(B \setminus F) \cap \{a, b, \bar{c}\} \neq \emptyset.$$

This implies $\bar{c} \notin F$, whence $c \in F$, since $\{c, \bar{c}\} \cap F \neq \emptyset$. F is also an ultrafilter, since for all $a \in B$ we have $\{a, \bar{a}\} \cap F \neq \emptyset$.

THEOREM 2.1. DT *and* $\mathrm{DT}_{2,3}$ *are equivalent to the Boolean prime ideal theorem.*

For the proof recall that FA and FA′ both are equivalent to the Boolean prime ideal theorem (op. cit.).

3. Equivalence and dependence of $\mathrm{DT}_{2,2}$**, LA, and LA′.** In this section we prove that $\mathrm{DT}_{2,2}$, LA, and LA′ are equivalent. Furthermore, we prove that these axioms follow from OEP; hence the Boolean prime ideal theorem is independent of LA, since it is independent of OEP. We show also the independence of LA of the ZF-axioms by proving that LA implies C_2 (op. cit.). Since, clearly, LA′ → LA, it is enough to prove

$$\mathrm{OEP} \to \mathrm{LA} \to \mathrm{DT}_{2,2} \to \mathrm{LA}' \quad \text{and} \quad \mathrm{DT}_{2,2} \to C_2.$$

We remark that for a linked system L in a Boolean algebra $\mathscr{B} = (B, \wedge, \vee, ^{-}, 0, 1)$ to be maximal it is necessary and sufficient that for all $a \in B$: $a \in L$ or $\bar{a} \in L$. Also, if L is a maximal linked system, then for all $a, b \in B$ with the property $a \vee b = 1$ we have $a \in L$ or $b \in L$.

PROPOSITION 3.1. OEP *implies* LA.

Proof. Let $\mathscr{B} = (B, \wedge, \vee, ^{-}, 0, 1)$ be a Boolean algebra. We prove that $\mathscr{B}$ has a maximal linked system. Let $\leqslant$ be the usual partial order on B, i.e. let

$$x \leqslant y \text{ if and only if } x \wedge \bar{y} = 0.$$

By OEP, there exists a total order $\preccurlyeq$ on B such that $x \leqslant y$ implies $x \preccurlyeq y$. Now let $M = \{x \mid \bar{x} \preccurlyeq x\}$. We prove that M is a maximal linked system.

M is a linked system, for suppose $a, b \in M$ and $a \wedge b = 0$, i.e. $a \leqslant \bar{b}$ and $b \leqslant \bar{a}$. Therefore, also $a \preccurlyeq \bar{b}$ and $b \preccurlyeq \bar{a}$. Since $a, b \in M$, we also have $\bar{a} \preccurlyeq a$ and $\bar{b} \preccurlyeq b$. Thus $a \preccurlyeq \bar{b} \preccurlyeq b \preccurlyeq \bar{a} \preccurlyeq a$, whence $a = \bar{a}$, which cannot be the case.

M is also a maximal linked system, since for all $a \in B$ we have $\bar{a} \preccurlyeq a$ or $a \preccurlyeq \bar{a}$, and hence $a \in M$ or $\bar{a} \in M$.

PROPOSITION 3.2. LA *implies* $\mathrm{DT}_{2,2}$.

Proof. Let X be a set and let $\mathscr{U}$ and $\mathscr{V}$ be subsets of $\mathscr{P}_2(X)$ such that $\mathrm{dt}(\mathscr{U}_0, \mathscr{V}_0)$ for all finite $\mathscr{U}_0 \subset \mathscr{U}$ and $\mathscr{V}_0 \subset \mathscr{V}$. As in the proof of Proposition 2.1 let $\mathscr{B} = (B, \wedge, \vee, ^-, 0, 1)$ be the Boolean algebra freely generated by X and let

$$G = \Big\{ \bigvee_{x\in U} x \mid U \in \mathscr{U}\Big\} \cup \Big\{ \bigvee_{x\in V} \bar{x} \mid V \in \mathscr{V}\Big\}.$$

Again, G generates a filter, say F. Now let $\mathscr{B}_1 = (B_1, \wedge, \vee, ^-, 0, 1)$ be the quotient algebra of $\mathscr{B}$ modulo the filter F. Let $[b]$ be the image of $b \in B$ in the quotient algebra. By LA, this quotient algebra has a maximal linked system, say M. Let X_1 be the set of all $x \in X$ such that $[x]$ is in M. Let X_2 be the set of all $x \in X$ such that $[\bar{x}]$ is in M. Then $X_1 \cap X_2 = \emptyset$. Also X_1 is a $\mathscr{U}$-transversal. For let $U \in \mathscr{U}$. Then

$$\bigvee_{x\in U} x \in G \subset F,$$

hence

$$[1] = \Big[\bigvee_{x\in U} x\Big] = \bigvee_{x\in U} [x].$$

Therefore, $[x] \in M$ for some $x \in U$, hence $U \cap X_1 \neq \emptyset$.

