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# A NOTE ON HOWARD'S VALUE DETERMINATION STEP

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## 1. INTRODUCTION

Let $P$ be an $N \times N$ Markov matrix whose $(i,j)$ element is $p_{ij}$ $(i,j=1,\dots,N)$, i.e., $p_{ij} \geq 0$ and $\sum_j p_{ij}=1$. Let $T$ be an $N$ component column vector whose ith element is $T_i$ where $T_i > 0$ for $i=1,\dots,N$, and let $q$ be an $N$ component column vector whose ith element is $q_i$ $(i=1,\dots,N)$. The triple $(P,T,q)$ can be thought of as a semi-Markov reward process with transition probabilities $p_{ij}$, expected transition times $T_i$ and one-transition rewards $q_i$. It is assumed that the Markov matrix $P$ has a single recurrent chain. Let state $N$ be a recurrent state of the Markov matrix $P$.

In each iteration of Howard's [2] well known policy-iteration algorithm a set of linear simultaneous equations must be solved. For the single chain case this set of equations is of the following type:

$$gT + v = q + Pv \,, \tag{1}$$

where $g$ is an unknown scalar and $v$ is an unknown $N$ component column vector whose ith element is $v_i$ $(i=1,\dots,N)$. It is important to have an efficient method for solving (1). For the case where $P$ is an aperiodic Markov matrix Morton [4] has given a simple iterative scheme to solve (1).

The purpose of this note is to demonstrate that a solution of (1) can be found by solving two sets of linear simultaneous equations which are more easy to tackle than (1). In our approach we need not require that $P$ is aperiodic. Despite the fact that our approach is implied in the paper of Derman and Veinott[1], the theorem below seems to have passed unnoticed.

## 2. RESULTS

We first introduce some notation. Let $T^*$ be the N-1 component column vector whose ith element is $T_i$, let $q^*$ be the N-1 component column vector whose ith element is $q_i$, and let R be the N-1 component row vector whose ith element is $p_{Ni}$ $(i=1,\ldots,N-1)$. Denote by Q the $(N-1)\times(N-1)$ matrix whose $(i,j)$ element is $p_{ij}$ $(i,j=1,\ldots,N-1)$. Observe that $Q^n \to 0$ as $n \to \infty$, since N is a recurrent state of the Markov matrix P.

We have the following theorem (cf. Derman and Veinott [1] and Theorem 1 of Morton [4])

THEOREM. *Let the column vector $x=(x_1,\ldots,x_{N-1})$ be the unique solution to*

$$x = q^* + Qx \ , \tag{2}$$

*and let the column vector $y=(y_1,\ldots,y_{N-1})$ be the unique solution to*

$$y = T^* + Qy \ . \tag{3}$$

*Define the scalar g by*

$$g=(q_N+Rx)/(T_N+Ry) \ , \tag{4}$$

*and define the N component column vector $v=(v_1,\ldots,v_N)$ by*

$$v_i = x_i - gy_i \quad \text{for } i=1,\ldots,N-1 \ , \qquad v_N = 0 \ . \tag{5}$$

*Then g,v satisfy equation (1).*

*Proof.* Let us first observe that both (2) and (3) have a unique solution, since $Q^n \to 0$ as $n \to \infty$. Denote by $v^*$ the N-1 component column vector whose ith element is $v_i$ $(i=1,\ldots,N-1)$. From (2), (3) and (5),

$$gT^* + v^* = gT^* + q^* + Qx-g(T^*+Qy) = q^* + Q(x-gy) = q^* + Qv^* ,$$

while from (4) and (5) it follows that

$$gT_N + v_N = q_N + Rx - gRy = q_N + R(x-gy) = q_N + Rv^* .$$

Using $v_N = 0$ the theorem now follows.

Observe that g in (4) can be interpreted as the ratio of the expected return earned during a cycle and the expected length of a cycle, where a cycle is defined as the time interval between two successive visits to the recurrent state N. It is well-known that this ratio equals the long-run average return.

*Remark*. Suppose that $p_{iN}=1-\alpha_i > 0$ for $i=1,\ldots,N-1$. Let $z_0$ be an arbitrary N-1 component column vector, and for $n\geq 1$ define $z_n$ by $z_n = b + Qz_{n-1}$, where b is a given N-1 component column vector. Let z be the unique solution to $z=b+Qz$. Define for any $n\geq 1$,

$$u_n'(i)=z_n(i)+(1-\alpha_i)^{-1} \min_j\{z_n(j)-z_{n-1}(j)\} \qquad \text{for } i=1,\ldots,N-1 ,$$

and

$$u_n''(i)=z_n(i)+(1-\alpha_i)^{-1} \max_j\{z_n(j)-z_{n-1}(j)\} \qquad \text{for } i=1,\ldots,N-1 .$$

Then, for any $n\geq 1$, $u_n'(i)\leq z(i)\leq u_n''(i)$ for $i=1,\ldots,N-1$, where $u_n'(i)$ is nondecreasing in n to z(i) and $u_n''(i)$ is nonincreasing in n to z(i) for all i. The proof of this assertion is a slight modification of proofs given by Macqueen [3] and is based on the following fact: If $Tu\leq Tw$ then $u\leq w$, where the transformation T is defined by $Tu=u-(b+Qu)$ for any N-1 component column vector u.

*Remark*. It is straightforward to extend the analysis above to the case of a general Markov matrix P; in this case the set of simultaneous equations g=Pg and gT+v=q+Pv has to be solved where g and v are unknown N component column vectors.

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