A GODUNOV-TYPE METHOD FOR CAPTURING WATER WAVES

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Abstract.

In spite of the absence of shock waves in most hydrodynamic applications, sufficient reason remains to employ Godunov-type schemes in this field. In the instance of two-phase flow, the shock capturing ability of these schemes may serve to maintain robustness and accuracy at the interface. Moreover, approximate Riemann solvers have greatly relieved the initial drawback of computational expensiveness of Godunov-type schemes. In the present work we develop an Osher-type approximate Riemann solver for application in hydrodynamics. Actual computations are left to future research.

1. Introduction

The advantages of Godunov-type schemes (Godunov, 1959) in hydrodynamic flow computations are not as widely appreciated as in gas dynamics applications. Admittedly, the absence of supersonic speeds and hence

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shock waves in incompressible flow (the prevailing fluid model in hydrodynamics) reduces the necessity of advanced shock capturing schemes. Nevertheless, many reasons remain to apply Godunov-type schemes in hydrodynamics: Firstly, these schemes have favorable robustness properties due to the inherent upwind treatment of the flow. Secondly, they feature a consistent treatment of boundary conditions. Thirdly, (higher-order accurate) Godunov-type schemes display low dissipative errors, which is imperative for an accurate resolution of boundary layers in viscous flow. Finally, the implementation of these schemes in conjunction with higher-order limited interpolation methods, to maintain accuracy and prevent oscillations in regions where large gradients occur (see, e.g., (Sweby, 1984; Spekreijse, 1987)), is relatively straightforward.

In addition, Godunov-type schemes can be particularly useful in hydrodynamics in case of two-phase flows, e.g., flows suffering cavitation and free surface flows. In these situations, an interface exists between the primary phase (water) and the secondary phase (air, damp, etc.) and fluid properties may vary discontinuously across the interface. In our opinion, the ability of Godunov-type schemes to capture discontinuities is then very useful to maintain robustness and accuracy at the interface. Examples of such interface capturing can be found in, for instance, (Mulder *et al.*, 1992; Chang *et al.*, 1996; Kelecy and Pletcher, 1997).

A disadvantage of the method originally proposed by Godunov is that it requires the solution of an associated Riemann problem with each flux evaluation. In practice, many such evaluations are performed during an actual computation. Consequently, the method is notorious for its high computational costs. To relieve this problem, several approaches have been suggested to reduce the computational costs of the flux evaluations involved, by approximating the Riemann solution. Examples of such approximate Riemann solvers are the flux difference splitting schemes (such as Roe's (Roe, 1981) and Osher's (Osher and Solomon, 1982)).

In the present work we develop an Osher-type flux-difference splitting scheme for the approximate solution of the Riemann problem and we investigate its application in hydrodynamics. Details are presented for the Euler equations for four types of fluids that are commonly used to model the behavior of water, viz., a genuinely compressible fluid, an artificially compressible fluid, a genuinely incompressible fluid, and a two-phase fluid. As a preliminary, we examine the Riemann problem. Next, we give an outline of Osher's approximate Riemann solver. Analysis shows that Osher's scheme suffers loss of accuracy in the presence of centered shock waves and therefore a modified scheme is proposed. Finally, we present the specifics for the aforementioned hydrodynamic applications. Actual computations are deferred to future research.

2. Riemann Problem

In this section we investigate the Riemann Problem:

Definition 1 *Let* $\mathbf{q} \in \mathbb{R}^n = (q_1, \dots, q_n)^T$, $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ *and* $\mathbf{f} \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. *Consider the Cauchy problem*

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{q})}{\partial x} = 0, \qquad \forall x \in \mathbb{R}, t \in \mathbb{R}^+, \tag{1a}$$

subject to the initial condition

$$\mathbf{q}(x,0) = \begin{cases} \mathbf{q}_L, & \text{if } \ x < 0, \\ \mathbf{q}_R, & \text{if } \ x > 0, \end{cases} \tag{1b}$$

with $\mathbf{q}_L$ *and* $\mathbf{q}_R$ *constant. The initial value problem (1a) and the initial condition (1b) define the Riemann problem.*

First, an introductory analysis is presented. Subsequently, we obtain the general solution to (1).

2.1. PRELIMINARY ANALYSIS

Let $\mathbf{A}(\mathbf{q})$ denote the Jacobian of $\mathbf{f}(\mathbf{q})$, $\mathbf{A}(\mathbf{q}) = \partial_{\mathbf{q}}\mathbf{f}(\mathbf{q})$, and let $\lambda_k(\mathbf{q})$, $k = 1, 2, \dots, n$, $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$, be its eigenvalues and $\mathbf{r}_k(\mathbf{q})$ the corresponding eigenvectors. Equation (1a) constitutes a hyperbolic system if the eigenvalues $\lambda_k(\mathbf{q})$ are real and nonzero. Then, the matrix $\mathbf{A}(\mathbf{q})$ can be decomposed with respect to a basis of its eigenvectors:

$$\mathbf{A}(\mathbf{q}) = \mathbf{R}(\mathbf{q}) \cdot \Lambda(\mathbf{q}) \cdot \mathbf{R}(\mathbf{q})^{-1}, \tag{2}$$

where $\Lambda(\mathbf{q}) = \text{diag}(\lambda_1(\mathbf{q}), \dots, \lambda_n(\mathbf{q}))$ and $\mathbf{R}(\mathbf{q}) = (\mathbf{r}_1(\mathbf{q}), \dots, \mathbf{r}_n(\mathbf{q}))$ contains the eigenvectors. From (Lax, 1957) we adopt the following classification of the eigenpairs $(\lambda_k(\mathbf{q}), \mathbf{r}_k(\mathbf{q}))$:

Definition 2 *Consider the matrix* $\mathbf{A}(\mathbf{q}) \in \mathbb{R}^{n \times n}$. *Let* $\lambda_k(\mathbf{q})$, $k = 1, 2, \dots, n$, *be its eigenvalues and* $\mathbf{r}_k(\mathbf{q})$ *the corresponding eigenvectors. An eigenvalue* $\lambda_k(\mathbf{q})$ *and an eigenvector* $\mathbf{r}_k(\mathbf{q})$ *are called genuinely nonlinear on a subdomain* $\Omega \subseteq \mathbb{R}^n$ *if*

$$\partial_{\mathbf{q}}\lambda_k(\mathbf{q}) \cdot \mathbf{r}_k(\mathbf{q}) \neq 0, \qquad \forall \mathbf{q} \in \Omega. \tag{3}$$

An eigenvalue $\lambda_k(\mathbf{q})$ *and an eigenvector* $\mathbf{r}_k(\mathbf{q})$ *are said to be linearly degenerate on* Ω *if*

$$\partial_{\mathbf{q}}\lambda_k(\mathbf{q}) \cdot \mathbf{r}_k(\mathbf{q}) = 0, \qquad \forall \mathbf{q} \in \Omega. \tag{4}$$

The eigenvalues that are genuinely nonlinear for all $\mathbf{q} \in \mathbb{R}^n$ are related to rarefaction waves and shocks in the solution of the Riemann problem. The eigenvalues that are linearly degenerate on $\mathbb{R}^n$ correspond to contact discontinuities in the solution. More complex contact phenomena can occur for eigenvalues that are neither genuinely nonlinear nor linearly degenerate on $\mathbb{R}^n$, see, e.g., (LeVeque, 1990, pages 48–50).

With each of the eigenpairs $(\lambda_k(\mathbf{q}), \mathbf{r}_k(\mathbf{q}))$ we associate two paths in state space. Firstly, the k-shock path:

Definition 3 *Consider hyperbolic system (1a). The k-shock path through $\mathbf{q}_L$ is the set*

$$\mathcal{S}_k(\mathbf{q}_L) = \{\mathbf{q} \in \mathbb{R}^n \mid s(\mathbf{q}; \mathbf{q}_L)(\mathbf{q} - \mathbf{q}_L) = \mathbf{f}(\mathbf{q}) - \mathbf{f}(\mathbf{q}_L)\}, \tag{5}$$

where $s(\mathbf{q}; \mathbf{q}_L)$ is referred to as the shock speed.