X_2 is a $\mathscr{V}$-transversal. For let $V \in \mathscr{V}$. Then

$$\bigvee_{x\in V} \bar{x} \in G \subset F,$$

hence

$$[1] = \Big[\bigvee_{x\in V} \bar{x}\Big] = \bigvee_{x\in V} [\bar{x}].$$

Therefore, $[\bar{x}] \in M$ for some $x \in V$, hence $V \cap X_2 \neq \emptyset$. Thus we obtain $\mathrm{dt}(\mathscr{U}, \mathscr{V})$.

PROPOSITION 3.3. $\mathrm{DT}_{2,2}$ *implies* LA′.

Proof. Let $\mathscr{B} = (B, \wedge, \vee, ^-, 0, 1)$ be a Boolean algebra and let $L \subset B$ be a linked system. We have to prove the existence of a maximal linked system containing L. Write

$$\mathscr{U} = \big\{\{x, \bar{x}\} \mid x \in B\big\} \cup \big\{\{x\} \mid x \in L\big\}$$

and

$$\mathscr{V} = \big\{\{x, y\} \mid x, y \in B \text{ and } x \wedge y = 0\big\}.$$

Take finite subsets $\mathscr{U}_0$ of $\mathscr{U}$ and $\mathscr{V}_0$ of $\mathscr{V}$. The elements of B occurring in $\mathscr{U}_0$ and $\mathscr{V}_0$ generate a finite sub-Boolean algebra $\mathscr{B}_0 = (B_0, \wedge, \vee, ^-, 0, 1)$ of $\mathscr{B}$. Let $L_0 = L \cap B_0$. Since B_0 is finite, there exists a maximal linked system M_0 in $\mathscr{B}_0$ containing L_0.

Now let $X_1 = M_0$ and $X_2 = B_0 \setminus M_0$. Then $X_1 \cap X_2 = \emptyset$, X_1 is a $\mathscr{U}_0$-transversal, and X_2 is a $\mathscr{V}_0$-transversal.

So for each finite subsets $\mathscr{U}_0$ of $\mathscr{U}$ and $\mathscr{V}_0$ of $\mathscr{V}$ we have $\mathrm{dt}(\mathscr{U}_0, \mathscr{V}_0)$; from $\mathrm{DT}_{2,2}$ it follows that $\mathrm{dt}(\mathscr{U}, \mathscr{V})$, that is, there are disjoint subsets X_1 and X_2 of B such that X_1 is a $\mathscr{U}$-transversal and X_2 is a $\mathscr{V}$-transversal.

Let $M = X_1$; then M is a maximal linked system containing L. M is a linked system, for suppose $x, y \in M$ and $x \wedge y = 0$; then $\{x, y\} \in \mathscr{V}$, hence $x \in X_2$ or $y \in X_2$. Since $X_1 \cap X_2 = \emptyset$, we have $x \notin X_1 = M$ or $y \notin X_1 = M$, contradicting our assumption. Also $L \subset M$, since for all x in L we have $\{x\} \cap M = \{x\} \cap X_1 \neq \emptyset$. Finally, M is maximal, since for all $x \in B$ we have $\{x, \bar{x}\} \cap X_1 \neq \emptyset$, i.e. $x \in M$ or $\bar{x} \in M$.

PROPOSITION 3.4. $\mathrm{DT}_{2,2}$ *implies* C_2.

Proof. Let $\mathscr{U}$ be a collection of 2-sets. We have to prove the existence of a function, assigning to each set in $\mathscr{U}$ an element of that set. Without restrictions on the generality we may suppose that the sets in $\mathscr{U}$ are pairwise disjoint.

