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A NOTE ON REDUCING A SYSTEM TO A SINGLE EQUATION

Preliminary report

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Summary

It is shown that a system of equations of (qualified) integer-valued functions is equivalent to a linear combination of the equations. As applied to linear functions in integer variables with integer coefficients the results improve two theorems by Mathews.

1. Introduction

Recently (1970) S.E. Elmaghraby and M.K. Wig [1] republished two theorems due to G.B. Mathews (1897) [3].

These theorems show that a pair of linear equations with strictly positive coefficients in non-negative integer variables is equivalent to a single equation, which is a linear combination of the original ones. By repeated application a system of such equations can either be reduced to a single equation or is shown to be infeasible.
In this way, a linear programming problem in integer variables is reduced to a knapsack problem, which might be less difficult to solve than the original problem. The coefficients in the knapsack-constraint, however, tend to be rather large.

In the present paper it is shown that, under certain conditions, a system of equations of integer valued functions is equivalent to a linear combination of the original equations. The result is applied to the linear case and leads to an equation with smaller coefficients than those obtained by Mathews.

2. General

Let $f_i(x)$ $(i = 1, \ldots, m)$ denote integer-valued functions, defined for $x \in D$, where D is an arbitrary domain.

Define:

$$S_i = \sup\,(f_i(x) \mid f_j(x) = 0 \quad (j=1, \ldots, i-1, i+1, \ldots, m))$$

$$s_i = \inf\,(f_i(x) \mid f_j(x) = 0 \quad (j=1, \ldots, i-1, i+1, \ldots, m)) \tag{1}$$

for $i = 1, 2, \ldots, m$.

Theorem 1

If the system of equations

$$f_i(x) = 0 \qquad (i=1, \ldots, m) \tag{2}$$

has a solution then

$$s_i \leq 0 \leq S_i \qquad (i=1, \ldots, m). \tag{3}$$

Proof

If x solves $f_i(x) = 0$ with either $0 < s_i$ or $0 > S_i$ then, by (1), $f_j(x) \neq 0$ for at least one j. This completes the proof.

Define

$$T_{m-1} \geq \sup\,(f_{m-1}(x) \mid f_j(x) = 0 \quad (j=1, \ldots, m-2))$$

$$t_{m-1} \leq \inf\,(f_{m-1}(x) \mid f_j(x) = 0 \quad (j=1, \ldots, m-2)) \,. \tag{4}$$

Theorem 2

If

$$-\infty < t_{m-1} \leq 0 \leq T_{m-1} < \infty \tag{5}$$

then for each integer λ_{m-1}

with $$\lambda_{m-1} > \max(-t_{m-1}, T_{m-1}) \tag{6}$$

the system

$$f_i(x) = 0 \qquad (i = 1, \ldots, m) \tag{2}$$

is equivalent to

$$\begin{aligned} & f_i(x) = 0 \qquad (i = 1, \ldots, m-2) \\ & f_{m-1}(x) + \lambda_{m-1} f_m(x) = 0 \end{aligned} \tag{7}$$

<u>Proof</u>

Note that $T_{m-1} \geq S_{m-1}$ and $t_{m-1} \leq s_{m-1}$. If the relation $t_{m-1} \leq 0 \leq T_{m-1}$ does not hold the system is infeasible and equivalent to any infeasible equation. So $t_{m-1} \leq 0 \leq T_{m-1}$ may be assumed without loss of generality.

Any x solving (2) evidently solves (7). Now assume x is a solution of (7). It is obvious that $f_m(x) = 0$ implies $f_{m-1}(x) = 0$ and conversely, thus the theorem holds if $x \in D$ implies $f_m(x) = 0$ or $f_{m-1}(x) = 0$. The only remaining case to be considered is

$$\begin{aligned} \text{both} \quad & f_{m-1}(x) \neq 0 \\ \text{and} \quad & f_m(x) \neq 0 \quad . \end{aligned} \tag{8}$$

Then (7) yields

$$f_m(x) = \frac{-f_{m-1}(x)}{\lambda_{m-1}} \neq 0 \text{ and integer} \tag{9}$$

hence

$$|f_{m-1}(x)| \geq \lambda_{m-1} \tag{10}$$

and, by (5) and (6),

$$\begin{aligned} \text{either} \quad & f_{m-1}(x) < t_{m-1} \\ \text{or} \quad & f_{m-1}(x) > T_{m-1}, \end{aligned} \tag{11}$$

contradicting (4), as x solves (7).

This completes the proof.