Secondly, we distinguish the k-path:

Definition 4 *Consider the hyperbolic system (1a). The k-path through $\mathbf{q}_L$ is the set*

$$\mathcal{R}_k(\mathbf{q}_L) = \{\mathbf{q} \in \mathbb{R}^n \mid \mathbf{q} = \mathbf{h}(\xi), \xi \in \mathbb{R}\}, \tag{6}$$

with $\mathbf{h}(\xi)$ the solution of the ordinary differential equation

$$\begin{aligned} \frac{\partial \mathbf{h}(\xi)}{\partial \xi} &= \mathbf{r}_k(\mathbf{h}(\xi)), \qquad \xi \in \mathbb{R}, \\ \mathbf{h}(\xi_L) &= \mathbf{q}_L, \end{aligned} \tag{7}$$

for some $\xi_L \in \mathbb{R}$.

Furthermore, to each k-path corresponds a set of functions which are invariant on $\mathcal{R}_k$:

Definition 5 *Consider the hyperbolic system (1a). Let $\mathbf{r}_k(\mathbf{q})$ denote the k^{th} eigenvector of the Jacobian $\mathbf{A}(\mathbf{q}) = \partial \mathbf{f}(\mathbf{q})/\partial \mathbf{q}$. A k-Riemann invariant is any function $\psi_k \in C^1(\mathbb{R}^n, \mathbb{R})$ satisfying*

$$\partial_{\mathbf{q}} \psi_k(\mathbf{q}) \cdot \mathbf{r}_k(\mathbf{q}) = 0, \qquad \forall \mathbf{q} \in \mathbb{R}^n. \tag{8}$$

There are at most $n - 1$ such k-Riemann invariants with linearly independent gradients in $\mathbb{R}^n$. Observe that for a linearly degenerate eigenpair $(\lambda_k(\mathbf{q}), \mathbf{r}_k(\mathbf{q}))$ the eigenvalue $\lambda_k(\mathbf{q})$ is a k-Riemann invariant.

2.2. SOLUTION

The general solution to (1) consists of regions in the (x,t)-domain where the solution is constant, separated by simple waves, contact discontinuities and shock waves. Before constructing the general solution, we first obtain the (weak) solution to (1) in the case that it contains only one of the aforementioned contact phenomena.

We establish that the (weak) solution to the Riemann problem can generally be written in similarity form (see, e.g., (Smoller, 1983)):

Theorem 1 *Suppose a unique solution* $\mathbf{q}(x,t)$ *to the Riemann Problem (1) exists. Then* $\mathbf{q}(x,t)$ *can be written in similarity form* $\mathbf{q}(x,t) = \mathbf{h}(x/t)$.

Proof: *Assume* $\mathbf{q}(x,t)$ *solves (1). Then for all* $\alpha \in \mathbb{R}$, $\mathbf{q}(\alpha x, \alpha t)$ *is also a solution:*

$$\frac{\partial \mathbf{q}(\alpha x, \alpha t)}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{q}(\alpha x, \alpha t))}{\partial x} = \alpha\left[\mathrm{D}_2\mathbf{q}(\alpha \mathrm{x}, \alpha \mathrm{t}) + \mathbf{A}(\mathbf{q}(\alpha \mathrm{x}, \alpha \mathrm{t}))\cdot \mathrm{D}_1\mathbf{q}(\alpha \mathrm{x}, \alpha \mathrm{t})\right] = 0, \quad (9)$$

where D_l *denotes differentiation with respect to the* l^{th} *function-argument. Because the solution is unique by assumption,* $\mathbf{q}(x,t) = \mathbf{q}(\alpha x, \alpha t)$. *Hence,* $\mathbf{q}(x,t) = \mathbf{h}(x/t)$. □

A (classical) simple wave solution of (1) exists if $\lambda_k(\mathbf{q})$ is a genuinely nonlinear eigenvalue, $\lambda_k(\mathbf{q}_L) < \lambda_k(\mathbf{q}_R)$ and $\mathbf{q}_R$ is on the k-path through $\mathbf{q}_L$. Note that this implies that the k-Riemann invariants are equal for $\mathbf{q}_L$ and $\mathbf{q}_R$, i.e., $\psi_k^m(\mathbf{q}_L) = \psi_k^m(\mathbf{q}_R)$, for $m \neq k$, $m = 1, \ldots, n$. Assuming that the genuinely nonlinear eigenvector in (7) is normalized such that

$$\partial_{\mathbf{q}}\lambda_k(\mathbf{q})\cdot \mathbf{r}_k(\mathbf{q}) = 1, \qquad \forall \mathbf{q} \in \mathbb{R}^n, \quad (10)$$

we find that $\mathbf{q}(x,t) = \mathbf{h}(x/t)$ according to (7) is the similarity solution in the simple wave region $\lambda_k(\mathbf{q}_L) < x/t < \lambda_k(\mathbf{q}_R)$ (see, e.g., (Smoller, 1983; Lax, 1973)):

Theorem 2 *Suppose* $\mathbf{h} \in C^1(\mathbb{R}, \mathbb{R}^n)$ *solves (7), with* $\mathbf{r}_k(\mathbf{q})$ *normalized according to (10), and* $\mathbf{q}_R \in \mathcal{R}_k(\mathbf{q}_L)$. *Then* $\mathbf{q}(x,t) = \mathbf{h}(x/t)$ *is the similarity solution of (1) in the simple-wave region* $\lambda_k(\mathbf{q}_L) < x/t < \lambda_k(\mathbf{q}_R)$.

Proof: *We will only show that* $\mathbf{q}(x,t) = \mathbf{h}(x/t)$ *solves (1a). Inserting* $\mathbf{q}(x,t) = \mathbf{h}(x/t)$ *in (1a), one obtains*

$$\frac{\partial \mathbf{h}(x/t)}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{h}(x/t))}{\partial x} = \frac{1}{t}\left(\mathbf{A}(\mathbf{h}(x/t)) - \mathbf{I}\frac{x}{t}\right)\cdot \mathrm{D}\mathbf{h}(\mathrm{x}/\mathrm{t}), \quad (11)$$

where $\mathbf{I}$ *stands for the* $\mathbb{R}^{n\times n}$ *identity matrix and* D *denotes differentiation with respect to the function-argument. The right-hand side term of (11) vanishes if* $x/t = \lambda_k(\mathbf{h}(x/t))$ *and* $\mathrm{D}\mathbf{h}(\mathrm{x}/\mathrm{t}) = \mathbf{r}_k(\mathbf{h}(\mathrm{x}/\mathrm{t}))$. *The latter trivially follows from (7), the former from (10). Hence,* $\mathbf{h}(x/t)$ *solves (1a).* □

Outside the wave region the solution remains unchanged. The Riemann solution $\mathbf{q}(x,t)$ in the case of a k-rarefaction wave is now simply composed of the solutions on the separate regions:

$$\mathbf{q}(x,t) = \begin{cases} \mathbf{q}_L, & \text{if} \quad x/t < \lambda(\mathbf{q}_L), \\ \mathbf{h}(x/t), & \text{if} \quad \lambda(\mathbf{q}_L) < x/t < \lambda(\mathbf{q}_R), \\ \mathbf{q}_R, & \text{if} \quad x/t > \lambda(\mathbf{q}_R). \end{cases} \tag{12}$$

Next, we derive the (weak) Riemann solution in the instance that it contains a single contact discontinuity. The states $\mathbf{q}_L$ and $\mathbf{q}_R$ are connected by a k-contact discontinuity if $(\lambda_k(\mathbf{q}), \mathbf{r}_k(\mathbf{q}))$ is a linearly degenerate eigenpair and $\mathbf{q}_R$ is on the k-path through $\mathbf{q}_L$. Then, by (4), $\lambda_k(\mathbf{q}_R) = \lambda_k(\mathbf{q}_L)$. The solution to the Riemann problem is now obtained immediately from (12):

$$\mathbf{q}(x,t) = \begin{cases} \mathbf{q}_L, & \text{if} \quad x/t < \lambda(\mathbf{q}_L) = \lambda(\mathbf{q}_R), \\ \mathbf{q}_R, & \text{if} \quad x/t > \lambda(\mathbf{q}_L) = \lambda(\mathbf{q}_R). \end{cases} \tag{13}$$

However, because (13) is discontinuous at $x/t = \lambda(\mathbf{q}_L) = \lambda(\mathbf{q}_R)$, it must be verified that (13) satisfies the weak form of (1a):

$$\oint_{\mathcal{C}} (\mathbf{q}n_t + \mathbf{f}(\mathbf{q})n_x)\, \mathrm{d}\mathcal{C} = 0. \tag{14}$$

Here $\mathcal{C}$ is any closed curve in (x,t) and $\mathbf{n} = (n_t, n_x)$ denotes the outward pointing unit normal on $\mathcal{C}$. It can easily be shown that (14) does indeed hold for (13), so that (13) is a valid weak solution.

