

Note

A Note on David Lubell's Article "Local Matchings in the Function Space of a Partial Order"

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A characterization is given of nonnegative functions defined on partially ordered sets with the property that the sum of the values on each antichain is at most 1. With the help of this characterization some theorems of Lubell can be proved.

In [1], Lubell defined the *function space* E of a finite partially ordered set (π, \leq) as the vector space of all real-valued functions on π ; as usual, the inner product of α and β in E is

$$(\alpha, \beta) = \sum_{x \in \pi} \alpha x \cdot \beta x.$$

By identifying a subset $\xi \subset \pi$ with its characteristic function, one can speak of subsets of π as elements of E , i.e., $\xi x = 0$ if $x \notin \xi$ and $\xi x = 1$ if $x \in \xi$, for $x \in \pi$ and $\xi \subset \pi$.

A *chain*, resp., *antichain*, is a subset ξ of π such that any two different elements of ξ are comparable, resp., not comparable (with respect to the ordering \leq). A *maximal chain* (*maximal antichain*) is a chain (antichain) not contained in another chain (antichain); a *maximum chain* (*maximum antichain*) is a chain (antichain) with maximal cardinality.

Let n be the length of a maximum chain. For $i = 0, 1, \dots, n - 1$ let μ_i , resp., ν_i , be the set of all elements of π of height i , resp., depth i ; this means:

$$\mu_i = \{x \in \pi \mid \text{the largest chain in } \pi \text{ with maximum } x \text{ has length } i\},$$

$$\nu_i = \{x \in \pi \mid \text{the largest chain in } \pi \text{ with minimum } x \text{ has length } i\}.$$

Finally, a *convex cone* in E is a nonempty subset $A \subset E$ such that if $\alpha, \beta \in A$

and $\lambda, \mu \in \mathbb{R}, \lambda \geq 0, \mu \geq 0$, then $\lambda\alpha + \mu\beta \in A$. The *polar* of a convex cone A is the convex cone

$$A^0 = \{\beta \in E \mid \text{for all } \alpha \in A, (\alpha, \beta) \leq 0\}.$$

Then A^{00} is the closure \bar{A} of A in the Euclidean space E (see, e.g., [2]). Let K be the convex cone generated by all maximal chains in π , i.e.,

$$K = \{\lambda_1\gamma_1 + \dots + \lambda_r\gamma_r \mid \lambda_1, \dots, \lambda_r \geq 0 \text{ and } \gamma_1, \dots, \gamma_r \text{ maximal chains in } \pi\};$$

this is the intersection of all convex cones in E containing all maximal chains (as vectors in E , of course). Since K is a closed set, we have $K^{00} = K$.

Writing $\alpha \ll 0$ if $(\alpha, \gamma) \leq 0$ for each maximal chain γ , we have also that $\psi \in K^{00}$ iff $\alpha \ll 0$ implies $(\psi, \alpha) \leq 0$.

The purpose of this note is to give more general theorems from which the following two theorems of Lubell follow.

THEOREM 3 (Lubell). *Let $(\psi, \mu_i) = 1$ for all $i = 0, 1, \dots, n-1$. Then $(\psi, \xi) \leq 1$ for every antichain ξ if and only if $\alpha \ll 0$ implies $(\psi, \alpha) \leq 0$ (i.e., $\psi \in K^{00} (=K)$).*

THEOREM 4 (Lubell). *Let $(\psi, \mu_i) = 1$ for all $i = 0, 1, \dots, n-1$, and $(\psi, \xi) \leq 1$ for every antichain ξ . Then $(\psi, \nu_i) = 1$ for all $i = 0, 1, \dots, n-1$.*

(Using Theorem 1 of Lubell this is clearly equivalent to the formulation in [1].)

We now give our two theorems.

THEOREM A. *Let $\psi \in E$ such that $\psi x \geq 0$ for all $x \in \pi$. Then $(\psi, \xi) \leq 1$ for each antichain ξ if and only if ψ is a convex combination of chains.*

Proof. Since for each chain γ and antichain ξ , $(\gamma, \xi) \leq 1$ holds, the “if” part of the theorem is trivial. For the “only if” part we use the (easy) theorem that in a partially ordered set the maximal cardinality of a chain equals the minimal cardinality of a collection of antichains which covers the partially ordered set.

Take $\psi \in E$ such that $(\psi, \xi) \leq 1$ for each antichain ξ . Furthermore, let $w \in E$ be such that wx is a nonnegative integer for all $x \in \pi$. If $\alpha_1, \dots, \alpha_m$ are antichains in π such that

$$wx \leq \sum_{i=1}^m \alpha_i x,$$

we have

$$(w, \psi) = \sum_{x \in \pi} wx \cdot \psi x \leq \sum_{x \in \pi} \sum_{i=1}^m \alpha_i x \cdot \psi x = \sum_{i=1}^m (\alpha_i, \psi) \leq m \cdot 1 = m.$$

Hence

$$(w, \psi) \leq \min \left\{ m \mid \text{there are antichains } \alpha_1, \dots, \alpha_m \text{ in } \pi \text{ such that} \right. \\ \left. wx \leq \sum_{i=1}^m \alpha_i x \text{ for all } x \text{ in } \pi \right\}.$$

