

Note

The Blocking Number of an Affine Space

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It is proved that the minimum cardinality of a subset of $AG(k, q)$ which intersects all hyperplanes is $k(q - 1) + 1$. In case $k = 2$ this settles a conjecture of J. Doyen.

Doyen [1] proved that the minimum cardinality of a subset of $PG(2, q)$ intersecting all lines equals $q + 1$, where this minimum is attained only if such a subset is a line. He also showed that in each affine plane $AG(2, q)$ there is a subset of cardinality $2q - 1$, intersecting all lines (by taking, e.g., the union of two intersecting lines). He conjectured that for all values of q there is no subset of $AG(2, q)$, intersecting all lines and with fewer than $2q - 1$ points, and verified this conjecture for $q \leq 5$. Hansen and Lorea [2] proved it for $q = 7$ by exhaustive computer search, and below we prove the conjecture for all q . This same result has been obtained by R. E. Jamison [3], but with a rather long and involved proof. He proves more than we do, viz.:

Let V be a vector space of dimension n over a finite field F with q elements. If $0 < k < n$, then any covering of V^\times with nonzero k -flats contains at least $q^{n-k} - 1 + k(q - 1)$ k -flats. Furthermore, a covering with this number of k -flats is always possible.

The theorem below is the special case $k = n - 1$.

THEOREM. *Let $AG(k, q)$ be the k -dimensional affine space over $GF(q)$. Then the minimum cardinality of a subset of $AG(k, q)$ which intersects all hyperplanes is $k(q - 1) + 1$.*

(Note that we do not have any results on non-Desarguesian affine planes.)

Proof. Let q be a prime-power and let $AG(k, q)$ be the k -dimensional affine space over $GF(q)$. We first observe that there is always a subset of

cardinality $k(q - 1) + 1$ intersecting all hyperplanes. For the union of k independent lines through one given point intersects all hyperplanes and has cardinality $k(q - 1) + 1$. Secondly suppose $A \subset \text{AG}(k, q)$ intersects all hyperplanes. We may suppose that $0 = (0, \dots, 0) \in A$; let $B = A \setminus \{0\}$. Then B intersects all hyperplanes not through 0. A hyperplane not through 0 is determined by an equation

$$w_1x_1 + \dots + w_kx_k = 1,$$

for some w_1, \dots, w_k in $\text{GF}(q)$, not all zero. Hence for all $(w_1, \dots, w_k) \neq 0$ there exists a $b = (b_1, \dots, b_k)$ in B such that $w_1b_1 + \dots + w_kb_k = 1$. Therefore, if we let

$$F(x_1, \dots, x_k) = \prod_{b \in B} (b_1x_1 + \dots + b_kx_k - 1),$$

then $F(w_1, \dots, w_k) = 0$ for all k -tuples $(w_1, \dots, w_k) \neq 0$.

Now (as one easily proves by induction on k) if $P(x_1, \dots, x_k)$ is a polynomial which only assumes the value zero then $P(x_1, \dots, x_k) \in (x_1^q - x_1, \dots, x_k^q - x_k)$, that is, there are polynomials $P_i(x_1, \dots, x_k)$ (for $i = 1, \dots, k$) such that

$$P(x_1, \dots, x_k) = P_1(x_1, \dots, x_k)(x_1^q - x_1) + \dots + P_k(x_1, \dots, x_k)(x_k^q - x_k).$$

Now let

$$F(x_1, \dots, x_k) = F_1(x_1, \dots, x_k)(x_1^q - x_1) + \dots + F_k(x_1, \dots, x_k)(x_k^q - x_k) + J(x_1, \dots, x_k),$$

such that the highest degree of x_i in $J(x_1, \dots, x_k)$ is at most $q - 1$ ($1 \leq i \leq k$). Since for each $i = 1, \dots, k$ the polynomial $x_i F(x_1, \dots, x_k)$ only assumes the value zero, also for each $i = 1, \dots, k$ the polynomial $x_i J(x_1, \dots, x_k)$ only assumes the value zero. Applying the above mentioned theorem and using the fact that the highest degree of each x_i in $J(x_1, \dots, x_k)$ is at most $q - 1$, it follows that for each $i = 1, \dots, k$:

$$(x_i^{q-1} - 1) \mid J(x_1, \dots, x_k),$$

or

$$\prod_{i=1}^k (x_i^{q-1} - 1) \mid J(x_1, \dots, x_k).$$

Since $F(0, \dots, 0) \neq 0$ and hence $J(0, \dots, 0) \neq 0$, it follows that the degree of $J(x_1, \dots, x_k)$ is $k(q - 1)$. This implies that the degree of $F(x_1, \dots, x_k)$ is at least $k(q - 1)$. Now, by definition, the degree of $F(x_1, \dots, x_k)$ equals $|B|$. Hence $|B| \geq k(q - 1)$ and $|A| \geq k(q - 1) + 1$, proving the theorem. ■

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