For each finite subset $\mathscr{U}_0$ of $\mathscr{U}$, there is a set X_0 such that $|X_0 \cap U| = 1$ for all U in $\mathscr{U}_0$. This implies $\mathrm{dt}(\mathscr{U}_0, \mathscr{U}_0)$ for all finite subsets $\mathscr{U}_0$ of $\mathscr{U}$ and, consequently, $\mathrm{dt}(\mathscr{U}_0, \mathscr{U}_0')$ for all finite $\mathscr{U}_0, \mathscr{U}_0' \subset \mathscr{U}$.

From $\mathrm{DT}_{2,2}$ we obtain $\mathrm{dt}(\mathscr{U}, \mathscr{U})$, i.e. there are disjoint sets X_1 and X_2 with the property that $|X_1 \cap U| = |X_2 \cap U| = 1$ for all $U \in \mathscr{U}$. Now assign to each set in $\mathscr{U}$ the unique element in $X_1 \cap U$. This clearly determines the required choice-function.

THEOREM 3.1. $\mathrm{DT}_{2,2}$, LA, *and* LA$'$ *are logically equivalent axioms*: LA *follows from* OEP *and* LA *itself implies* C_2.

This follows straightforwardly from the foregoing propositions and the trivial observation LA$' \to$ LA.

4. Some combinatorial aspects of $\mathrm{DT}_{2,2}$. It is obvious that DT and $\mathrm{DT}_{n,m}$ have combinatorial aspects. In particular, $\mathrm{DT}_{2,2}$ gives rise to a question in the theory of graphs. In this we define a *graph* as a pair $(X, \mathscr{U})$, where $\mathscr{U}$ is a subset of $\mathscr{P}_2(X)$ and $\emptyset \notin \mathscr{U}$. The elements of X are called *vertices* and the elements of $\mathscr{U}$ are called *edges* of the graph. Sometimes we shall speak shortly of the graph $\mathscr{U}$ instead of $(X, \mathscr{U})$. A graph $(X, \mathscr{U})$ is said to be *bicolourable* or *bipartite* if we can "colour" the vertices with two colours (i.e. partition the set of vertices X into two classes) such that no edge U in $\mathscr{U}$ is monocoloured (i.e. no edge U in $\mathscr{U}$ is contained in one of the classes of the partition). Hence the graph $(X, \mathscr{U})$ is bicolourable if and only if $\mathrm{dt}(\mathscr{U}, \mathscr{U})$. So, by $\mathrm{DT}_{2,2}$ we have

P_2. *A graph $(X, \mathscr{U})$ is bicolourable if and only if each finite subgraph $\mathscr{U}_0$ is bicolourable.*

This axiom P_2 is equivalent to C_2 (op. cit.).

Suppose that we have now two graphs; if we have a characterization of pairs of finite graphs $\mathscr{U}_0$ and $\mathscr{V}_0$ with the property $\mathrm{dt}(\mathscr{U}_0, \mathscr{V}_0)$, then using $\mathrm{DT}_{2,2}$ we can extend this characterization to pairs of arbitrary graphs $\mathscr{U}$ and $\mathscr{V}$. We now give such a characterization for finite graphs. For this we define the notion of an alternating path. Let $(X, \mathscr{U}_i)$ be a graph for each $i \in I$ (I is some index set). Let $i \in I$ and $j \in I$. An *alternating (i, j)-path from x to y* is a sequence

$$x = x_0, i = i_0, x_1, i_1, x_2, \ldots, x_{n-2}, i_{n-2}, x_{n-1}, j = i_{n-1}, y = x_n \quad (n \geqslant 1)$$

such that

(i) $x_0, x_1, \ldots, x_n \in X$ and $i_0, i_1, \ldots, i_{n-1} \in I$;

(ii) $i_k \neq i_{k+1}$ for $k = 0, 1, \ldots, n-2$;

(iii) $\{x_k, x_{k+1}\} \in \mathscr{U}_{i_k}$ for $k = 0, 1, \ldots, n-1$.

One may consider an alternating (i, j)-path from x to y as a path from x to y in the union of the graphs, in which path the first edge is an edge of $\mathscr{U}_i$ and the last edge is an edge of $\mathscr{U}_j$ and in which two succeeding edges belong to different graphs (in a sense made more precise above). The characterization for finite graphs is as follows (here $I = \{0, 1\}$).