Now define, for i = 1, ..., m-1,

$$\begin{aligned} T_i &\geq \sup\ (f_i(x) \mid f_j(x) = 0 \quad (j=1, \ldots, i-1)) \\ t_i &\leq \inf\ (f_i(x) \mid f_j(x) = 0 \quad (j=1, \ldots, i-1)). \end{aligned} \tag{12}$$

Theorem 3

If $$-\infty < t_i \leq 0 \leq T_i < \infty \qquad (i=1, \ldots, m-1) \tag{13}$$

then for each set of integers

$$\lambda_i > \max\ (-t_i, T_i) \qquad (i=1, \ldots, m-1), \tag{14}$$

the system

$$f_i(x) = 0 \qquad (i=1, \ldots, m) \tag{2}$$

is equivalent to the single equation

$$\sum_{i=1}^{m} M_i\ f_i(x) = 0 \tag{15}$$

where

$$M_i = \prod_{j=1}^{i-1} \lambda_j\ . \tag{16}$$

Proof

The proof by induction is straightforward, and omitted.

It should be noted that, by taking $\lambda_i = \lambda$ (i=1, ..., m-1) with

$$\lambda > \max(\max(-t_i, T_i) \mid i = 1, \ldots, m-1) \tag{17}$$

equation (15) becomes

$$\sum_{i=1}^{m} \lambda^{i-1} f_i(x) = 0. \tag{18}$$

If the functions $f_i(x)$ $(i=1, \ldots, m-1)$ are bounded and

$$R_i = \max f_i(x) - \min f_i(x) \quad (i=1, \ldots, m-1) \tag{19}$$

then

$$\lambda > \max (R_i \mid i=1, \ldots, m-1) \tag{20}$$

satisfies (17).

The values of the M_i are minimized by minimizing the λ_i, but will, in general, depend on the ordering of the equations. If $x \in D$ implies

$$s_i \leq f_i(x) \leq S_i \qquad (i=1, \ldots, m-1) \tag{21}$$

$T_i = S_i$ and $t_i = s_i$ can be used. Then the M_i are independent of the ordering of the equations. The M_i are absolutely minimal if $x \in D$ implies $f_i(x) = 0$ $(i=1, \ldots, m-1)$.

The righthand sides of (12) seem independent of the equations $f_j(x) = 0$ $(j=i, \ldots, m)$. However, implications of these equations may be used in the definition of D. These implications must be considered again while solving (15).

It should also be noted that theorem 3 requires $f_1(x)$ to be bounded on D, $f_2(x)$ must be bounded on $D \cap \{x \mid f_1(x) = 0\}$ etc. However, $f_m(x)$ may be unbounded on $D \cap \{x \mid f_1(x) = \ldots = f_{m-1}(x) = 0\}$.

Finally, it is clear that any integer valued function $\mu_i(x) \geq \lambda_i$ for $x \in D$ can be used instead of λ_i.

3. The linear case

3.1 General

Consider a system of m linear equations in n variables, with integer coefficients. The variables are bounded and required to be integers.

$$f_i(x) = \sum_{j=1}^{n} a_{ij} x_j = a_{i0} \quad (i=1, \ldots, m), \tag{22}$$

$$D = \{(x_1, \ldots, x_n) \mid \alpha_j \leq x_j \leq \beta_j, \; x_j \text{ integer} \}. \tag{23}$$

Define, for i=1, ..., m,

$$U_i = \sum_{j=1}^{n} (a_{ij} \beta_j \mid a_{ij} > 0) + \sum_{j=1}^{n} (a_{ij} \alpha_j \mid a_{ij} < 0)$$

$$u_i = \sum_{j=1}^{n} (a_{ij} \beta_j \mid a_{ij} < 0) + \sum_{j=1}^{n} (a_{ij} \alpha_j \mid a_{ij} > 0). \tag{24}$$

Clearly, $u_i \leq f_i(x) \leq U_i \quad (i=1, \ldots, m)$,

and for each set of integers $\lambda_i > \max (U_i - a_{i0}, a_{i0} - u_i)$ the system (22) is equivalent to

$$\sum_{j=1}^{n} a_j x_j = a_0 \tag{25}$$

where

$$a_j = \sum_{i=1}^{m} a_{ij} \lambda_1 \lambda_2 \ldots \lambda_{i-1} \quad (j=0, 1, \ldots, n) \tag{26}$$

or, if $\lambda \geq \max_i \lambda_i$

$$a_j = \sum_{i=1}^{m} a_{ij} \lambda^{i-1} \quad (j=0, 1, \ldots, n). \tag{27}$$

The equivalence is not certain if, for any $i \neq m$, $a_{i0} < u_i$ or $U_i < a_{i0}$.

