Grid Approximation of a Singularly Perturbed Boundary Value Problem Modelling Heat Transfer in the Case of Flow over a Flat Plate with Suction of the Boundary Layer *

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1. Introduction

Numerical analysis of laminar flows of incompressible fluid for large Reynolds and/or Péclet numbers often leads to the consideration of boundary value problems for boundary layer equations. These quasilinear equations are singularly perturbed, with two perturbation parameters ε_R and ε_P defined by $\varepsilon_R = Re^{-1}$ and $\varepsilon_P = Pe^{-1}$, where Re and Pe are the Reynolds and Péclet numbers; $Pe = Re\,Pr$, Pr is the Prandtl number. Parabolic and regular layers are typical for such problems [1, 2]. Singularities of the same type occur in problems modelling heat transfer processes for flow past surfaces in the case of boundary layers controlled by suction of some amount of the flowing fluid (see, for example, [1, Chapter 14]).

The presence of parabolic boundary and/or interior layers in such problems results in large errors (for small values of the perturbation parameters ε_1, ε_2 multiplying the space derivatives involved in the equations) if we apply classical methods for finding numerical solutions. Thus, it necessarily requires to develop special numerical methods whose errors do not depend on the value of the vector-parameter $\varepsilon = (\varepsilon_1, \varepsilon_2)$, i.e. methods which converge ε-uniformly. Possible approaches to constructing such methods and also some special schemes are given, for example, in [3–9]; see also references therein).

In the present paper we consider a boundary value problem on a semiaxis $(0, \infty)$ for a singularly perturbed parabolic equation with the two perturbation parameters ε_1 and ε_2 multiplying the derivatives with respect to the space variable. Depending on the value of the parameter ε_2 multiplying the first derivative in x, the differential equation can be either of reaction-diffusion type (for $\varepsilon_2 < \varepsilon_1^{1/2}$) or of convection-diffusion type (for $\varepsilon_2 \gg \varepsilon_1^{1/2}$). Correspondingly, the boundary layer can be either parabolic or regular. Model problems of such type appear in the mathematical modeling of heat transfer processes for flow past a flat plate with continuous suction of fluid out of the boundary layer (see, for example, Section 3). Errors of classical numerical methods applied to the problem in question can be unsatisfactorily large for small values of the parameter ε_1. Standard methods allow one to obtain satisfactory numerical approximations to the solution only under the very restrictive condition imposed on the number of mesh points $N^{-1} \ll \varepsilon_1(\varepsilon_1^{1/2} + \varepsilon_2^2)^{-1}$, where N defines the number of nodes in the space mesh on a unit interval

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(see condition (4.6) in Section 4). At the same time, the technique for constructing ε-uniformly convergent schemes based on a fitted operator method turns out to be inapplicable to such problems due to the presence of parabolic boundary layers in the solution (see Remark 1 in Section 4). For the problem under consideration we construct a monotone finite difference scheme (on piecewise uniform meshes) which converges ε-uniformly with the rate $\mathcal{O}\left(N^{-1}\ln N + N_0^{-1}\right)$, where N_0 defines the number of nodes in the time mesh. We present the results of numerical experiments illustrating the efficiency of the constructed scheme.

Note that special difference schemes for the problem studied in this paper, which converge ε-uniformly (in the maximum norm), are unknown in the literature.

2. Problem formulation. Aim of research

1. On the set $\overline{G}$, where

$$\overline{G} = G \cup S, \quad G = D \times (0,T], \quad D = (0,\infty), \tag{2.1}$$

we consider the following boundary value problem for the singularly perturbed parabolic equation

$$L\,u(x,t) \equiv \left\{\varepsilon_1 a(x,t)\frac{\partial^2}{\partial x^2} + \varepsilon_2 b(x,t)\frac{\partial}{\partial x} - c(x,t) - p(x,t)\frac{\partial}{\partial t}\right\} u(x,t) = f(x,t), \quad (x,t) \in G,$$

$$u(x,t) = \Phi(x,t), \quad (x,t) \in S. \tag{2.2}$$

Here the parameters ε_1 and ε_2, which are the components of the vector-parameter ε (or, shortly, of the parameter ε), take arbitrary values from the half-interval $(0,1]$ and the segment $[0,1]$ respectively. The coefficients $a(x,t)$, $b(x,t)$, $c(x,t)$, $p(x,t)$ and the right-hand side $f(x,t)$ are sufficiently smooth functions on $\overline{G}$ satisfying the condition [1]