Finally, we consider the solution to (1) when it comprises a single shock. A shock occurs if $\lambda_k(\mathbf{q})$ is a genuinely nonlinear eigenvalue, $\lambda_k(\mathbf{q}_L) > \lambda_k(\mathbf{q}_R)$ and $\mathbf{q}_R$ is on the k-shock path through $\mathbf{q}_L$. A solution of the form (12) is then necessarily multiple-valued and must therefore be discarded. Instead, the weak solution reads

$$\mathbf{q}(x,t) = \begin{cases} \mathbf{q}_L, & \text{if} \quad x/t < s(\mathbf{q}_L; \mathbf{q}_R), \\ \mathbf{q}_R, & \text{if} \quad x/t > s(\mathbf{q}_L; \mathbf{q}_R), \end{cases} \tag{15}$$

where $s(\mathbf{q}_L; \mathbf{q}_R)$ denotes the shock speed, determined by the Rankine-Hugoniot relation

$$s(\mathbf{q}_L; \mathbf{q}_R)(\mathbf{q}_L - \mathbf{q}_R) = \mathbf{f}(\mathbf{q}_L) - \mathbf{f}(\mathbf{q}_R). \tag{16}$$

Expression (16) is in fact equivalent to (14). Hence, (15) is a valid weak solution of (1).

The general solution to the Riemann problem consists of $n+1$ (possibly empty) regions Ω_l where the solution is constant, separated by simple waves, contact discontinuities and shock waves. Define $\mathbf{q}_0 = \mathbf{q}_L$, $\mathbf{q}_1 = \mathbf{q}_R$ and let $\mathbf{q}_{l/n}, l = 0, \ldots, n$, be the solution in Ω_l. Assuming that $\mathbf{q}_{(l-1)/n}$ is connected to $\mathbf{q}_{l/n}$ by a simple wave, we denote by $\mathbf{h}_l(x/t)$ the similarity solution in the wave region. Conversely, if $\mathbf{q}_{(l-1)/n}$ is connected to $\mathbf{q}_{l/n}$ by a shock wave, we designate s_l the appropriate shock speed. Then, in succinct form:

$$\mathbf{q}(x,t) = \begin{cases} \mathbf{q}_0, & \text{if} \quad x/t < \sigma_0^+, \\ \mathbf{q}_{l/n}, & \text{if} \quad \sigma_l^- < x/t < \sigma_l^+, \quad l = 1,\dots,n-1, \\ \mathbf{h}_l(x/t), & \text{if} \quad \sigma_{l-1}^+ < x/t < \sigma_l^-, \quad l = 1,\dots,n-1, \\ \mathbf{q}_1, & \text{if} \quad x/t > \sigma_n^-, \end{cases} \tag{17a}$$

where $\sigma_l^{\pm}$ denotes the contact speed

$$\sigma_l^{\pm} = \begin{cases} \lambda_{l+(1\pm1)/2}(\mathbf{q}_{l/n}) & \text{if } \pm\lambda_{l+(1\pm1)/2}(\mathbf{q}_{l/n}) < \pm\lambda_{l+(1\pm1)/2}(\mathbf{q}_{(l\pm1)/n}), \\ s_{l+(1\pm1)/2} & \text{otherwise.} \end{cases} \tag{17b}$$

The general solution (17) is schematically depicted in Figure 1. The figure illustrates the contiguity of regions connected by shock waves and contact discontinuities, for instance, $\Omega_{(l-1)/n}$ and $\Omega_{l/n}$, and the separation of regions connected by rarefaction waves, e.g., $\Omega_{l/n}$ and $\Omega_{(l+1)/n}$.

As a side-note, we mention that for general $\mathbf{f}(\mathbf{q})$ and sufficiently large $\|\mathbf{q}_L - \mathbf{q}_R\|$, a solution to (1) can be non-existent (see, e.g., (Smoller, 1983)).

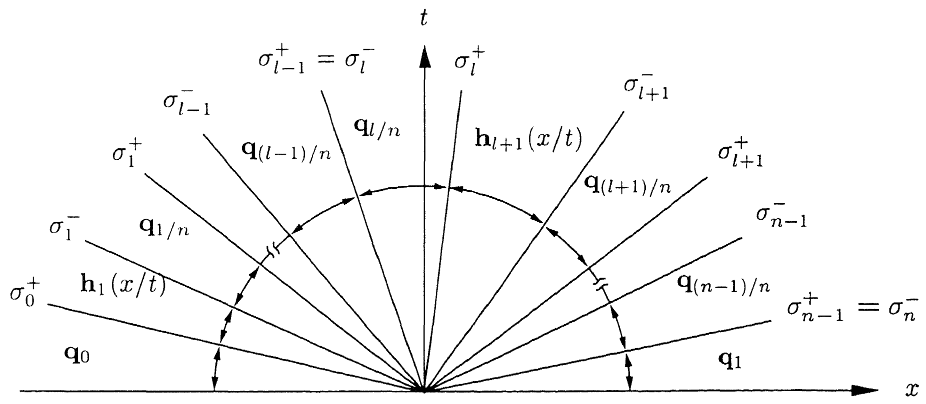

Figure 1. General solution to the Riemann problem

3. Approximate Riemann Solution

In the previous section we established that the solution to the Riemann problem can generally be written in similarity form, $\mathbf{h}(x/t)$. Denoting by

$\mathbf{h}(x/t; \mathbf{q}_L, \mathbf{q}_R)$ the similarity solution for given $\mathbf{q}_L$ and $\mathbf{q}_R$, $\mathbf{f}(\mathbf{h}(0; \mathbf{q}_L, \mathbf{q}_R))$ expresses the corresponding centered flux, next indicated by $\mathbf{f}(\mathbf{q}_L, \mathbf{q}_R)$. This flux is of particular importance in computational applications: following Godunov's approach, it can be interpreted as the flux between two adjacent cells in the discretized domain. Unfortunately, solving the Riemann problem exactly is computationally expensive and it is therefore necessary to revert to approximate solution techniques.

In this section, we investigate Osher's approximate Riemann solver and a modified Osher-type scheme. We will first present a general outline of the Osher scheme. Subsequently, the approximate Riemann solution employed in Osher's scheme is examined and the computed flux approximation is compared to the exact solution. Finally, we shall propose the modified scheme, based on the preceding analysis.

3.1. OSHER'S SCHEME

In the scheme developed by Osher (Osher and Solomon, 1982; Osher and Chakravarthy, 1983), the centered flux, $\mathbf{f}(\mathbf{q}_L, \mathbf{q}_R) = \mathbf{f}(\mathbf{h}(0; \mathbf{q}_L, \mathbf{q}_R))$, is approximated by:

$$\tilde{\mathbf{f}}(\mathbf{q}_L, \mathbf{q}_R) = \frac{1}{2}\mathbf{f}(\mathbf{q}_L) + \frac{1}{2}\mathbf{f}(\mathbf{q}_R) - \frac{1}{2}\int_{\mathbf{q}_L}^{\mathbf{q}_R} |\mathbf{A}(\mathbf{w})| \cdot \mathrm{d}\mathbf{w}, \tag{18}$$

with the absolute value of the Jacobian matrix defined by $|\mathbf{A}(\mathbf{q})| \equiv \mathbf{R}(\mathbf{q}) \cdot |\Lambda(\mathbf{q})| \cdot \mathbf{R}(\mathbf{q})^{-1}$. Here, $|\Lambda(\mathbf{q})| = \mathrm{diag}(|\lambda_1(\mathbf{q})|, \dots, |\lambda_n(\mathbf{q})|)$. Clearly, the integral term is the upwind contribution to the centered flux approximation.