But then the system apparently is an infeasible one. This qualification is not repeated in the sequel.

If, for any i, $u_i = U_i$ the i-th equation is superfluous, because $x \in D$ implies that the equation holds, and can be deleted.

Sharper bounds on $f_i(x)$ can be obtained by using (12):

Define, for i=1, ..., m,

$$V_i = \text{maximum of } \sum_{j=1}^{n} a_{ij}\, x_j \tag{28}$$

subject to

$$\begin{aligned} &\sum_{j=1}^{n} a_{kj}\, x_j = a_{k0} \qquad (k=1, \ldots, i-1), \\ &\alpha_j \leq x_j \leq \beta_j \qquad (j=1, \ldots, n), \end{aligned} \tag{29}$$

and

$$v_i = \text{minimum of } \sum_{j=1}^{n} a_{ij}\, x_j,$$

also subject to (29).

To compute V_i and v_i a number of linear programming problems must be solved, but the relations between the problems can be exploited. If any of the problems is infeasible the original problem also is infeasible. If all problems are feasible it is worthwhile to optimize an arbitrary objective function subject to (22) and $\alpha_j \leq x_j \leq \beta_j$.

The sharperbounds on $f_i(x)$ yield smaller lower bounds for the λ_i. However, $\lambda_i \geq 2$ can be assumed without loss of generality, because $\lambda_i = 1$ implies that the corresponding equation is superfluous. Consequently, the multipliers $M_i = \lambda_1 \lambda_2 \ldots \lambda_{i-1} \geq 2^{i-1}$ and the coefficients a_j may turn out to be rather large. If the a_j have greatest common divisor $\neq 1$ division will result in smaller coefficients.

The λ_i are bounded from below only, and might be selected to lead to a reducible equation (25). If the system of equations was obtained by adding slack variables to a system of inequalities, reduction of the system certainly results in an irreducible equation. It is also possible to multiply some of the eqautions by -1 to obtain both positive and negative coefficients in a column. If an equation is multiplied by α, $|\alpha| > 1$, the lower bound for λ_i increases.
It might be worthwhile to divide each equation by the greatest common divisor of its coefficients before introducing slacks and reducing the system.
Finally, system (22) might imply sharper bounds on x_j than those specified in (23), the sharper bounds can be substituted cf. Zionts [4].

3.2. Positive coefficients

Any system of linear equations in bounded variables is equivalent to one with $\alpha_j = 0$ and $a_{ij} \geq 0$. The transformation to obtain $\alpha_j = 0$ is obvious.
If $a_{ij} < 0$, with $j \neq 0$ the introduction of a variable $y_j \geq 0$ and an additional equation $x_j + y_j = \beta_j$, together with the substitution $a_{ij}\, x_j = a_{ij}\, \beta_j - a_{ij}\, y_j$ give the desired result.
As the system

$$\begin{aligned} \sum_{j=1}^{n} a_{1j}\, x_j &= b_1 \\ \sum_{j=1}^{n} a_{2j}\, x_j &= b_2 \end{aligned} \tag{30a}$$

is equivalent to

$$\begin{aligned} \sum_{j=1}^{n} (a_{1j} + t a_{2j})\, x_j &= b_1 + t b_2 \\ \sum_{j=1}^{n} (s a_{1j} + a_{2j})\, x_j &= s b_1 + b_2 \end{aligned} \tag{30b}$$

provided $st \neq 1$, even an equivalent system with strictly positive coefficients can be obtained.

Mathews [3], as cited by Elmaghraby and Wig [1], gave two theorems on (30a) with positive coefficients and $\alpha_j = 0$:

1. If the system is feasible the inequality $b_2 \frac{a_{1j}}{a_{2j}} \geq b_1$ must hold for at least one j.

2. For any positive integer λ subject to $\lambda > \max_j b_2 \frac{a_{1j}}{a_{2j}} \geq b_1$ the system (30a) is equivalent to
$\sum_{j=1}^{n} (a_{1j} + \lambda a_{2j})x_j = b_1 + \lambda b_2$.

Indeed, if x_j is not restricted to be integer the problem

$$\begin{aligned} &\text{maximize} && \sum_{j=1}^{n} a_{1j}\, x_j \\ &\text{subject to} && \sum_{j=1}^{n} a_{2j}\, x_j = b_2 \\ & && x_j \geq 0 \qquad (j=1, \ldots, n) \end{aligned} \tag{31}$$

is solved by taking

$$x_j = \begin{cases} \frac{b_2}{a_{2j}} & \text{if } j = j_1 \\ 0 & \text{if } j \neq j_1 \end{cases} \tag{32}$$

where $\frac{a_{1j_1}}{a_{2j_1}} = \max_j \frac{a_{1j}}{a_{2j}}$.