$$a_0 \le a(x,t) \le a^0, \quad b_0 \le b(x,t) \le b^0, \quad 0 \le c(x,t) \le c^0, \quad p_0 \le p(x,t) \le p^0, \tag{2.3a}$$
$$|\, f(x,t)\,| \le M, \quad (x,t) \in \overline{G}, \qquad a_0,\, b_0,\, p_0 > 0,$$

the boundary function $\Phi(x,t) = \Phi(x,t;\varepsilon)$ for a fixed value of the parameter ε is sufficiently smooth on the sets $\overline{S}^{\,L}$ and S_0 and continuous on S, moreover,

$$|\,\Phi(x,t)\,| \le M, \quad (x,t) \in S; \tag{2.3b}$$

$S = S^L \cup S_0$, S^L and S_0 are the lateral and bottom parts of the boundary S; $S^L = \Gamma \times (0,T]$, $S_0 = \overline{D} \times \{t=0\}$, $\Gamma = \overline{D} \setminus D$.

The solution of the boundary value problem is regarded as a function $u \in C^{2,1}(G) \cap C(\overline{G})$ (bounded on $\overline{G}$) satisfying the differential equation on G and the boundary condition on S.

For simplicity, we suppose that on the set $S^c = \overline{S}^{\,L} \cap S_0$, i.e. at the "corner" points, the compatibility conditions (see, e.g., [10]) are satisfied which ensure the required smoothness of the solution of the problem for each fixed value of the parameter ε.

2. We now discuss more precise conditions imposed on the function $\Phi(x,t)$.

In the case when the following condition holds:

$$\left|\frac{\partial^k}{\partial x^k}\,\Phi(x,t)\right| \le M, \quad (x,t) \in S_0, \tag{2.4}$$

$$\left|\frac{\partial^{k_0}}{\partial t^{k_0}}\Phi(x,t)\right| \le M, \quad (x,t) \in \overline{S}^{\,L}, \quad k \le K, \quad k_0 \le K_0,$$

[1] Here and below M, M_i (or m) denote sufficiently large (small) positive constants which do not depend on ε and on the discretization parameters. Throughout the paper, the notation $L_{(j.k)}$ ($M_{(j.k)}$, $G_{h(j.k)}$) means that these operators (constants, meshes) are introduced in equation $(j.k)$.

where $K, K_0 > 0$ are sufficiently large numbers, a boundary layer appears in a neighbourhood of the set $\overline{S}^L$ as the parameter ε_1 tends to zero. This layer is parabolic if the condition $\varepsilon_2 = \mathcal{O}(\varepsilon_1^{1/2})$ holds, and regular under the condition $\varepsilon_1 = o(\varepsilon_2^2)$.

If the data of the problem are sufficiently smooth (when the derivatives of the function $\Phi(x,t)$ are ε-uniformly bounded, for example, in the case (2.4) for $K = 7,\ K_0 = 2$), the solution of the problem can be decomposed into a sum of its regular and singular components

$$u(x,t) = U(x,t) + V(x,t), \quad (x,t) \in \overline{G}. \tag{2.5}$$

Suppose the function $\Phi(x,t)$ for $t = 0$ can also be written as a sum of the regular and singular components

$$\Phi(x,t) = \Phi_U(x,t) + \Phi_V(x,t), \quad (x,t) \in S_0. \tag{2.6a}$$

The singular component $\Phi_V(x,t)$ has the same singularities as the component $V(x,t)$ for the case of boundary value problem (2.2), (2.1), (2.4) with $K \geq 7,\ K_0 \geq 2$. In this case the singular component $V(x,t)$ of the solution of problem (2.2), (2.1) retains the character of the singularity (see, e.g., the estimates of theorem 3 and Remark 3). This decomposition of the solution into its regular and singular components in a number of cases allows us to construct and to study ε-uniform numerical methods (see, e.g., [6, 7] in the case of regular initial conditions).