The integral in (18) is evaluated along a path $\Gamma = \{\mathbf{q}(s) : 0 \le s \le 1\} \subset \mathbb{R}^n$ in state space, satisfying $\mathbf{q}(0) = \tilde{\mathbf{q}}_0$ and $\mathbf{q}(1) = \tilde{\mathbf{q}}_1$, with $\tilde{\mathbf{q}}_0 = \mathbf{q}_L$ and $\tilde{\mathbf{q}}_1 = \mathbf{q}_R$ or vice versa. This path is composed of sub-paths Γ_l, $l = 1, 2, \dots, n$, where each of the sub-paths connects two adjacent states $\tilde{\mathbf{q}}_{(l-1)/n}$ and $\tilde{\mathbf{q}}_{l/n}$. Moreover, Γ_l is tangential to an eigenvector $\mathbf{r}_{k(l)}$, where $k : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ is a bijective mapping. It should be appreciated here that Γ_l is thus a section of the $k(l)$-path through $\tilde{\mathbf{q}}_{(l-1)/n}$, connecting $\tilde{\mathbf{q}}_{(l-1)/n}$ and $\tilde{\mathbf{q}}_{l/n}$. Usual choices for the ordering of the sub-paths are the O-variant $k(l) = n - l$ and the P-variant $k(l) = l$.

Rewriting the integral in (18) as a summation of contributions of the integral over each of the sub-paths,

$$\int_{\mathbf{q}_L}^{\mathbf{q}_R} |\mathbf{A}(\mathbf{w})| \cdot \mathrm{d}\mathbf{w} = \sum_{l=1}^{n} \int_{\Gamma_l} |\mathbf{A}(\mathbf{w}(\xi))| \cdot \mathbf{r}_{k(l)}(\mathbf{w}(\xi))\, \mathrm{d}\xi = \sum_{l=1}^{n} \int_{\Gamma_l} \mathrm{sign}(\lambda_{k(l)}(\mathbf{w}))\mathbf{A}(\mathbf{w}) \cdot \mathrm{d}\mathbf{w}. \tag{19}$$

Obviously, if $\lambda_{k(l)}$ does not change sign along Γ_l, then the sub-integral can be evaluated to $[\mathbf{f}(\tilde{\mathbf{q}}_{l/n}) - \mathbf{f}(\tilde{\mathbf{q}}_{(l-1)/n})]\,\mathrm{sign}(\lambda_{\mathrm{k(l)}})$. Then, by (4), if $\lambda_{k(l)} = \lambda_{k(l+1)} = \ldots = \lambda_{k(l+\mu)}$ is a linearly degenerate eigenvalue, the sum in (19) concatenates and we simply obtain

$$\sum_{i=0}^{\mu} \int_{\Gamma_{l+i}} |\mathbf{A}(\mathbf{w})| \cdot d\mathbf{w} = \mathrm{sign}(\lambda_{\mathrm{k(l)}}(\mathbf{q}_{\mathrm{l/n}}))[\mathbf{f}(\mathbf{q}_{\mathrm{l/n}}) - \mathbf{f}(\mathbf{q}_{(\mathrm{l}+\mu)/\mathrm{n}})]. \qquad (20)$$

Hence, the intermediate stages $\tilde{\mathbf{q}}_{(l+i)/n}$, $i = 1, 2, \ldots, \mu - 1$ are of no consequence and can be eliminated from the composed path Γ.

As a result of the choice of the sub-paths Γ_l, the intermediate $\tilde{\mathbf{q}}_{l/n}$, $l = 1, 2, \ldots, n-1$ can be conveniently determined by means of the Riemann invariants: Because the sub-path $\Gamma_l \subset \mathcal{R}_{k(l)}(\tilde{\mathbf{q}}_{(l-1)/n})$,

$$\psi^m_{k(l)}(\tilde{\mathbf{q}}_{(l-1)/n}) = \psi^m_{k(l)}(\tilde{\mathbf{q}}_{l/n}), \qquad l, m = 1, 2, \ldots, n, \quad m \neq k(l), \qquad (21)$$

see section 2.1. If it is assumed that the k-Riemann invariants in (21) have linearly independent gradients, then by the implicit function theorem, (21) constitutes a solvable system of equations from which the $\tilde{\mathbf{q}}_{l/n}$, $l = 1, 2, \ldots, n$ can be extracted. In many practical cases the intermediate stages can then be solved explicitly from (21). Once the intermediate states $\tilde{\mathbf{q}}_{l/n}$ have been obtained, the flux approximation $\tilde{\mathbf{f}}(\mathbf{q}_L, \mathbf{q}_R)$ can be computed using (18), (19).

3.2. ACCURACY

The flux computed by means of the Osher scheme, $\tilde{\mathbf{f}}(\mathbf{q}_L, \mathbf{q}_R)$, relies on an approximate solution of the Riemann problem. Because the approximation can again be written in similarity form, it is useful to introduce the notation $\tilde{\mathbf{f}}(\mathbf{q}_L, \mathbf{q}_R) = \mathbf{f}(\tilde{\mathbf{h}}(0; \mathbf{q}_L, \mathbf{q}_R))$, where $\tilde{\mathbf{h}}(x/t; \mathbf{q}_L, \mathbf{q}_R)$ stands for the approximate similarity solution. In this section we investigate the accuracy of the approximate similarity solution and of the corresponding centered flux approximation.

To evaluate the accuracy of the approximate solution, we examine the inherent representation of simple waves, contact discontinuities and shock waves. In section 3.1 it was emphasized that the sub-paths, Γ_l, in Osher's scheme are actually sections of $k(l)$-paths. Referring to section 2.2, it follows that the intermediate states $\tilde{\mathbf{q}}_{l/n}$, $l = 0, \ldots, n$, in the approximate solution are connected by simple-waves only. Clearly, this representation is correct for simple waves and contact discontinuities. However, shock waves in the actual solution are then replaced by so-called *overturned simple waves*, see (van Leer, 1984). We will now show that this representation is accurate for weak shocks. From (Smoller, 1983) we adopt:

Lemma 1 *Suppose* $\mathbf{q}_L$ *and* $\mathbf{q}_R$ *are connected by a weak k-shock with shock strength* ϵ, *i.e.,* $\mathbf{q}_R \in \mathcal{S}_k(\mathbf{q}_L)$ *and* $\lambda_k(\mathbf{q}_L) = \lambda_k(\mathbf{q}_R) + \epsilon$, *with* ϵ *a small positive number. Then the change in a k-Riemann invariant across the k-shock is of order* $\mathcal{O}(\epsilon^3)$.

Proof: *Proof is omitted here, but can be found in (*Smoller, 1983, pages 326–333*).* □

Then, we obtain:

Theorem 3 *Suppose* $\mathbf{q}_R \in \mathcal{S}_k(\mathbf{q}_L)$ *and* $\lambda_k(\mathbf{q}_L) = \lambda_k(\mathbf{q}_R) + \epsilon$. *Then a* $\tilde{\mathbf{q}}_R \in \mathcal{R}_k(\mathbf{q}_L)$ *exists such that* $\lambda_k(\tilde{\mathbf{q}}_R) = \lambda_k(\mathbf{q}_R)$ *and* $|\tilde{\mathbf{q}}_R - \mathbf{q}_R|$ *is of order* $\mathcal{O}(\epsilon^3)$.

