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A NOTE ON HOWARD'S VALUE DETERMINATION STEP

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1. INTRODUCTION

Let $P$ be an $N \times N$ Markov matrix whose $(i,j)$ element is $p_{ij}$ ($i,j=1,\ldots,N$), i.e., $p_{ij} \geq 0$ and $\sum_{j} p_{ij} = 1$. Let $T$ be an $N$ component column vector whose ith element is $T_i$ where $T_i > 0$ for $i=1,\ldots,N$, and let $q$ be an $N$ component column vector whose ith element is $q_i$ ($i=1,\ldots,N$). The triple $(P,T,q)$ can be thought of as a semi-Markov reward process with transition probabilities $p_{ij}$, expected transition times $T_i$ and one-transition rewards $q_i$. It is assumed that the Markov matrix $P$ has a single recurrent chain. Let state $N$ be a recurrent state of the Markov matrix $P$.

In each iteration of Howard's [2] well known policy-iteration algorithm a set of linear simultaneous equations must be solved. For the single chain case this set of equations is of the following type:

$$ gT + v = q + Pv, \quad (1) $$

where $g$ is an unknown scalar and $v$ is an unknown $N$ component column vector whose ith element is $v_i$ ($i=1,\ldots,N$). It is important to have an efficient method for solving (1). For the case where $P$ is an aperiodic Markov matrix Morton [4] has given a simple iterative scheme to solve (1).

The purpose of this note is to demonstrate that a solution of (1) can be found by solving two sets of linear simultaneous equations which are more easy to tackle than (1). In our approach we need not require that $P$ is aperiodic. Despite the fact that our approach is implied in the paper of Derman and Veinott[1], the theorem below seems to have passed unnoticed.
2. RESULTS

We first introduce some notation. Let \( T^* \) be the \( N-1 \) component column vector whose ith element is \( T_i \), let \( q^* \) be the \( N-1 \) component column vector whose ith element is \( q_i \), and let \( R \) be the \( N-1 \) component row vector whose ith element is \( p_{Ni} \) (\( i=1,\ldots,N-1 \)). Denote by \( Q \) the \((N-1)\times(N-1)\) matrix whose (i,j) element is \( p_{ij} \) (\( i,j=1,\ldots,N-1 \)). Observe that \( q^* \to 0 \) as \( n \to \infty \), since \( N \) is a recurrent state of the Markov matrix \( P \).

We have the following theorem (cf. Derman and Veinott [1] and Theorem 1 of Morton [4]):

**THEOREM.** Let the column vector \( x=(x_1,\ldots,x_{N-1}) \) be the unique solution to

\[
x = q^* + Qx ,
\]

and let the column vector \( y=(y_1,\ldots,y_{N-1}) \) be the unique solution to

\[
y = T^* + Qy .
\]

Define the scalar \( g \) by

\[
g = (q_N + Rx)/(T_N + Ry) ,
\]

and define the \( N \) component column vector \( v=(v_1,\ldots,v_N) \) by

\[
v_i = x_i - gy_i \quad \text{for} \quad i=1,\ldots,N-1 , \quad v_N = 0 .
\]

Then \( g,v \) satisfy equation (1).

**Proof.** Let us first observe that both (2) and (3) have a unique solution, since \( Q^n \to 0 \) as \( n \to \infty \). Denote by \( v^* \) the \( N-1 \) component column vector whose
The $i$th element is $v_i$ ($i=1,\ldots,N-1$). From (2), (3) and (5),

$$gT^* + v^* = gT^* + q^* + Qx - g(T^* + qy) = q^* + Q(x - gy) = q^* + Qv^* ,$$

while from (4) and (5) it follows that

$$gT_N^* + v_N = q_N^* + Rx - gry = q_N^* + R(x - gy) = q_N^* + Rv^* .$$

Using $v_N = 0$ the theorem now follows.

Observe that $g$ in (4) can be interpreted as the ratio of the expected return earned during a cycle and the expected length of a cycle, where a cycle is defined as the time interval between two successive visits to the recurrent state $N$. It is well-known that this ratio equals the long-run average return.

Remark. Suppose that $p_{iN} = 1 - x_i > 0$ for $i=1,\ldots,N-1$. Let $z_0$ be an arbitrary $N$-1 component column vector, and for $n \geq 1$ define $z_n$ by $z_n = b + Qz_{n-1}$, where $b$ is a given $N$-1 component column vector. Let $z$ be the unique solution to $z = b + Qz$. Define for any $n \geq 1$,

$$u_n'(i) = z_n(i) + (1 - x_i)^{-1} \min_j (z_n(j) - z_{n-1}(j)) \quad \text{for } i=1,\ldots,N-1 ,$$

and

$$u_n''(i) = z_n(i) + (1 - x_i)^{-1} \max_j (z_n(j) - z_{n-1}(j)) \quad \text{for } i=1,\ldots,N-1 .$$

Then, for any $n \geq 1$, $u_n'(i) < z(i) < u_n''(i)$ for $i=1,\ldots,N-1$, where $u_n'(i)$ is nondecreasing in $n$ to $z(i)$ and $u_n''(i)$ is nonincreasing in $n$ to $z(i)$ for all $i$. The proof of this assertion is a slight modification of proofs given by Macqueen [3] and is based on the following fact: If $Tu \leq Tw$ then $u \leq w$, where the transformation $T$ is defined by $Tu = u - (b + Qu)$ for any $N$-1 component column vector $u$. 
Remark. It is straightforward to extend the analysis above to the case of a general Markov matrix $P$; in this case the set of simultaneous equations $g = Pg$ and $gt + v = q + Pv$ has to be solved where $g$ and $v$ are unknown $N$ component column vectors.

REFERENCES.