We assume throughout that the function $\Phi(x,t)$ and its components from (2.6a) satisfy the condition

$$\left|\frac{\partial^k}{\partial x^k}\,\Phi_U(x,t)\right| \leq M, \tag{2.6b}$$

$$\left|\frac{\partial^k}{\partial x^k}\,\Phi_V(x,t)\right| \leq M \left\{\begin{array}{ll} \varepsilon_1^{-k/2}\exp(-m_1\varepsilon_1^{-1/2}x) & \text{for } \varepsilon_2 \leq M_0\varepsilon_1^{1/2} \\ \varepsilon_2^k\varepsilon_1^{-k}\exp(-m_2\varepsilon_2\varepsilon_1^{-1}x) & \text{for } \varepsilon_2 > M_0\varepsilon_1^{1/2} \end{array}\right\}, \quad (x,t) \in S_0;$$

$$\left|\frac{\partial^{k_0}}{\partial t^{k_0}}\,\Phi(x,t)\right| \leq M, \quad (x,t) \in \overline{S}^L, \quad k \leq K, \quad k_0 \leq K_0,$$

where m_1 is any constant, m_2 is a constant from the interval $(0, m_0)$, $m_0 = \min_{\overline{G}}[a^{-1}(x,t)\,b(x,t)]$, and where K, K_0 are sufficiently large numbers.

3. Our goal is to construct a finite difference scheme which is ε-uniformly convergent for the singularly perturbed boundary value problem (2.2), (2.1) with the singularly perturbed initial function satisfying condition (2.6).

Note that, for problem (2.2), (2.1) corresponding to the heat transfer problem (3.3) in the case of flow past a flat plate with suction of the boundary layer [1], we have $\varepsilon_1 = \varepsilon_T$ and $\varepsilon_2 = \varepsilon_R^{1/2} + v_0$, where $\varepsilon_T = Pe^{-1}$, $\varepsilon_R = Re^{-1}$, $v_0 \geq 0$ is the intensity of suction.

3. Motivation of the research

In this section we consider a boundary value problem for boundary layer equations in the case of a bounded domain, which describes heat transfer for the flow of a viscous fluid past a flat plate. Let a semi-infinite flat plate be disposed on the semiaxis $P = \{(x,y):\ x \geq 0,\ y = 0\}$. The problem is symmetric with respect to the plane $y = 0$; we examine the steady flow of an incompressible fluid on both sides of P, which is laminar and parallel to the plate. We consider the solution of this problem on the bounded set

$$\overline{G}, \quad \text{where} \quad G = \{(x,y):\ x \in (d_1, d_2],\ y \in (0, d_0)\}, \quad d_1 > 0. \tag{3.1}$$

Let $G^0 = \{(x,y):\ x \in [d_1, d_2],\ y \in (0, d_0]\}$; $\overline{G}^0 = \overline{G}$. Assume $S = \overline{G} \setminus G$, $S = \cup S_j$, $j = 0,1,2$, where $S_0 = \{(x,y):\ x \in [d_1, d_2],\ y = 0\}$, $S_1 = \{(x,y):\ x = d_1,\ y \in (0, d_0]\}$, $S_2 = \{(x,y):\ x \in$

$(d_1, d_2]$, $y = d_0\}$, $\overline{S_0} = S_0$; $S^0 = \overline{G} \setminus G^0 = S_0$. On the set $\overline{G}$, it is necessary to find the solution $U(x,y) = (u(x,y),\, v(x,y))$ of the following Prandtl problem:

$$L^1 U(x,y) \equiv \left\{ \varepsilon_R \frac{\partial^2}{\partial y^2} - u(x,y)\frac{\partial}{\partial x} - v(x,y)\frac{\partial}{\partial y} \right\} u(x,y) = 0, \quad (x,y) \in G, \tag{3.2a}$$

$$L^2 U(x,y) \equiv \frac{\partial}{\partial x} u(x,y) + \frac{\partial}{\partial y} v(x,y) = 0, \quad (x,y) \in G^0, \tag{3.2b}$$

$$u(x,y) = \varphi(x,y), \quad (x,y) \in S, \tag{3.2c}$$

$$v(x,y) = \psi(x,y), \quad (x,y) \in S^0. \tag{3.2d}$$

Here ε_R is the viscosity in the case when $U(x,y)$ and x, y are dimensional quantities, and $\varepsilon_R = Re^{-1}$ when $U(x,y)$ and x, y are dimensionless ones. The parameter ε_R takes arbitrary values from $(0,1]$.

The solution of problem (3.2), (3.1) exists and is sufficiently smooth if the functions $\varphi(x,y)$ and $\psi(x,y)$ are sufficiently smooth and satisfy appropriate compatibility conditions respectively on the sets $S^* = \overline{S}_1 \cap \{S_0 \cup \overline{S}_2\}$ (i.e. at the corner points adjoining to the side $\overline{S}_1$) and $S^{0*} = \overline{S}_1 \cap S^0$ [2].