Proof: *By definition 5,* $\psi_k^m(\mathbf{q}_L) = \psi_k^m(\tilde{\mathbf{q}}_R)$, $k = 1, 2, \ldots, n$, $k \neq m$. *Then, by lemma 1,*

$$\psi_k^m(\tilde{\mathbf{q}}_R) = \psi_k^m(\mathbf{q}_R) + \mathcal{O}(\epsilon^3). \tag{22}$$

System (22) can be augmented with $\lambda_k(\tilde{\mathbf{q}}_R) = \lambda_k(\mathbf{q}_R)$ *to obtain* n *equations for* $\tilde{\mathbf{q}}_R$. *Because* $\mathrm{rank}(\partial_{\mathbf{q}}\psi_k^1, \ldots, \partial_{\mathbf{q}}\psi_k^n) = n - 1$ *and* $\partial_{\mathbf{q}}\lambda_k \in (\partial_{\mathbf{q}}\psi_k^1, \ldots, \partial_{\mathbf{q}}\psi_k^n)^{\perp}$, *it follows that* $\det(\partial_{\mathbf{q}}\psi_k^1, \ldots, \partial_{\mathbf{q}}\psi_k^n, \partial_{\mathbf{q}}\lambda_k) \neq 0$. *The result now simply follows by a Taylor expansion around* $\mathbf{q}_R$ *of the terms in* $\tilde{\mathbf{q}}_R$ *of the augmented system.* □

From Theorem 3 it may be inferred that the intermediate states obtained by a rarefaction-waves-only approximation are $\mathcal{O}(\epsilon_{\max}^3)$ accurate, with

$$\epsilon_{\max} = \max_{l=1\ldots n} \left(\lambda_l(\mathbf{q}_{(l-1)/n}) - \lambda_l(\mathbf{q}_{l/n}), 0\right) \tag{23}$$

the strength of the strongest shock.

Although the computed intermediate states are accurate even in the presence of (weak) shocks, the flux approximation $\tilde{\mathbf{f}}(\mathbf{q}_L, \mathbf{q}_R)$ is not necessarily so. By (19), if $\tilde{\mathbf{q}}_R \in \mathcal{R}_k(\mathbf{q}_L)$ and $\lambda_k(\mathbf{q}_L) > 0 > \lambda_k(\tilde{\mathbf{q}}_R)$,

$$\tilde{\mathbf{f}}(\mathbf{q}_L, \tilde{\mathbf{q}}_R) = \mathbf{f}(\mathbf{q}_L) + \mathbf{f}(\tilde{\mathbf{q}}_R) - \mathbf{f}(\mathbf{q}^\star), \tag{24}$$

with $\mathbf{q}^\star \in \mathcal{R}_k(\mathbf{q}_L)$ such that $\lambda_k(\mathbf{q}^\star) = 0$. In contrast, the actual flux corresponding to the *k-shock* reads $\mathbf{f}(\mathbf{q}_L)$ if $s(\mathbf{q}_R; \mathbf{q}_L) > 0$ and $\mathbf{f}(\mathbf{q}_R)$ if $s(\mathbf{q}_R; \mathbf{q}_L) < 0$. Consequently, the error in the approximate flux in the case of a centered shock with strength ϵ can be of $\mathcal{O}(\epsilon)$. An instance of such failure of Osher's flux is discussed in (van Leer, 1984).

3.3. MODIFIED OSHER SCHEME

In view of the above, a modification of the scheme is advocated. The rarefaction-waves-only approximation of the similarity solution is maintained. However, the centered flux approximation is obtained differently, to avoid loss of accuracy due to centered shock waves.

We propose to extract the intermediate states in the approximate solution to the Riemann problem from

$$\psi_l^m(\tilde{\mathbf{q}}_{(l-1)/n}) = \psi_l^m(\tilde{\mathbf{q}}_{l/n}), \qquad l, m = 1, 2, \ldots, n, \quad m \neq l, \tag{25}$$

with $\tilde{\mathbf{q}}_0 = \mathbf{q}_L$ and $\tilde{\mathbf{q}}_1 = \mathbf{q}_R$. This is in fact equivalent to (21) with a presumed P-variant ordering of the sub-paths. Next, approximate contact speeds $\tilde{\sigma}_l^\pm$ are obtained:

$$\tilde{\sigma}_l^\pm = \begin{cases} \lambda_{l+(1\pm1)/2}(\tilde{\mathbf{q}}_{l/n}) & \text{if } \pm\lambda_{l+(1\pm1)/2}(\tilde{\mathbf{q}}_{l/n}) < \pm\lambda_{l+(1\pm1)/2}(\tilde{\mathbf{q}}_{(l\pm1)/n}), \\ \tilde{s}_{l+(1\pm1)/2} & \text{otherwise}, \end{cases} \tag{26a}$$

with

$$\tilde{s}_{l+(1\pm1)/2} = \frac{1}{2}\lambda_{l+(1\pm1)/2}(\tilde{\mathbf{q}}_{l/n}) + \frac{1}{2}\lambda_{l+(1\pm1)/2}(\tilde{\mathbf{q}}_{(l\pm1)/n}). \tag{26b}$$

Estimate (26b) of the shock speed is justified by the following theorem, taken from (Smoller, 1983):

Theorem 4 *Suppose* $\mathbf{q}_R \in \mathcal{S}_k(\mathbf{q}_L)$ *and* $\lambda_k(\mathbf{q}_L) = \lambda_k(\mathbf{q}_R) + \epsilon$, $\epsilon > 0$. *Then the speed of the k-shock connecting* $\mathbf{q}_L$ *and* $\mathbf{q}_R$ *satisfies* $s(\mathbf{q}_L; \mathbf{q}_R) = \frac{1}{2}\lambda_k(\mathbf{q}_L) + \frac{1}{2}\lambda_k(\mathbf{q}_R) + \mathcal{O}(\epsilon^2)$.

Proof: *Proof can be found in (*Smoller, 1983, pages 326–333*).* □

Once the intermediate states and contact speeds have been established, the approximate Riemann solution can be constructed in a manner similar to (17a). However, considering that our purpose is to compute an approximation to the centered flux, we only need to obtain the central part of the approximate solution:

$$\tilde{\mathbf{h}}(0; \mathbf{q}_L, \mathbf{q}_R) = \begin{cases} \tilde{\mathbf{q}}_0, & \text{if} \quad \tilde{\sigma}_0^+ > 0, \\ \tilde{\mathbf{q}}_{l/n}, & \text{if} \quad \tilde{\sigma}_l^- < 0 < \tilde{\sigma}_l^+, \quad l \in \{1, \ldots, n-1\}, \\ \tilde{\mathbf{q}}^*, & \text{if} \quad \tilde{\sigma}_{l-1}^+ < 0 < \tilde{\sigma}_l^-, \quad l \in \{1, \ldots, n-1\}, \\ \tilde{\mathbf{q}}_1, & \text{if} \quad \sigma_n^- < 0, \end{cases} \tag{27}$$

with $\tilde{\mathbf{q}}^* \in \mathcal{R}_l(\tilde{\mathbf{q}}_{(l-1)/n})$ such that $\lambda_l(\tilde{\mathbf{q}}^*) = 0$ in case of a centered rarefaction wave. The centered flux approximation is now simply $\tilde{\mathbf{f}}(\mathbf{q}_L, \mathbf{q}_R) = \mathbf{f}(\tilde{\mathbf{h}}(0; \mathbf{q}_L, \mathbf{q}_R))$.

4. Applications in Hydrodynamics

In the previous section we presented a flux-difference splitting scheme that gives an accurate approximation of the centered flux in the Riemann problem, even in the presence of (weak) centered shock waves. A prerequisite for the flux evaluation is the derivation of the intermediate states $\tilde{\mathbf{q}}_{l/n}$, $l = 1, \ldots, n$. Once these states have been obtained, the flux calculation proceeds via straightforward operations.

In this section we derive the intermediate states for the one-dimensional Euler equations for three types of fluids that are commonly used to model the behavior of water. These fluids are, successively, a genuinely compressible fluid, an artificially compressible fluid and an incompressible fluid. Furthermore, we obtain the intermediate states for the Euler equations in the case of an immiscible, compressible two-phase flow.

4.1. COMPRESSIBLE FLUID

Suppose that u, v and w denote the x, y and z components of a fluid velocity $\mathbf{u} \in \mathbb{R}^3$ in a Cartesian coordinate system, respectively, and that $\rho \in \mathbb{R}^+$ denotes the density of the fluid. Consider the hyperbolic system (1a) with $\mathbf{q} = (\rho u, \rho v, \rho w, \rho)^T$ and $\mathbf{f}(\mathbf{q})$ given by

$$\mathbf{f}(\mathbf{q}) = \left(q_1^2/q_4 + p(q_4), q_1 q_2/q_4, q_1 q_3/q_4, q_1\right)^T . \tag{28}$$

Then equations (1a) are the Euler equations for a compressible fluid in one dimension. In this section it is assumed that the pressure is related to the density via an equation of state of the form $p = p(\rho)$, with $p \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ an increasing function. An example is Tait's equation of state, which is often used to model the behavior of water:

$$p(\rho) = \alpha \rho^\gamma + \beta, \tag{29}$$

where $\alpha, \gamma \in]0, \infty[$ and $\beta \in \mathbb{R}$ are given constants. Our objective now is to obtain the approximate intermediate states for the Euler equations (1a), (28).

