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THE RUSH IN A DIRECTED GRAPH

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1. Introduction

In several applications of graph theory it is of interest to compare the 'importance' of the vertices of a graph. Evidently, the definition of 'importance' should depend upon the specific application.
The centrality of a vertex, i.e. the sum of the distances from a vertex to the other vertices, is a well-known example.
In this note the concept of 'rush' is introduced, applicable to the vertices as well as to the arcs of a directed graph.
The rush in an element is the total flow through that element, resulting from a flow between each pair of vertices.

2. Definitions

Only finite directed graphs without loops are consired, undirected graphs can be included by interpreting them as symmetric digraphs.

A path from vertex u to vertex z is a sequence

$$u,\ (u,v),\ v,\ (v,w),\ w,\ \ldots,\ y,\ (y,z),\ z$$

where

u, v, w, ... denote vertices

and

(u,v), (v,w), ... denote arcs, directed from u to v, from v to w, etc.

If a path from u to z exists then z is a descendant of u, whereas u is an ascendant of z.

The length of a path is defined as the number of arcs constituting the path. If a path from u to z exists there also exist one or more minpaths, i.e. paths of minimal length. The length of a minpath from u to z is the distance from u to z.

From the matrix (a_{ij}) associated with a graph, i.e.

$$a_{ij} = \begin{cases} 1 & \text{if an arc from } x_i \text{ to } x_j \text{ exists,} \\ 0 & \text{otherwise,} \end{cases}$$

the matrix (d_{ij}) of distances, i.e.

$$d_{ii} = 0$$

$$d_{ij} = \left.\begin{cases} \text{distance from } x_i \text{ to } x_j\text{, if } x_j \text{ is a descendant of } x_i\text{,} \\ \infty \quad \text{otherwise,} \end{cases}\right\} \qquad i \neq j$$

is easily found using Floyds algorithm [1].

It is not difficult to construct the minpaths from (d_{ij}).

To define the rush, one unit of flow is sent from each vertex x_i to each vertex x_j, provided $0 < d_{ij} < \infty$. The flow is sent along minpaths only, if e_{ij} minpaths from x_i to x_j exist then $1/e_{ij}$ units are sent along each path.

The rush in an element (vertex or arc) is defined as the total flow through that element, resulting from the flows defined above.

It should be noted that the flow originating from x_i and the flow with destination x_i do not belong to the rush in x_i.

The definition of rush can be extended in at least two directions. Instead of one unit, f_{ij} units of flow can be sent from x_i to x_j. Instead of length one, length l_{ij} can be assigned to arc (x_i, x_j). The length of a path is then defined at the total length (sum) of its constituent arcs.

As an example, consider the graph depicted in figure 1.

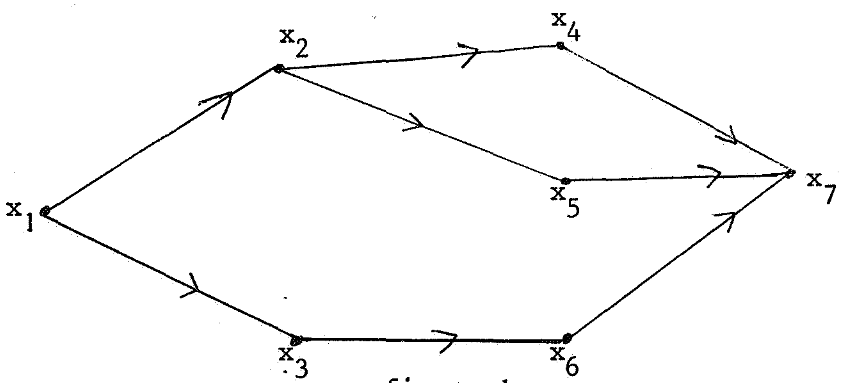

figure 1

Table 1 gives the minpaths with length > 1, only the constituent vertices are listed.

from	to	minpaths
1	4	1, 2, 4
1	5	1, 2, 5
1	6	1, 3, 6
1	7	1, 2, 4, 7 ; 1, 2, 5, 7 ; 1, 3, 6, 7
2	7	2, 4, 7 ; 2, 5, 7
3	7	3, 6, 7

table 1

Table 2 contains the rush in each element of the graph.

vertex	rush	arc	rush
1	0	1 2	$3\frac{2}{3}$
2	$2\frac{2}{3}$	1 3	$2\frac{1}{3}$
3	$1\frac{1}{3}$	2 4	$2\frac{5}{6}$
4	$\frac{5}{6}$	2 5	$2\frac{5}{6}$
5	$\frac{5}{6}$	3 6	$3\frac{1}{3}$
6	$1\frac{1}{3}$	4 7	$1\frac{5}{6}$
7	0	5 7	$1\frac{5}{6}$
		6 7	$2\frac{1}{3}$