In the case of heat transfer between the plate and the fluid (under the assumption that the Archimedean body force is equal to zero, and that the viscosity is independent of the temperature), in addition to the system of equations (3.2), we have the following heat equation with appropriate boundary conditions [1]

$$L^3 T(x,y) \equiv \left\{ \varepsilon_T \frac{\partial^2}{\partial y^2} - u(x,y)\frac{\partial}{\partial x} - v(x,y)\frac{\partial}{\partial y} \right\} T(x,y) = -\varepsilon_R \left(\frac{\partial}{\partial y} u(x,y) \right)^2, \quad (x,y) \in G, \tag{3.3a}$$

$$T(x,y) = \varphi_T(x,y), \quad (x,y) \in S. \tag{3.3b}$$

Here ε_T is the temperature conduction coefficient if the problem is considered in dimensional variables, and $\varepsilon_T = Pe^{-1}$ in the case of dimensionless variables; Pe is the Péclet number, $Pe = Pr\, Re$.

The solution of this problem in an infinite domain (including also the leading edge of the plate) for large Re and/or Pe has singularities of the boundary layer kind in a neighbourhood of the plate (for $x > 0$), and also an additional singularity in a neighbourhood of the leading edge due to the incompatibility of the problem data at the leading edge.

Since we are interested first of all in finding approximations to the solution of the problem in the neighbourhood of the boundary layer, we consider the heat transfer problem for flow around the flat plate in a bounded subdomain which adjoins the plate and contains the boundary layer, but outside some neighbourhood of the leading edge.

In the absence of suction and blowing the solutions of problem (3.2), (3.1) and (3.2), (3.3), (3.1) have such typical singularity as a parabolic boundary layer. For example, in the case of a self-similar solution of the Prandtl problem on flow past an infinite plate (see [1]) the function $v(x,y)$ satisfies the estimate

$$|\, v(x,y) \,| \le M \varepsilon_R^{1/2}, \quad (x,y) \in \overline{G};$$

in this case the thickness of the boundary layer is of order $\varepsilon_R^{1/2}$. The estimate for the function $v(x,y)$ in such a form allows us to use the technique of constructing ε-uniformly convergent schemes developed in [6, 7] for the case of problem (3.2), (3.1) (see, e.g., [9]). It seems that the same technique is also applicable for problem (3.2), (3.3), (3.1) provided that $Pr \approx 1$.

However, in the case of the problem of flow past a plate with the boundary layer controllable by means of suction, the function $v(x,y)$ may essentially exceed the quantity $\varepsilon_R^{1/2}$. For example, if suction with the intensity $v_0(x) = const > 0$ occurs, we obtain the following estimate for the function $u(x,y)$:

$$|\, u(x,y) - U_\infty \,| \le M \exp(-m\, v_0\, \varepsilon_R^{-1}\, y), \quad (x,y) \in \overline{G}$$

where U_∞ is the flow velocity at infinity. Thus, the thickness of the boundary layer becomes of order $v_0^{-1}\, \varepsilon_R$, that is, much less (for $v_0 \gg \varepsilon_R^{1/2}$) than above in the case of the passive plate. The boundary layer in this case becomes regular.

A similar behaviour of the controllable boundary layers is observed also in the case of problem (3.3), (3.1) under the condition

$$\varepsilon_T^{1/2} \ll v_0^{-1}\, \varepsilon_R.$$

Therefore, it is of urgent interest to construct ε-uniformly convergent numerical methods in the case of boundary layers which can be (depending on the parameter v_0) both parabolic and regular.

4. Classical difference schemes

We first introduce a classical difference scheme for problem (2.2), (2.1) and discuss problems arising in the numerical solution for small values of the parameter ε.

On the set $\overline{G}$ we introduce the mesh

$$\overline{G}_h = \overline{\omega} \times \overline{\omega}_0, \tag{4.1}$$

where $\overline{\omega}$ and $\overline{\omega}_0$ are meshes on the sets $\overline{D}$ and $[0,T]$ respectively; $\overline{\omega}$ and $\overline{\omega}_0$ are meshes with any distribution of the nodes satisfying only the condition $h \le MN^{-1}$, $h_t \le MN_0^{-1}$, where $h = \max_i h^i$, $h^i = x^{i+1} - x^i$, x^i, $x^{i+1} \in \overline{\omega}$, $h_t = \max_j h_t^j$, $h_t^j = t^{j+1} - t^j$, t^j, $t^{j+1} \in \overline{\omega}_0$. Here $N+1$ and N_0+1 are the minimal number of nodes on an interval of unit length on the set $\overline{D}$ and the number of nodes in the mesh $\overline{\omega}_0$ respectively. It is of interest to consider schemes on the simplest meshes

$$\overline{G}_h^u, \tag{4.2}$$

where $\overline{\omega}$ and $\overline{\omega}_0$ are uniform meshes with the step-sizes $h = N^{-1}$ and $h_t = TN_0^{-1}$.