In order to compute the intermediate states from (25), k-Riemann invariants for the system under consideration have to be derived first. The Jacobian of the flux vector (28) reads

$$\mathbf{A}(\mathbf{q}) = \begin{pmatrix} 2q_1/q_4 & 0 & 0 & -q_1^2/q_4^2 + c^2(q_4) \\ q_2/q_4 & q_1/q_4 & 0 & -q_1 q_2/q_4 \\ q_3/q_4 & 0 & q_1/q_4 & -q_3 q_1/q_4 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{30}$$

where $c(\rho) = \sqrt{\partial p/\partial \rho}$ denotes the speed of sound. Computation of the eigenvalues of $\mathbf{A}(\mathbf{q})$ and the corresponding eigenvectors then yields

$$\lambda_1 = q_1/q_4 - c(q_4), \quad \lambda_{2,3} = q_1/q_4, \quad \lambda_4 = q_1/q_4 + c(q_4), \tag{31}$$

and

$$\begin{aligned}
\mathbf{r}_1 &= (q_1/q_4 - c(q_4), q_2/q_4, q_3/q_4, 1)^T, \\
\mathbf{r}_2 &= (0,1,0,0)^T, \\
\mathbf{r}_3 &= (0,0,1,0)^T, \\
\mathbf{r}_4 &= (q_1/q_4 + c(q_4), q_2/q_4, q_3/q_4, 1)^T.
\end{aligned} \tag{32}$$

Notice that the eigenvalue λ_k and the eigenvector $\mathbf{r}_k$ are genuinely nonlinear for $k = 1, 4$ and linearly degenerate for $k = 2, 3$. Riemann invariants are then obtained by solving partial differential equations (8), with the eigenvectors according to (32). The details are omitted here, but it can easily be verified that (33) constitutes a complete set of k-Riemann invariants:

$$\begin{aligned}
&\psi_1^2 = q_1/q_4 + \phi(q_4), && \psi_1^3 = q_2/q_4, && \psi_1^4 = q_3/q_4, \\
&\psi_2^1 = q_1, && \psi_2^3 = q_3, && \psi_2^4 = q_4, \\
&\psi_3^1 = q_1, && \psi_3^2 = q_2, && \psi_3^4 = q_4, \\
&\psi_4^1 = q_1/q_4 - \phi(q_4), && \psi_4^2 = q_2/q_4, && \psi_4^3 = q_3/q_4,
\end{aligned} \tag{33a}$$

where $\phi(\rho)$ is defined by

$$\phi(\rho) = \int_0^\rho \frac{c(\eta)}{\eta} \mathrm{d}\eta. \tag{33b}$$

The intermediate states can now be extracted from (25), (33). In view of the linear degeneracy of the eigenvalues λ_2 and λ_3 and the arguments presented in section 3.1, we ignore $\tilde{\mathbf{q}}_{1/2}$. We then find that $\mathbf{q}_0$ is connected to $\mathbf{q}_1$ via two approximate intermediate states $\tilde{\mathbf{q}}_{1/3}$ and $\tilde{\mathbf{q}}_{2/3}$:

$$\tilde{\mathbf{q}}_{1/3} = \phi^{-1}\left(\tfrac{1}{2}\psi_1^2(\mathbf{q}_0) - \tfrac{1}{2}\psi_4^1(\mathbf{q}_1)\right) \begin{pmatrix} \tfrac{1}{2}\psi_1^2(\mathbf{q}_0) + \tfrac{1}{2}\psi_4^1(\mathbf{q}_1) \\ \psi_1^3(\mathbf{q}_0) \\ \psi_1^4(\mathbf{q}_0) \\ 1 \end{pmatrix} \tag{34}$$

and

$$\tilde{\mathbf{q}}_{2/3} = \phi^{-1}\left(\tfrac{1}{2}\psi_1^2(\mathbf{q}_0) - \tfrac{1}{2}\psi_4^1(\mathbf{q}_1)\right) \begin{pmatrix} \tfrac{1}{2}\psi_1^2(\mathbf{q}_0) + \tfrac{1}{2}\psi_4^1(\mathbf{q}_1) \\ \psi_4^2(\mathbf{q}_1) \\ \psi_4^3(\mathbf{q}_1) \\ 1 \end{pmatrix}, \tag{35}$$

where $\phi^{-1}(\psi)$ denotes the inverse of $\phi(\psi)$.

For a fluid that is described by Tait's equation of state, the intermediate states can be determined by substituting (29) in equations (33) to (35). The intermediate velocity components $\tilde{v}_{1/3}$, $\tilde{v}_{2/3}$, $\tilde{w}_{1/3}$ and $\tilde{w}_{2/3}$ are immediately obtained from (33):

$$\begin{aligned} \tilde{v}_{1/3} &= v_0, & \tilde{v}_{2/3} &= v_1, \\ \tilde{w}_{1/3} &= w_0, & \tilde{w}_{2/3} &= w_1. \end{aligned} \tag{36}$$

From (33) it is also clear that $\tilde{u}_{1/3} = \tilde{u}_{2/3} \equiv \tilde{u}_{1/2}$ and $\tilde{\rho}_{1/3} = \tilde{\rho}_{2/3} \equiv \tilde{\rho}_{1/2}$. To determine $\tilde{u}_{1/2}$ and $\tilde{\rho}_{1/2}$, it is necessary to distinguish between the cases $\gamma = 1$ and $\gamma \neq 1$. For $\gamma = 1$ one obtains

$$\begin{aligned} \tilde{u}_{1/2} &= \frac{1}{2}(u_0 + u_1) + \frac{\sqrt{\alpha}}{2}\ln(\rho_0/\rho_1), \\ \tilde{\rho}_{1/2} &= \sqrt{\rho_0\rho_1}\exp\left(\frac{u_0 - u_1}{2\sqrt{\alpha}}\right). \end{aligned} \tag{37}$$

In case $\gamma \neq 1$, it is convenient to express the density in terms of the speed of sound:

$$\begin{aligned} \tilde{u}_{1/2} &= \frac{1}{2}(u_0 + u_1) + \frac{1}{\gamma - 1}[c(\rho_0) - c(\rho_1)], \\ c(\tilde{\rho}_{1/2}) &= \frac{\gamma - 1}{4}(u_0 - u_1) + \frac{1}{2}[c(\rho_0) + c(\rho_1)]. \end{aligned} \tag{38}$$

4.2. ARTIFICIALLY COMPRESSIBLE FLUID

Assume that u, v and w again denote the x, y and z components of a fluid velocity $\mathbf{u} \in \mathbb{R}^3$ in a Cartesian coordinate system, respectively, and that $p \in \mathbb{R}^+$ denotes the fluid pressure. Consider hyperbolic system (1a) with $\mathbf{q} = (u, v, w, p)^T$. Let $\mathbf{f}(\mathbf{q})$ be

$$\mathbf{f}(\mathbf{q}) = \left(q_1^2 + q_4, q_1q_2, q_1q_3, c^2q_1\right)^T, \tag{39}$$

with c constant. Equations (1a), (39) are the Euler equations for an artificially compressible fluid in one dimension. Notice that the $\partial p/\partial t$ term that occurs in (1a) in this case, implies compressibility of the fluid.