<u>table 2</u>

3. Computation

Two ALGOL-60 procedures are presented.
The procedure minpaths (d, e, n) calculates, from the n x n matrix d of distances, the n x n matrix e, where e_{ij} = the number of minpaths from x_i to x_j.
The procedure rush (d, e, r, n) calculates from the matrices d and e, the rush-matrix r, where r_{ii} = the rush in x_i and, for $i \neq j$, r_{ij} = the rush in arc (x_i, x_j) if this arc exists, zero otherwise.
Both procedures are straightforward, and no claim of efficiency is made.

The procedures are based upon the relation

$$e_{ij} = \begin{cases} 1 & \text{if} \quad d_{ij} = 1 \\ \sum\limits_h (e_{hj} \mid d_{ih} = 1,\ d_{ij} = d_{ih} + d_{hj}) & \text{if } 1 < d_{ij} < \infty. \end{cases}$$

```
procedure minpaths(d,e,n); value n; integer n;
integer array d,e;
begin   integer i,j;
        integer procedure paths(i,j); value i,j; integer i,j;
        begin
                if e[i,j] = -1 then
                begin   integer a,h;
                        a:= d[i,j] - 1; e[i,j]:= 0;
                        if a = 0 then e[i,j]:= 1 else
                        for h:= 1 step 1 until n do
                        if d[i,h] = 1 ∧ d[h,j] = a then
                        e[i,j]:= e[i,j] + paths(h,j)
                end;
                paths:= e[i,j]
        end paths;
        for i:=1 step 1 until n do
        for j:=1 step 1 until n do e[i,j]:=-1;
        for i:=1 step 1 until n do
        for j:=1 step 1 until n do
        if d[i,j] = 0 ∨ d[i,j] ≥ n then e[i,j]:= 0 else
        if e[i,j] = -1 then paths(i,j)
end minpaths;
```

```
procedure rush(d,e,r,n); value n; integer n;
integer array d,e; real array r;
begin   integer i,j;
        procedure add(b,i,j); value b,i,j;
        integer i,j; real b;
        begin   integer a,h,p;
                real c;
                a:= d[i,j] - 1; p:= e[i,j];
                if a = 0 then r[i,j]:= r[i,j] + b else
                for h:= 1 step 1 until n do
                if d[i,h] = 1 ∧ d[h,j] = a then
                begin
                        c:=b × e[h,j] / p;
                        r[i,h]:= r[i,h] + c;
                        r[h,h]:= r[h,h] + c;
                        add(c,h,j)
                end
        end add;

        for i:= 1 step 1 until n do
        for j:= 1 step 1 until n do r[i,j]:= 0;
        for i:= 1 step 1 until n do
        for j:= 1 step 1 until n do
        if d[i,j] > 0 ∧ d[i,j] < n
        then add(1,i,j)
end rush;
```

4. An Application

The concept of rush has been applied in a study of relations between committees. Table 3 describes the constitution of 9 committees.

person \ committee	1	2	3	4	5	6	7	8	9
a	1	1	1						
b			1	1					
c				1	1				
d				1		1			
e						1	1		
f							1	1	
g							1		1

table 3

In figure 2 a graph is depicted, which describes the relations between the committees.
Each vertex corresponds to a committee, two vertices are connected if the committees have a member in common.

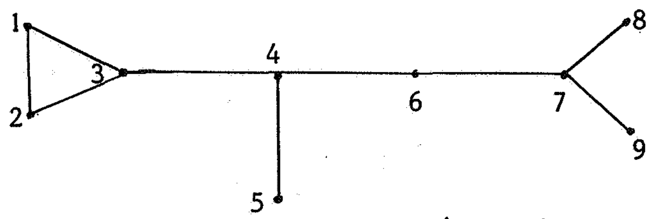

figure 2

It is very easy to compute the rush in this graph, table 4 contains the rush in each element.

vertex	rush	edge	rush		person
1	0	1,2	2		
2	0	1,3	14	30	a
3	24	2,3	14		
4	38	3,4	36		b
5	0	4,5	16		c
6	30	4,6	40		d
7	22	6,7	36		e
8	0	7,8	14		f
9	0	7,9	14		g

table 4

From table 4 it might be concluded that committee 4 is very important for the exchange of information between the committees, and that person d has an important position.

5. Reference

R.W. Floyd,
Algorithm 97
Comm. ACM 5(1962) 345.