Problem (2.2), (2.1) is approximated by the implicit difference scheme [11]

$$\Lambda\, z(x,t) \equiv \left\{ \varepsilon_1 a(x,t)\delta_{\overline{x}\widehat{x}} + \varepsilon_2 b(x,t)\delta_x - c(x,t) - p(x,t)\delta_{\overline{t}} \right\} z(x,t) = f(x,t), \quad (x,t) \in G_h,$$

$$z(x,t) = \Phi(x,t), \quad (x,t) \in S_h. \tag{4.3}$$

Here $\delta_{\overline{x}\widehat{x}}\, z(x,t)$ and $\delta_x\, z(x,t)$, $\delta_{\overline{t}}\, z(x,t)$ are the second and first (forward and backward) difference derivatives; $\delta_{\overline{x}\widehat{x}}\, z(x,t) = 2\, \left(h^i + h^{i-1}\right)^{-1} \{\, \delta_x - \delta_{\overline{x}} \,\}\, z(x,t), \;\; x = x^i$.

For the difference scheme (4.3), (4.1) the maximum principle is valid [11].

Taking into account *a priori* estimates of the solution of problem (2.2), (2.1) (see Section 6), we find the following estimate for the solution of scheme (4.3), (4.1)

$$|\, u(x,t) - z(x,t) \,| \le \tag{4.4}$$

$$\le M \left\{ \begin{array}{ll} \left[\left(\varepsilon_1^{1/2} + N^{-1}\right)^{-1} N^{-1} + N_0^{-1} \right] & \text{for } \varepsilon_2 \le M_0\, \varepsilon_1^{1/2} \\ \left[\varepsilon_2^3 \varepsilon_1^{-2}\, N^{-1} + N_0^{-1} \right] & \text{for } \varepsilon_2 > M_0\, \varepsilon_1^{1/2} \end{array} \right\}, \quad (x,t) \in \overline{G}_h.$$

On the mesh (4.2) we have the estimate

$$|\,u(x,t)-z(x,t)\,| \le \tag{4.5}$$

$$\le M \left\{ \begin{array}{ll} \left[\left(\varepsilon_1^{1/2}+N^{-1}\right)^{-1} N^{-1}+N_0^{-1}\right] & \text{for } \ \varepsilon_2 \le M_0\, \varepsilon_1^{1/2} \\ \left[\left(\varepsilon_2^{-2}\varepsilon_1+N^{-1}\right)^{-1} N^{-1}+N_0^{-1}\right] & \text{for } \ \varepsilon_2 > M_0\, \varepsilon_1^{1/2} \end{array} \right\}, \quad (x,t)\in \overline{G}_h^u;$$

estimate (4.5) is unimprovable with respect to the entering values of N, N_0, ε_1, ε_2.

Thus, the condition

$$N^{-1} = o\Big(\min\left[\varepsilon_1^{1/2},\ \varepsilon_2^{-2}\,\varepsilon_1\right]\Big) \tag{4.6}$$

is necessary and sufficient for the convergence of scheme (4.3), (4.2); schemes (4.3), (4.1) and (4.3), (4.2) do not converge ε-uniformly.

Theorem 1 *Let the data of the boundary value problem* (2.2), (2.1) *satisfy conditions* (2.3), (2.6), *and also* $a, b, c, p, f \in C^{l_1+\alpha}(\overline{G})$, $\varphi \in C^{l_0+\alpha}(\overline{S}^L) \cap C^{l_1+\alpha}(S_0)$, *and let* $u \in C^{3+\alpha,\,2+\alpha}(\overline{G})$, $K_{(2.6)} = l_1 = 7$, $K_{0(2.6)} = l_0 = 2$, $\alpha > 0$. *Then the condition* (4.6) *is necessary (necessary and sufficient) for the convergence of the difference scheme* (4.3) *on the mesh* (4.1) *(on the mesh* (4.2)*). For the mesh solutions the estimates* (4.4) *and* (4.5) *are valid; estimate* (4.5) *is unimprovable with respect to the values of* N, N_0, ε_1, ε_2.