To determine the intermediate states, we first derive Riemann invariants for (1a), (39). For the Jacobian of $\mathbf{f}(\mathbf{q})$ we simply obtain

$$\mathbf{A}(\mathbf{q}) = \begin{pmatrix} 2q_1 & 0 & 0 & 1 \\ q_2 & q_1 & 0 & 0 \\ q_3 & 0 & q_1 & 0 \\ c^2 & 0 & 0 & 0 \end{pmatrix}. \tag{40}$$

The eigenvalues and corresponding eigenvectors of $\mathbf{A}(\mathbf{q})$ follow by straightforward computation:

$$\lambda_1 = q_1 - \sqrt{q_1^2 + c^2}, \quad \lambda_{2,3} = q_1, \quad \lambda_4 = q_1 + \sqrt{q_1^2 + c^2}, \tag{41}$$

and

$$\begin{aligned}
\mathbf{r}_1 &= \left(1, -q_2/\sqrt{q_1^2 + c^2}, -q_3/\sqrt{q_1^2 + c^2}, -q_1 - \sqrt{q_1^2 + c^2}\right)^T, \\
\mathbf{r}_2 &= (0, 1, 0, 0)^T, \\
\mathbf{r}_3 &= (0, 0, 1, 0)^T, \\
\mathbf{r}_4 &= \left(1, q_2/\sqrt{q_1^2 + c^2}, q_3/\sqrt{q_1^2 + c^2}, -q_1 + \sqrt{q_1^2 + c^2}\right)^T.
\end{aligned} \tag{42}$$

The eigenpairs $(\lambda_1, \mathbf{r}_1)$ and $(\lambda_4, \mathbf{r}_4)$ are genuinely non-linear, whereas the eigenpairs $(\lambda_2, \mathbf{r}_2)$ and $(\lambda_3, \mathbf{r}_3)$ are linearly degenerate. Riemann invariants are now obtained by solving (8), (42):

$$\begin{aligned}
&\psi_1^2 = q_2\lambda_4, &&\psi_1^3 = q_3\lambda_4, &&\psi_1^4 = \lambda_4 \exp\left([2q_4 + q_1\lambda_4]/c^2\right), \\
&\psi_2^1 = q_1, &&\psi_2^3 = q_3, &&\psi_2^4 = q_4, \\
&\psi_3^1 = q_1, &&\psi_3^2 = q_2, &&\psi_3^4 = q_4, \\
&\psi_4^1 = q_2\lambda_1, &&\psi_4^2 = q_3\lambda_1, &&\psi_4^3 = \lambda_1 \exp\left([2q_4 + q_1\lambda_1]/c^2\right).
\end{aligned} \tag{43}$$

The foregoing invariants have linearly independent gradients. Hence, the intermediate states can be obtained from (25), (43).

Considering the linear degeneracy of λ_2, λ_3, we only need to obtain $\tilde{\mathbf{q}}_{1/3}$ and $\tilde{\mathbf{q}}_{2/3}$. Unfortunately, in this instance we have not succeeded in deriving a closed form expression for these intermediate states. However, from (43) it immediately follows that $\tilde{u}_{1/3} = \tilde{u}_{2/3} \equiv \tilde{u}_{1/2}$ and $\tilde{\rho}_{1/3} = \tilde{\rho}_{2/3} \equiv \tilde{\rho}_{1/2}$. Then, using the expressions for ψ_1^4 and ψ_4^3, one finds that $\tilde{u}_{1/2}$ is determined by the implicit relation:

$$\left(\frac{\tilde{u}_{1/2} + \sqrt{\tilde{u}_{1/2}^2 + c^2}}{\tilde{u}_{1/2} - \sqrt{\tilde{u}_{1/2}^2 + c^2}}\right) \exp\left(\frac{2\tilde{u}_{1/2}\sqrt{\tilde{u}_{1/2}^2 + c^2}}{c^2}\right) = \frac{\psi_1^4(\mathbf{q}_0)}{\psi_4^3(\mathbf{q}_1)}. \tag{44}$$

Once $\tilde{u}_{1/2}$ has been solved from (44), $\tilde{\mathbf{q}}_{1/3}$ and $\tilde{\mathbf{q}}_{2/3}$ are simply obtained from (43).

4.3. INCOMPRESSIBLE FLUID

We consider the Euler equations for an incompressible flow. Assume that $\mathbf{u} \in \mathbb{R}^3$ denotes the fluid velocity and that the fluid pressure divided by the (constant) fluid density is designated $p \in \mathbb{R}^+$. Next, let $\mathbf{uu} \in C^1(\mathbb{R}^3, \mathbb{R}^{3\times3})$ be the convective momentum flux tensor. The Euler equations for an incompressible fluid read

$$\frac{\partial \mathbf{u}}{\partial t} + \operatorname{div} \mathbf{uu} + \boldsymbol{\nabla} \mathrm{p} = 0, \tag{45a}$$

$$\operatorname{div} \mathbf{u} = 0. \tag{45b}$$

Due to the absence of a time derivative in (45b), equations (45) do not constitute a hyperbolic system. However, equation (45a) can trivially be recast into an inhomogeneous hyperbolic system governing $\mathbf{u}$:

$$\frac{\partial \mathbf{u}}{\partial t} + \operatorname{div} \mathbf{uu} = -\boldsymbol{\nabla} \mathrm{p}. \tag{46}$$

Equation (45b) is then interpreted as a constraint on $\mathbf{u}$ and p serves as a Lagrangian multiplier to enforce the constraint. Solving the Euler equations for an incompressible fluid now requires the resolution of the hyperbolic system (46) subject to the constraint (45b). Here, we shall only concern ourselves with the hyperbolic part of the operator. Furthermore, in the following section we will only consider the homogeneous system in one dimension, i.e., we shall neglect the forcing term $-\boldsymbol{\nabla} p$ and (assuming a Cartesian coordinate system is employed) the flux gradients in the y and z direction. We then retrieve an expression of the form (1a), with $\mathbf{q} = (u, v, w)^T$, where u, v, w again denote the x, y, z components of the fluid velocity $\mathbf{u} \in \mathbb{R}^3$ in a Cartesian coordinate system, respectively, and $\mathbf{f}(\mathbf{q})$ given by

$$\mathbf{f}(\mathbf{q}) = \left(q_1^2, q_1 q_2, q_1 q_3\right)^T. \tag{47}$$

We acknowledge that the first equation of (1a), (47) is decoupled from the remaining system and can therefore be treated separately. However, for completeness we refrain from doing so.

To obtain the approximate intermediate states for (1a), (47), we first determine Riemann invariants for this system. The Jacobian of $\mathbf{f}(\mathbf{q})$ reads

$$\mathbf{A}(\mathbf{q}) = \begin{pmatrix} 2q_1 & 0 & 0 \\ q_2 & q_1 & 0 \\ q_3 & 0 & q_1 \end{pmatrix}, \tag{48}$$

with the eigenvalues

$$\lambda_1 = q_1^2, \quad \lambda_{2,3} = q_1, \tag{49}$$

and the corresponding eigenvectors

$$\begin{aligned}\mathbf{r}_1 &= (q_1, q_2, q_3)^T,\\ \mathbf{r}_2 &= (0,1,0)^T,\\ \mathbf{r}_3 &= (0,0,1)^T.\end{aligned} \tag{50}$$

The first eigenpair is neither linearly degenerate nor genuinely nonlinear: the gradient of $\lambda_1(\mathbf{q})$ in the direction of $\mathbf{r}_1(\mathbf{q})$ vanishes for $q_1 = 0$, but is nonzero otherwise. Nevertheless, for our purposes it is sufficient to treat $(\lambda_1, \mathbf{r}_1)$ as a genuinely nonlinear eigenpair, because the eigenvalue vanishes only if $q_1 = 0$ and, therefore, the eigenvalue can change sign only once along $\mathcal{R}_1(\mathbf{q}_L)$. The second and third eigenpair are linearly degenerate. Riemann invariants are obtained by solving (8), (50):

$$\begin{aligned}\psi_1^2 &= q_1/q_2, & \psi_1^3 &= q_1/q_3,\\ \psi_2^1 &= q_1, & \psi_2^3 &= q_3,\\ \psi_3^1 &= q_1, & \psi_3^2 &= q_2.\end{aligned} \tag{51}$$

These invariants have linearly independent gradients in $\mathbb{R}^3$.