THEOREM 4.1. *Let $(X, \mathscr{U}_0)$ and $(X, \mathscr{U}_1)$ be finite graphs (that is, $\mathscr{U}_0$ and $\mathscr{U}_1$ are finite). Then $\mathrm{dt}(\mathscr{U}_0, \mathscr{U}_1)$ holds (i.e. there are two disjoint sets X_0 and X_1 such that $X_0 \cap U \neq \emptyset$ for each $U \in \mathscr{U}_0$ and $X_1 \cap U \neq \emptyset$ for each $U \in \mathscr{U}_1$) if and only if there is no $x \in X$ such that there is an alternating $(0, 0)$-path from x to x and an alternating $(1, 1)$-path from x to x.*

Proof. We first prove the sufficiency. Suppose that $\mathrm{dt}(\mathscr{U}_0, \mathscr{U}_1)$ holds, i.e. there are disjoint sets X_0 and X_1 such that $X_0 \cap U \neq \emptyset$ for all $U \in \mathscr{U}_0$ and $X_1 \cap U \neq \emptyset$ for all $U \in \mathscr{U}_1$. Suppose, furthermore, that for some $x \in X$ there is an alternating (0, 0)-path from x to x, say

$$x = x_0, 0, x_1, 1, x_2, \ldots, 1, x_{n-1}, 0, x = x_n,$$

and an alternating $(1, 1)$-path from x to x, say

$$x = x_0', 1, x_1', 0, x_2', \ldots, 0, x_{m-1}', 1, x = x_m'.$$

We prove that $x \in X_0 \cap X_1$, which is a contradiction, since this set is empty. Suppose that $x \notin X_0$. Since $\{x_0, x_1\} \in \mathcal{U}_0$, and hence $\{x_0, x_1\} \cap X_0 \neq \emptyset$, we have $x_1 \in X_0$. This implies $x_1 \notin X_1$. Now $\{x_1, x_2\} \in \mathcal{U}_1$, so $\{x_1, x_2\} \cap X_1 \neq \emptyset$ and $x_2 \in X_1$. This implies $x_2 \notin X_0$.

By repeating these arguments one finds $x_n \in X_0$, or $x \in X_0$.

By a similar reasoning one finds $x \in X_1$. Hence $x \in X_0 \cap X_1$.

Second, we prove the necessity. Suppose that there is no $x \in X$ such that there are an alternating $(0, 0)$-path from x to x and an alternating $(1, 1)$-path from x to x. We proceed by induction on $|X|$, which we may suppose to be finite. If $X = \emptyset$, then the theorem is clearly valid. Suppose that $X \neq \emptyset$ and for each pair of graphs $(X', \mathcal{U}_0')$ and $(X', \mathcal{U}_1')$ with $|X'| < |X|$ we have proved the theorem. Choose $x \in X$ arbitrarily. Now there are two possibilities (which do not exclude each other).

(1) There is no alternating $(1, 1)$-path from x to x.

Let

$$A_0 = \{x\} \cup \{y \mid \text{there is an alternating } (1, 0)\text{-path from } x \text{ to } y\}$$

and

$$A_1 = \{y \mid \text{there is an alternating } (1, 1)\text{-path from } x \text{ to } y\}.$$

A_0 and A_1 are disjoint, for suppose $y \in A_0 \cap A_1$. Then there are an alternating $(1, 0)$-path from x to y and an alternating $(1, 1)$-path from y to x, and hence an alternating $(1, 1)$-path from x to x. This contradicts our assumption. Let

$$X' = X \setminus (A_0 \cup A_1),$$

$$\mathcal{U}_0' = \{U \in \mathcal{U}_0 \mid U \subset X'\} \quad \text{and} \quad \mathcal{U}_1' = \{U \in \mathcal{U}_1 \mid U \subset X'\}.$$

Now again, for the pair of graphs $(X', \mathcal{U}_0')$ and $(X', \mathcal{U}_1')$, there is no $x \in X'$ with an alternating $(0, 0)$-path from x to x and an alternating $(1, 1)$-path from x to x. Hence, by induction, since $|X'| < |X|$, we know $\mathrm{dt}(\mathcal{U}_0', \mathcal{U}_1')$, that is, there are disjoint subsets X_0' and X_1' of X' with the properties $X_0' \cap U \neq \emptyset$ for all $U \in \mathcal{U}_0'$ and $X_1' \cap U \neq \emptyset$ for all $U \in \mathcal{U}_1'$.