Remark 1 To construct ε-uniformly convergent difference schemes for problem (2.2), (2.1), one could use a fitted operator method (see the description of this method, e.g., in [3, 5–7]). But in the case of the condition $\varepsilon_2 = \mathcal{O}\,(\varepsilon_1^{1/2})$ the solution of the problem has a singularity of the parabolic layer kind. Using the technique given in [6, 7, 12], we can show that there are no fitted operator schemes convergent ε-uniformly under the above condition.

5. Special difference scheme

In order to construct schemes which are ε-uniformly convergent, in this section we use meshes condensing in a neighbourhood of the boundary layer.

On the set $\overline{G}$ we introduce the mesh

$$\overline{G}_h = \overline{\omega}^* \times \overline{\omega}_0, \tag{5.1a}$$

where $\overline{\omega}_0 = \overline{\omega}_{0(4.2)}$, $\overline{\omega}^* = \overline{\omega}^*(\sigma)$ is a *piecewise uniform* mesh on $\overline{D}$. The step-sizes of the mesh $\overline{\omega}^*$ are constant on the sets $[0,\sigma]$ and $[\sigma,\infty)$ and equal to $h^{(1)} = 2\,\sigma\, N^{-1}$ and $h^{(2)} = 2(1-\sigma)N^{-1}$ respectively. The value of σ is chosen to satisfy the condition

$$\sigma = \sigma(\varepsilon, N) = \left\{ \begin{array}{ll} \min\left[\,2^{-1},\ M_1\, \varepsilon_1^{1/2} \ln N\,\right] & \text{for } \ \varepsilon_2 \le M_0\, \varepsilon_1^{1/2}, \\ \min\left[2^{-1},\ M_2\, \varepsilon_2^{-2} \varepsilon_1 \ln N\right] & \text{for } \ \varepsilon_2 > M_0\, \varepsilon_1^{1/2}, \end{array} \right. \tag{5.1b}$$

where $M_1 = m^{-1}_{1(2.6)}$, $M_2 = m^{-1}_{2(2.6)}$. The mesh $\overline{G}_h$ has been constructed.

Using the majorant function technique from [6, 7] and taking into account the *a priori* estimates of the solution of problem (2.2), (2.1), we find the following estimate for the solution of scheme (4.3), (5.1):

$$|\,u(x,t)-z(x,t)\,| \le \tag{5.2}$$

$$\le M \left\{ \begin{array}{ll} \left[N^{-1} \min\left[\,\ln N,\ \varepsilon_1^{-1/2}\right] + N_0^{-1}\,\right] & \text{for } \ \varepsilon_2 \le M_0\, \varepsilon_1^{1/2} \\ \left[N^{-1} \min\left[\,\ln N,\ \varepsilon_2^{2}\, \varepsilon_1^{-1}\right] + N_0^{-1}\,\right] & \text{for } \ \varepsilon_2 > M_0\, \varepsilon_1^{1/2} \end{array} \right\}, \quad (x,t)\in \overline{G}_h.$$

The following ε-uniform estimate is also valid:

$$|\,u(x,t)-z(x,t)\,| \le M\,\left[N^{-1}\ln N + N_0^{-1}\right], \qquad (x,t)\in \overline{G}_h; \tag{5.3}$$

estimates (5.2) and (5.3) are unimprovable with respect to the entering values of N, N_0, ε_1, ε_2 and N, N_0 respectively.

Theorem 2 *Let the hypothesis of Theorem* 1 *be fulfilled. Then the solution of the difference scheme* (4.3), (5.1) *converges ε-uniformly. The mesh solutions satisfy the estimates* (5.2) *and* (5.3), *which are unimprovable with respect to the values N, N_0, ε_1, ε_2 and N, N_0 respectively.*

Remark 2 Although the solution of problem (2.2), (2.1) has the singularity only for $\varepsilon_1 \to 0$ (the solution of the problem is regular for $\varepsilon_1 \ge m$; see, e.g., estimates (6.8), (6.10) below), the character of the boundary layer depends essentially on the vector-parameter ε. Such behaviour of the singular component of the solution does not allow us to construct an ε-uniformly convergent scheme in the case when the value of $\sigma_{(5.1)}$ is independent of the parameter ε_2.