Thus $0 \leq \sum_{j=1}^{n} a_{1j}\, x_j \leq b_2 \frac{a_{1j_1}}{a_{2j_1}}$

and both theorems follow immediately from the theorems of the present paper.

Interchanging the row-indices in $b_2 \frac{a_{1j}}{a_{2j}} \geq b_1$ leads to the statement that $b_1 \frac{a_{2j}}{a_{1j}} \geq b_2$ must hold for at least one j, or, if $\min_j \frac{a_{1j}}{a_{2j}} \leq \frac{b_1}{b_2} \leq \max_j \frac{a_{1j}}{a_{2j}}$ does not hold then

(30a) has no integer solution.

Equations (30a) imply $x_j \leq \beta_j^1 = \min_{i=1,2} (\frac{b_i}{a_{ij}})$

and $x_j \geq \alpha_j^1 = \max_{i=1,2} (\frac{1}{a_{ij}} (b_i - \sum_{\substack{k=1 \\ k=j}}^{n} a_{ij} \beta_k^1), 0)$ with α_j^1 and β_j^1

rounded to the appropriate integer. It is not difficult to solve (31) and its minimizing equivalent with the additional constraints $\alpha_j^1 \leq x_j \leq \beta_j^1$.

The additional constraints $x_j + y_j = \beta_j$, if introduced, may be ignored during the reduction of the system. One of the substitutions $x_j = \beta_j - y_j$ or $y_j = \beta_j - x_j$ will transform the final equation into an equivalent one with non-negative coefficients and the original number of variables.

3.3 Final remarks

Now consider system (22) again and assume $a_{ij} \geq 0$. Relation (27) admits the following interpretation:

$(a_{1j},\ldots,a_{mj})$ represents a_j in the system with base λ.

Thus computation of the a_j is superfluous, or amounts to nothing but their representation in a different system. Consequently, any method to solve (25) with $a_j \geq 0$, can be used to solve (22). If other operations than comparing a_j's are necessary a value for λ must be determined.
The same remarks apply to (26), where the a_j are represented in a possible unfamiliar system, with a different number of 'digits' for each position.

If $a_{ij} \geq 0$ it may be assumed that $a_{0j} \geq 1$,

so $a_0 \geq \sum_{i=1}^{m} \lambda_1 \ldots \lambda_{i-1} \geq \sum_{i=1}^{m} 2^{i-1}$.

The conclusion is that a system with $a_{ij} \geq 0$ leads to a_j of such a magnitude that multi-length arithmetic is required while solving (25). But then the representation $a_j = (a_{1j}, \ldots, a_{mj})$ might be used. The multiplication of selected rows by -1 will not help much if the system was obtained from a system of inequalities. In this case the slack variable in the m-th row will have $\lambda_1 \lambda_2 \ldots \lambda_{m-1} \geq 2^{m-1}$ as coefficients.

If the original problem has all $a_{ij} \geq 0$ negative elements should be introduced. At least two methods are available, multiplication of a row by -1, and the substitution $a_{ij} x_j = a_{ij} \beta_j - a_{ij} y_j$. (with $\alpha_j = 0$). The substitution $x_j = \beta_j - y_j$ has no effects.

It is clear that a slight modification of the problem, e.g. $a_{ij} := a_{ij} + 1$ for some elements, may have considerable effects on the coefficients of the final equation. This possibly sheds some light on experiences with integer linear programming algorithms.

The linear case may be extended to systems of polynomial equations with integer coefficients and integer variables.
Extention to systems of inequalities is not straightforward, $x \leq 1$ and $y \leq 1$ imply $x + 10y \leq 11$ but the converse is not true.
In general, the lexicographical ordering is not applicable if inequality in each separate component is required.

Polynomials in bivalent (zero-one) variables are the Pseudo-Boolean functions. Bivalent polynomials in bivalent variables are the Boolean functions in Pseudo-Boolean representation. In case of Boolean equations the present approach is related to the methods of Fortet and Camion, cf. Hammer and Rudeanu [2], chap. III, §4.