Let $X_0 = A_0 \cup X_0'$ and $X_1 = A_1 \cup X_1'$. Then, clearly, X_0 and X_1 are disjoint. We prove that $X_0 \cap U \neq \emptyset$ for all $U \in \mathcal{U}_0$ and $X_1 \cap U \neq \emptyset$ for all $U \in \mathcal{U}_1$. Suppose that $U \in \mathcal{U}_0$ and

$$X_0 \cap U = (A_0 \cup X_0') \cap U = \emptyset.$$

Then $U \not\subset X'$, since, otherwise, $U \in \mathcal{U}_0'$ and $U \cap X_0' \neq \emptyset$. Hence $U \cap (A_0 \cup A_1) \neq \emptyset$. Since $U \cap A_0 = \emptyset$, we have $U \cap A_1 \neq \emptyset$. Suppose that $u \in U \cap A_1$ and $U = \{u, v\}$ (possibly $u = v$). Since $u \in A_1$, there exists an alternating $(1, 1)$-path from x to u. Now $\{u, v\} \in \mathcal{U}_0$, hence there is an alternating $(1, 0)$-path from x to v. But this means that $v \in A_0$ and $\{u, v\} \cap A_0 \neq \emptyset$, which is a contradiction. Hence $U \cap X_0 \neq \emptyset$.

In the same manner one proves that X_1 is a $\mathscr{U}_1$-transversal.

(2) There is no alternating $(0, 0)$-path from x to x.

This case is treated similarly to case (1).

Since, by assumption, each $x \in X$ is in at least one of both cases, we can always, use our induction step.

As a corollary we have

THEOREM 4.2. *Let* $(X, \mathscr{U}_0)$ *and* $(X, \mathscr{U}_1)$ *be graphs. Under the assumption of the axiom* $\mathrm{DT}_{2,2}$ *we have: there are disjoint sets* X_0 *and* X_1 *such that* $X_0 \cap U \neq \emptyset$ *for all* $U \in \mathscr{U}_0$ *and* $X_1 \cap U \neq \emptyset$ *for all* $U \in \mathscr{U}_1$ *if and only if there is no* $x \in X$ *with an alternating* $(0, 0)$*-path from* x *to* x *and an alternating* $(1, 1)$*-path from* x *to* x.

Proof. Since the condition of the non-existence of the two paths holds for two graphs if and only if it holds for each pair of finite subgraphs of these graphs, the theorem follows easily from $\mathrm{DT}_{2,2}$ and the foregoing theorem.

The second corollary generalizes Theorem 4.2 to arbitrary collections of graphs:

THEOREM 4.3. *Let* $(X, \mathscr{U}_i)$ *be a graph for each* $i \in I$ (I *is an index set). Under the assumption of the axiom* $\mathrm{DT}_{2,2}$ *we have: there are pairwise disjoint subsets* X_i *of* X ($i \in I$) *such that each* $i \in I$ *has* $X_i \cap U \neq \emptyset$ *for all* $U \in \mathscr{U}_i$ *if and only if there is no* $x \in X$ *and no two different* i *and* j ($i, j \in I$) *such that there are an alternating* (i, i)*-path from* x *to* x *and an alternating* (j, j)*-path from* x *to* x.

Proof. Let $\bar{X} = X \times I$, and

$$\overline{\mathscr{U}_0} = \{\{(u, i), (v, i)\} \mid \{u, v\} \in \mathscr{U}_i, i \in I\},$$
$$\overline{\mathscr{U}_1} = \{\{(x, i), (x, j)\} \mid x \in X,\ i \in I, j \in I, i \neq j\}.$$

Now we leave it to the reader to verify that

(i) There are disjoint subsets X_i of X ($i \in I$) such that for each $i \in I$ we have $X_i \cap U \neq \emptyset$ for all $U \in \mathscr{U}_i$ if and only if $\mathrm{dt}(\overline{\mathscr{U}_0}, \overline{\mathscr{U}_1})$.

(ii) There is an $\bar{x} \in \bar{X}$ such that in $(\overline{\mathscr{U}_0}, \overline{\mathscr{U}_1})$ there are an alternating $(0, 0)$-path from $\bar{x}$ to $\bar{x}$ and an alternating $(1, 1)$-path from $\bar{x}$ to $\bar{x}$ if and only if there are $x \in X$, $i, j \in I$, $i \neq j$, with an alternating (i, i)-path from x to x and an alternating (j, j)-path from x to x.

Thus Theorem 4.3 follows from Theorem 4.2.

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