6. *A-priori* estimates

In this section we give *a-priori* estimates used for the construction; the technique from [6, 10, 13, 14] is applied for deriving the estimates. Using the comparison theorems, we find

$$|\,u(x,t)\,| \le M, \qquad (x,t)\in \overline{G}. \tag{6.1}$$

Let the condition

$$\Phi(x,t)=\varphi(x,t), \qquad (x,t)\in S \tag{6.2}$$

be satisfied, where $\varphi(x,t)$ is independent of the parameter.

1. At first we find estimates of the solution in the case when

$$\varepsilon_2 \le M\,\varepsilon_1^{1/2}; \tag{6.3}$$

we use *a-priori* estimates up to the boundary [10]. The boundary value problem (2.2), (2.1) in the new variables $\xi=\varepsilon_1^{-1/2}x$ transforms into the problem

$$\widetilde{L}\widetilde{u}(\xi,t)=\widetilde{f}(\xi,t), \qquad (\xi,t)\in\widetilde{G}, \tag{6.4a}$$

$$\widetilde{u}(\xi,t)=\widetilde{\varphi}(\xi,t), \qquad (\xi,t)\in\widetilde{S}. \tag{6.4b}$$

Here $\widetilde{v}(\xi,t)=v(x(\xi),t)$, $v(x,t)$ is one of the functions $u(x,t),\ldots,\varphi(x,t)$; $\widetilde{G}^0=\{(\xi,t):\ \xi=\xi(x),\ (x,t)\in G^0\}$, G^0 is one of the sets G, S. The differential equation (6.4a) in the domain $\widetilde{G}$ and the boundary condition (6.4b) on $\widetilde{S}$ are regular with respect to the entering parameters ε_i. Using *a-priori* estimates up to the boundary, we find

$$\left|\frac{\partial^{k+k_0}}{\partial\xi^k\partial t^{k_0}}\widetilde{u}(\xi,t)\right| \le M, \qquad (\xi,t)\in\overline{\widetilde{G}}.$$

In the variables x,t we have

$$\left|\frac{\partial^{k+k_0}}{\partial x^k\partial t^{k_0}}u(x,t)\right| \le M\,\varepsilon_1^{-k/2}, \qquad (x,t)\in\overline{G}. \tag{6.5}$$

In fact we need a more accurate estimate than (6.5). We represent the solution of problem (2.2), (2.1) as a sum of the two functions

$$u(x,t)=U(x,t)+V(x,t), \qquad (x,t)\in\overline{G}, \tag{6.6}$$

where $U(x,t)$ and $V(x,t)$ are the regular and singular parts of the solution. The function $U(x,t)$ is the restriction on $\overline{G}$ of the function $U^*(x,t)$, $(x,t) \in \overline{G}^*$, where $U^*(x,t)$ is the solution of the problem

$$L^*U^*(x,t) = f^*(x,t), \quad (x,t) \in G^*, \quad U^*(x,t) = \varphi^*(x,t), \quad (x,t) \in S^*.$$

Here $S^* = S(G^*)$; the domain G^* is the extension of G beyond the set $\overline{S}^L$, G^* contains G together with its m-neighbourhood; the coefficients of the operator L^* and the function $f^*(x,t)$ are smooth continuations of the corresponding data of problem (2.2); $\varphi^*(x,t)$ is some smooth function, where $\varphi^*(x,t) = \varphi(x,t)$, $(x,t) \in S_0$. The function $V(x,t)$ is the solution of the problem

$$LV(x,t) = 0, \quad (x,t) \in G, \quad V(x,t) = \varphi(x,t) - U(x,t), \quad (x,t) \in S.$$

The function $U^*(x,t)$, $(x,t) \in \overline{G}^*$ can be decomposed into a sum of the functions

$$U^*(x,t) = \sum_{i=0}^{2} \varepsilon_1^{i/2} U_i^*(x,t) + v_U(x,t), \quad (x,t) \in \overline{G}^*. \tag{6.7}$$

Here $U_i^*(x,t)$ is the solution of the problem

$$L^{*0}U_0^*(x,t) \equiv \left\{ -c^*(x,t) - p^*(x,t)\frac{\partial}{\partial t} \right\} U_0^*(x,t) = f^*(x,t), \quad (x,t) \in \overline{G}^* \setminus S_0^*,$$

$$U_0^*(x,t) = \varphi^*(x,t), \quad (x,t) \in S_0^*;$$

$$L^{*0}U_i^*(x,t) = \left\{ \varepsilon_1 a^*(x,t)\frac{\partial^2}{\partial x^2} - \varepsilon_2 b^*(x,t)\frac{\partial}{\partial x} \right\} U_{i-1}^*(x,t), \quad (x,t) \in \overline{G}^* \setminus S_0^*,$$