Let π' be the partially ordered set arising from π by replacing each x by a chain of w_x elements. Then

$$\begin{aligned} (w, \psi) &\leq \min \left\{ m \mid \text{there are antichains } \alpha_1, \dots, \alpha_m \text{ in } \pi \text{ such that} \right. \\ &\quad \left. wx \leq \sum_{i=1}^m \alpha_i x \text{ for all } x \text{ in } \pi \right\} \\ &= \min \left\{ m \mid \text{there are antichains } \alpha'_1, \dots, \alpha'_m \text{ in } \pi' \text{ such that} \right. \\ &\quad \left. 1 \leq \sum_{i=1}^m \alpha'_i x' \text{ for all } x' \text{ in } \pi' \right\}; \end{aligned}$$

but this last equals, by the theorem mentioned,

$$\max\{|\gamma'| \mid \gamma' \text{ chain in } \pi'\},$$

and this is the same as

$$\max\{(w, \gamma) \mid \gamma \text{ chain in } \pi\}.$$

So we have proved

$$(w, \psi) \leq \max\{(w, \gamma) \mid \gamma \text{ chain in } \pi\}, \quad \text{for each } w: \pi \rightarrow \mathbb{Z}_+.$$

Then also

$$(w, \psi) \leq \max\{(w, \gamma) \mid \gamma \text{ chain in } \pi\}, \quad \text{for each } w: \pi \rightarrow \mathbb{Q}_+,$$

and this holds even for each $w: \pi \rightarrow \mathbb{R}_+$. This last result means that ψ is contained in all half-spaces bounded by positive hyperplanes and containing all chains (note that a subset of a chain is itself a chain, and, in particular, \emptyset is a chain). Thus, since $\psi x \geq 0$ for all $x \in \pi$, ψ is a convex combination of chains. ■

Remark. If we exchange the terms “chain” and “antichain” in Theorem A, the theorem is also true (we then have to use Dilworth’s theorem (cf. [3])).

From Theorem A, Theorem B follows.

THEOREM B. *Let $\psi \in E$ such that $(\psi, \xi) \leq 1$ for all antichains $\xi \subset \pi$; then the following assertions are equivalent:*

- (i) $(\psi, \mu_i) = 1$ for all $i = 0, \dots, n - 1$;
- (ii) there exists a partition of π into antichains $\alpha_1, \dots, \alpha_m$ such that for $i = 1, \dots, m$, we have $(\psi, \alpha_i) = 1$;
- (iii) ψ is a convex combination of maximum chains;
- (iv) for each minimum partition of π into antichains $\alpha_1, \dots, \alpha_n$ we have, for all $i = 1, \dots, n$, $(\psi, \alpha_i) = 1$.

Proof. (i) \rightarrow (ii) and (iv) \rightarrow (i) are easily seen, since μ_0, \dots, μ_{n-1} is a minimum partition of π into antichains.

(ii) \rightarrow (iii). Let $\alpha_1, \dots, \alpha_m$ be a partition of π into antichains such that $(\psi, \alpha_i) = 1$ for $i = 1, \dots, m$. By Theorem A, ψ is a convex combination of chains, say

$$\psi = \sum_{\gamma \text{ chain}} f(\gamma) \cdot \gamma, \quad \text{such that } \sum_{\gamma} f(\gamma) = 1,$$

and for each chain γ , $f(\gamma) \geq 0$.

Let $f(\gamma_0) > 0$; then γ_0 meets every α_i . For suppose $\gamma_0 \cap \alpha_i = \emptyset$. Then, since $(\gamma_0, \alpha_i) = 0$, we have

$$\begin{aligned} 1 = (\psi, \alpha_i) &= \sum_{\gamma} f(\gamma) \cdot (\gamma, \alpha_i) \\ &= \sum_{\gamma \neq \gamma_0} f(\gamma) \cdot (\gamma, \alpha_i) \leq \sum_{\gamma \neq \gamma_0} f(\gamma) < \sum_{\gamma} f(\gamma) = 1, \end{aligned}$$

which is a contradiction. Hence $|\gamma_0| = m$, and γ_0 is a maximum chain (and, consequently, $\alpha_1, \dots, \alpha_m$ is a minimum partition of π into antichains).

(iii) \rightarrow (iv). Let $\alpha_1, \dots, \alpha_n$ be an antichain partition of π with minimum cardinality; then $(\gamma, \alpha_i) = 1$ for all maximum chains γ and $i = 1, \dots, n$. Hence, also, $(\psi, \alpha_i) = 1$ for all convex combinations ψ of maximum chains and $i = 1, \dots, n$. ■

It is easy to see that $\psi \in K$ and $(\psi, \mu_i) = 1$ for all $i = 0, \dots, n - 1$ implies that ψ is a convex combination of maximum chains. Using this and the Theorems A and B one can easily deduce the Theorems 3 and 4 of Lubell.

REFERENCES

1. D. LUBELL, Local matchings in the function space of a partial order, *J. Combinatorial Theory A* **19** (1975), 154–159.
2. N. BOURBAKI, “Éléments de Mathématique,” Livre V, Chap. IV, Sect. 3, Herman, Paris, 1964.
3. M. HALL, JR., “Combinatorial Theory.” Blaisdell, Waltham, Mass., 1967.