4. Examples

4.1 Internal stability

An undirected graph consists of a set of vertices (nodes) and a set of edges (lines). Each edge connects (is incident to) exactly two vertices. For each pair of vertices at most one edge incident to both vertices exists.
Label the vertices 1, 2, ..., n and label the edges 1, 2, ..., m.
The edge-node incidence matrix of the graph is defined as

$$a_{ij} = \begin{cases} 1 \text{ if edge } i \text{ incident to vertex } j \\ 0 \text{ otherwise,} \end{cases} \tag{36}$$

thus a_{ij} has exactly two positive entries in each row.
A subset of the set of vertices is called internally stable if no edge connects two vertices in the subset.
Any solution of

$$\begin{aligned} \sum_{j=1}^{n} a_{ij}\, x_j &\leq 1 \qquad (i=1, \ldots, m), \\ x_j &\in \{0, 1\} \qquad (j=1, \ldots, n), \end{aligned} \tag{37}$$

describes an internally stable set.

A maximal internally stable set is found by solving

$$\text{maximize} \quad \sum_{j=1}^{n} x_j \quad ,$$

subject to

$$\sum_{j=1}^{n} \left(\sum_{i=1}^{m} 3^{i-1} a_{ij} \right) x_j + \sum_{i=1}^{m} 3^{i-1} s_i = \sum_{i=1}^{m} 3^{i-1} \quad , \tag{38}$$

$$x_j \in \{0, 1\} \quad (j=1, \ldots, n) \qquad s_i \in \{0, 1\} \quad (i=1, \ldots, m).$$

4.2 A numerical example

Consider the problem

$$
\begin{aligned}
f_1 &= -x_1 + 3x_2 - 5x_3 - x_4 + 4x_5 + x_6 &&= -2 \\
f_2 &= 2x_1 - 6x_2 + 3x_3 + 2x_4 - 2x_5 + x_7 &&= 0 \\
f_3 &= x_2 - 2x_3 + x_4 + x_5 + x_8 &&= -1 \qquad (39)
\end{aligned}
$$

$$x_j \in \{0, 1\} \ (j=1, \ldots, 5) , \quad x_j \geq 0 \quad (j=6, 7, 8).$$

Evidently

$$
\begin{aligned}
-7 = -1 - 5 - 1 &\leq f_1 \leq 3 + 4 + 5 = 12 , \\
-8 = -6 - 2 &\leq f_2 \leq 2 + 3 + 2 + 8 = 15 , \qquad (40) \\
-2 = -2 &\leq f_3 \leq 1 + 1 + 1 + 1 = 4 ,
\end{aligned}
$$

as the system implies $x_6 \leq 5$, $x_7 \leq 8$ and $x_8 \leq 1$.

These bounds are rather crude, because the implication $x_3 = 1$ is ignored and each bound is independent of the remaining constraints.

It follows that $\lambda_1 = 15$, $\lambda_2 = 16$ and $\lambda_3 = 5$ can be used. One of them is superfluous, after rearranging the constraints the system is equivalent to

$$f_3 + 5f_1 + 75f_2 = -1 - 14,$$

or

$$145x_1 - 434x_2 + 198x_3 + 146x_4 - 129x_5 + 5x_6 + 75x_7 + x_8 = -15.$$

In [1] the same problem leads to

$$\begin{aligned} & 55284\ x_1 + 14092\ y_1 + \\ + & 43359\ x_2 + 165852\ y_2 + \\ + & 82926\ x_3 + 45526\ y_3 + \\ + & 56367\ x_4 + 14092\ y_4 + \\ + & 57451\ x_5 + 55284\ y_5 + \\ + & 14092\ x_6 + 27642\ x_7 + \\ + & 1083\ x_8 = 292679\ . \end{aligned}$$

This equation is irreducible, $1083 = 3 \times 19 \times 19$, 14092 contains neither factor 3 nor 19.

The substitutions

$y_1 = 1 - x_1$, $x_2 = 1 - y_2$, $y_3 = 1 - x_3$, $y_4 = 1 - x_4$, $y_5 = 1 - x_5$,
lead to

$$\begin{aligned} & 41196\ x_1 + 142493\ y_2 + 37400\ x_3 + 42275\ x_4 \\ & 2167\ x_5 + 14092\ x_6 + 27642\ x_7 + 1083\ x_8 = 120326. \end{aligned}$$

4.3 Another example

Consider the system

$$f = 2x + 2y = 3$$

$$g = 2x + 3y = 4$$

$$h = 3x + 4y = 5$$

$$0 \leq x \leq 1$$

$$0 \leq y \leq 1$$

$$x, y = \text{integer.}$$

This system has no solution, even if x and y are not required to be integers.

Subject to $f = 3$, $g_{min} = 3\frac{1}{2}$ and $g_{max} = 4$.

Thus the system reduces to

$$f = 2x + 2y = 3$$

$$g + h = 5x + 7y = 9$$

and

$$f + 4(g+h) = 22x + 30y = 39.$$

5. References

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