$$U_i^*(x,t) = 0, \quad (x,t) \in S_0^*, \quad i = 1,2.$$

Taking into account estimates for the components from (6.7), we find the estimates for the components from the representation (6.6)

$$\begin{aligned} &\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} U(x,t) \right| \le M, \\ &\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} u(x,t) \right| \le M\,\varepsilon_1^{-k/2} \exp(-m_1 \varepsilon_1^{-1}), \quad (x,t) \in \overline{G}, \quad k + k_0 = K, \quad k_0 \le K_0, \end{aligned} \tag{6.8}$$

where m_1 is any positive constant, $K = 3$, $K_0 = 2$.

2. Let

$$\varepsilon_2 \ge m\varepsilon_1^{1/2}. \tag{6.9}$$

In this case we pass to the variables $\xi = \varepsilon_1^{-1}\varepsilon_2 x$, $\tau = \varepsilon_1^{-1}\varepsilon_2, t$.

We represent the function $U^*(x,t)$, $(x,t) \in \overline{G}^*$ as a sum of the functions

$$U^*(x,t) = \sum_{i=0}^{2} \varepsilon_1^i U_i^*(x,t) + v_U(x,t), \quad (x,t) \in \overline{G}^*.$$

where $U_i^*(x,t)$ are the solutions of the problems

$$L^{*1}U_0^*(x,t) \equiv \left\{ \varepsilon_2 b^*(x,t)\frac{\partial}{\partial x} - c^*(x,t) - p^*(x,t)\frac{\partial}{\partial t} \right\} U_0^*(x,t) = f^*(x,t), \quad (x,t) \in \overline{G}^* \setminus S_0^*,$$

$$U_0^*(x,t) = \varphi^*(x,t), \quad (x,t) \in S_0^*;$$

$$L^{*1}U_i^*(x,t) = -a^*(x,t)\,\frac{\partial^2}{\partial x^2}\,U_{i-1}^*(x,t), \quad (x,t) \in \overline{G}^* \setminus S_0^*,$$

$$U_i^*(x,t) = 0, \quad (x,t) \in S_0^*, \quad k = 1,2.$$

After having estimated the function $U^*(x,t)$, for the components from the representation (6.6) we obtain the estimates

$$\left|\frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} U(x,t)\right| \le M,$$
$$\left|\frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} V(x,t)\right| \le M\,\varepsilon_2^k \varepsilon_1^{-k} \exp(-m_2 \varepsilon_2 \varepsilon_1^{-1} x), \quad (x,t) \in \overline{G}, \quad k+k_0 \le K, \quad k_0 \le K_0, \tag{6.10}$$

where m_2 is an arbitrary constant from the interval $(0, m_0)$, $m_0 = \min_{\overline{G}}[a^{-1}(x,t)\, b(x,t)]$, $K = 3$, $K_0 = 2$.

When deducing estimates (6.8), (6.10), we supposed that the data of the boundary value problem satisfy the condition

$$a,\, b,\, c,\, p,\, f \in C^{l_1+\alpha, l_0+\alpha}(\overline{G}), \quad \varphi \in C^{l_0+\alpha}(\overline{S}^L) \bigcap C^{l_1+\alpha}(S_0), \quad l_0 \ge 2, \quad l_1 \ge 7, \quad \alpha > 0. \tag{6.11}$$

Moreover, the compatibility conditions [10] on the set S^c are satisfied which ensure the inclusion

$$u \in C^{3+\alpha, 2+\alpha}(\overline{G}) \tag{6.12}$$

for each fixed set of the parameters ε_i.

Theorem 3 *Let the data of the boundary value problem* (2.2), (2.1) *satisfy conditions* (2.3), (6.2), (6.11), *and let the condition* (6.12) *be fulfilled for the solution of the problem. Then the solution of the problem and its components from the representation* (6.6) *satisfy the estimate* (6.1) *and also estimates* (6.8) *and* (6.10) *in the case of conditions* (6.3) *and* (6.9) *respectively.*

Remark 3 Let the function $\Phi(x,t)$ have the singularity of the same type as the function $u(x,t)$. We consider that the function $\Phi(x,t)$ for $t = 0$ can be written as a sum of functions in the form (2.6a), moreover, this function itself and its components from (2.6a) satisfy condition (2.6b), where $K = 7$, $K_0 = 2$. In this case the conclusion of Theorem 3 remains valid for the solution of problem (2.2), (2.1).

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