Because the second and third eigenpair are linearly degenerate, $\mathbf{q}_0$ and $\mathbf{q}_1$ are connected via a single intermediate state $\tilde{\mathbf{q}}_{1/2}$. This intermediate state is immediately obtained from (25), (51):

$$\tilde{\mathbf{q}}_{1/2} = \begin{cases} \mathbf{0}, & \text{if } u_0 = 0,\\ \dfrac{u_1}{u_0}\mathbf{q}_0, & \text{otherwise.}\end{cases} \tag{52}$$

4.4. TWO-PHASE FLOW

In this section we derive the intermediate states for the Euler equations for an immiscible, compressible two-phase flow. The phases are supposed to be separated by a moving interface, which is described by the time-dependent set $\mathcal{I}(t) = \{\mathbf{x} \in \mathbb{R}^3 \mid \theta(\mathbf{x}, t) = 0\}$. Furthermore, we assume $\theta(\mathbf{x}, t)$ to be negative in one phase and positive in the other. As a result of the immiscibility of the phases, the following kinematic condition applies:

$$\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \vec{\nabla}\theta = 0, \tag{53}$$

where $\mathbf{u} \in \mathbb{R}^3$ again denotes the fluid velocity. Employing the continuity equation for compressible fluids, we can restate kinematic condition (53) in conservation form:

$$\frac{\partial \rho\theta}{\partial t} + \vec{\nabla} \cdot \rho\theta\mathbf{u} = \rho\left(\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \vec{\nabla}\theta\right) + \theta\left(\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \rho\mathbf{u}\right). \tag{54}$$

The first term in parentheses vanishes due to (53), the second due to continuity. Hence, $\rho\theta$ is a conserved quantity. Suppose that throughout the entire fluid volume the pressure is related to the density via an equation of state of the form $p = p(\theta, \rho)$. Then, again using u, v, w to designate the velocity components relative to a Cartesian coordinate system and ignoring spatial derivatives in y and z direction, we retrieve (1a), with $\mathbf{q} = (\rho u, \rho v, \rho w, \rho\theta, \rho)^T$ and

$$\mathbf{f}(\mathbf{q}) = \left(q_1^2/q_5 + p(q_4/q_5, q_5), q_1q_2/q_5, q_1q_3/q_5, q_1q_4/q_5, q_1\right)^T. \tag{55}$$

Equations (1a), (55) constitute the one-dimensional Euler equations for an immiscible, compressible two-phase flow.

Our first objective now is to derive Riemann invariants for (1a), (55). We define $c_1 = c_1(\theta, \rho) = \sqrt{\partial p/\partial\theta}$ and $c_2 = c_2(\theta, \rho) = \sqrt{\partial p/\partial\rho}$. Then, the Jacobian of (55) reads:

$$\mathbf{A}(\mathbf{q}) = \begin{pmatrix} 2q_1/q_5 & 0 & 0 & c_1^2/q_5 & -q_1^2/q_5^2 - c_1^2q_4/q_5^2 + c_2^2 \\ q_2/q_5 & q_1/q_5 & 0 & 0 & -q_2q_1/q_5^2 \\ q_3/q_5 & 0 & q_1/q_5 & 0 & -q_3q_1/q_5^2 \\ q_4/q_5 & 0 & 0 & q_1/q_5 & -q_4q_1/q_5^2 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{56}$$

The eigenvalues and eigenvectors of $\mathbf{A}(\mathbf{q})$ are

$$\lambda_1 = q_1/q_5 - c_2, \quad \lambda_{2,3,4} = q_1/q_5, \quad \lambda_5 = q_1/q_5 + c_2, \tag{57}$$

and

$$\begin{aligned} \mathbf{r}_1 &= (q_1/q_5 - c_2, q_2/q_5, q_3/q_5, q_4/q_5, 1)^T, \\ \mathbf{r}_2 &= (0, 1, 0, 0, 0)^T, \\ \mathbf{r}_3 &= (0, 0, 1, 0, 0)^T, \\ \mathbf{r}_4 &= \left(q_1c_1^2, 0, 0, -c_2^2q_5^2 + c_1^2q_4, q_5c_1^2\right)^T, \\ \mathbf{r}_5 &= (q_1/q_5 + c_2, q_2/q_5, q_3/q_5, q_4/q_5, 1)^T. \end{aligned} \tag{58}$$

The eigenvalue λ_k and the eigenvector $\mathbf{r}_k$ are genuinely nonlinear for $k = 1, 5$ and linearly degenerate for $k = 2, 3, 4$. Riemann invariants can now be obtained by solving (8), (58):

$$\begin{array}{llll} \psi_1^2 = q_1/q_5 + \phi, & \psi_1^3 = q_2/q_5, & \psi_1^4 = q_3/q_5, & \psi_1^5 = q_4/q_5, \\ \psi_2^1 = q_1/q_5, & \psi_2^3 = q_3, & \psi_2^4 = q_4, & \psi_2^5 = p, \\ \psi_3^1 = q_1/q_5, & \psi_3^2 = q_2, & \psi_3^4 = q_4, & \psi_3^5 = p, \\ \psi_4^1 = q_1/q_5, & \psi_4^2 = q_2, & \psi_4^3 = q_3, & \psi_4^5 = p, \\ \psi_5^1 = q_1/q_5 - \phi, & \psi_5^2 = q_2/q_5, & \psi_5^3 = q_3/q_5, & \psi_5^4 = q_4/q_5, \end{array} \tag{59a}$$

with $p = p(\theta, \rho)$ and $\phi = \phi(\theta, \rho)$ defined by

$$\phi(\theta, \rho) = \int_0^\rho \frac{c_2(\theta, \eta)}{\eta} \mathrm{d}\eta. \tag{59b}$$

Observe that θ is a k-Riemann invariant for $k \in \{1, 5\}$. Hence, it may be inferred that the phase transition is a contact discontinuity. Moreover, because both u and p are k-Riemann invariants for $k \in \{2, 3, 4\}$, the pressure and the normal velocity component are continuous across the interface.

The intermediate states can now be obtained from (25), (59). Because the linearly degenerate eigenvalue q_1/q_5 has algebraic multiplicity 3, only two intermediate states have to be distinguished. Trivially,

$$\begin{pmatrix} \tilde{v}_{1/3} \\ \tilde{w}_{1/3} \\ \tilde{\theta}_{1/3} \end{pmatrix} = \begin{pmatrix} v_0 \\ w_0 \\ \theta_0 \end{pmatrix}, \qquad \begin{pmatrix} \tilde{v}_{2/3} \\ \tilde{w}_{2/3} \\ \tilde{\theta}_{2/3} \end{pmatrix} = \begin{pmatrix} v_1 \\ w_1 \\ \theta_1 \end{pmatrix}, \tag{60}$$

and $\tilde{u}_{1/3} = \tilde{u}_{2/3} \equiv \tilde{u}_{1/2}$. Then, $\tilde{\rho}_{1/3}$ and $\tilde{\rho}_{2/3}$ are determined by

$$\begin{array}{rcl} \phi(\theta_0, \tilde{\rho}_{1/3}) + \phi(\theta_1, \tilde{\rho}_{2/3}) & = & u_0 - u_1 + \phi(\theta_0, \rho_0) + \phi(\theta_1, \rho_1), \\ p(\theta_0, \tilde{\rho}_{1/3}) & = & p(\theta_1, \tilde{\rho}_{2/3}). \end{array} \tag{61}$$

We refrain from a further reduction of these expressions and suffice by stating that once the intermediate densities have been obtained, $\tilde{u}_{1/2}$ follows by straightforward computation.

5. Conclusions

In spite of the absence of shock waves in most hydrodynamic applications, sufficient reason remains to employ Godunov-type schemes in this field. The shock capturing ability of these schemes renders them notably useful in the case of two-phase flow. In the present work we developed an Osher-type Riemann solver and we investigated several of its applications in the field of hydrodynamics. First, the Riemann problem was examined. Subsequently, Osher's approximate Riemann solver was discussed. It was shown that this scheme employs a rarefaction-waves-only approximate Riemann solution and that this approximation is accurate even in the presence of (weak) shocks. Then, it was demonstrated that the centered flux approximation obtained by means of Osher's scheme is not necessarily accurate and, therefore, a modified scheme was proposed. Finally, details were presented for several types of fluid-models commonly used in hydrodynamics.

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