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THE OPTIMALITY OF (s,S) INVENTORY POLICIES FOR THE AVERAGE COST CRITERION

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The optimality of (s,S) inventory policies for the average cost criterion

Summary. The infinite period stationary inventory model is considered. There is a constant lead time, a nonnegative set-up cost, a linear purchase cost, a holding and shortage cost function and total backlogging of unfilled demand. The optimality criterion is the average expected cost per period. Under the assumption that the negatives of the one period expected holding and shortage costs are unimodal,*) the existence of an optimal (s,S) policy is proved. As a by-product of the proof, upper and lower bounds on the optimal values of both s and S are found.

1. Introduction.

We consider the infinite period stationary inventory model in which demands for a single product in periods 1,2,... are independent, identically distributed random variables. At the beginning of each period an order may be placed for any nonnegative quantity of stock. There is a constant lead time, a fixed set-up cost, a linear purchase cost, a holding and shortage cost function and total backlogging of unfilled demand.

For the finite period model Scarf [9] has shown that if the one period expected holding and shortage costs are convex, then an (s,S) policy exists which minimizes the total expected cost (as noted in [10], the proof carries over to the nonstationary model). Under Scarf's assumptions Iglehart [3,4] has examined the infinite period stationary model. In [3] it is proved that an (s,S) policy exists which minimizes the total expected discounted cost and in [4] the existence of an (s,S) policy minimizing the average expected cost per period is shown.

Veinott [12] has proved the existence of an optimal (s,S) policy in the finite period nonstationary model under assumptions which do not imply and are not implied by Scarf's hypotheses. However for the infinite period stationary model Veinott's assumptions are weaker than Scarf's. Veinott has replaced Scarf's hypothesis that the one period expected holding and shortage costs are convex by the weaker assumption that the negatives of these costs are unimodal.

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*) This assumption can be weakened slightly (see remark 5.4, pp. 16).

In this paper we shall consider the infinite period stationary model and our optimality criterion is the average expected cost per period. Under Veinott's assumption that the negatives of the one period expected holding and shortage costs are unimodal, we shall prove the existence of an optimal (s,S) policy. The existence proof is based on the ingenious idea of Iglehart [4] to construct a solution of a functional equation known from Markov programming. We give the proof for the discrete demand case. However the proof carries over directly to the continuous demand case. We note that in [5] another way is indicated to prove the existence of an optimal (s,S) policy. However the approach suggested in [5] seems typically for the discrete demand case.

As a by-product of the existence proof we shall find bounds on the optimal values of s and S, which are already established in a quite different way in [11] for the case in which the one period expected holding and shortage costs are convex.

2. Model formulation.

We consider the infinite period stationary inventory model in which the demands $\underline{\xi}_1, \underline{\xi}_2, \ldots$ for a single item in periods 1,2,... are independent, nonnegative, discrete random variables with the common probability distribution $p_j = P\{\underline{\xi}_t = j\}$, ($j=0,1,\ldots$; $t=1,2,\ldots$). It is assumed that $\mu = \mathcal{E}\, \underline{\xi}_t$ is finite and positive. At the beginning of each period the stock on hand plus on order is reviewed. An order may be placed for any nonnegative, integral quantity of stock. An order placed in period t is delivered at the beginning of period $t+\lambda$, where λ is a known nonnegative integer. The demand is assumed to take place at the end of each period. All unsatisfied demand is backlogged and there is no obsolescence of stock.

The following costs are considered. In any period the cost of ordering z units is $K\delta(z)+cz$, where $K \geq 0$, $\delta(0) = 0$, and $\delta(z) = 1$ for $z > 0$. Let $g(i)$ be the holding and shortage cost in a period when i is the amount of stock on hand at the beginning of that period just after an eventual delivery.

Let $\underline{T}_0 = 0$ and let $\underline{T}_n = \underline{\xi}_1+\ldots+\underline{\xi}_n$, $n \geq 1$. Define $p_j^{(n)} = P\{\underline{T}_n=j\}$, ($j \geq 0$; $n \geq 0$). Assume that for each integer k

$$L(k) = \sum_{j=0}^{\infty} g(k-j)p_j^{(\lambda)} \tag{2.1}$$

exists and is finite. If at the beginning of the present period t the stock on hand plus on order, just after ordering in that period, is k, then at the beginning of period t+λ, just after delivery for period t+λ, the stock on hand is $k-\underline{T}_\lambda$. Hence L(k) is the expected holding and shortage cost in period t+λ when k is the stock on hand plus on order just after ordering in period t. The following conditions are imposed on L(k).

(i) A finite integer S_0 exists, such that $L(i) \leq L(j)$ for $j \leq i \leq S_0$ and $L(i) \geq L(j)$ for $i \geq j \geq S_0$.

(ii) $\lim_{|k|\to\infty} L(k) > L(S_0)+K$.

Because of (ii) we may assume that S_0 is the largest integer for which property (i) holds. Let s_1 be the smallest integer for which

(2.2) $$L(s_1) \leq L(S_0)+K$$

and let S_1 be the largest integer for which

(2.3) $$L(S_1) \leq L(S_0)+K \;.$$

The existence of the finite integers $s_1(\leq S_0)$ and $S_1(\geq S_0)$ is ensured by (ii).

Let us define the state of the system in a period as the stock on hand plus on order just before ordering in that period. We take the set I of all integers as the set of all possible states. An order can be placed only at the beginning of each period and every ordering decision is based on the stock on hand plus on order. Every ordering decision can be represented by the stock on hand plus on order just after that decision. Let us say that in state i decision $k(k \geq i)$ is made when k-i units are ordered. An order, placed at the beginning of period t, cannot influence the holding and shortage cost incurred between the beginnings of period t and period t+λ. Therefore we assign to decision k in state i the cost

$$(2.4) \qquad w(i,k) = K\delta(k-i) + (k-i)c + L(k) \;, \qquad i \in I,\; k \geqq i.$$

We impose the following mild restrictions on the choice of an ordering decision. There are finite integers $M_1 \leqq s_1$ and $M_2 \geqq S_1$, such that nothing is ordered if the stock on hand plus on order $i > M_2$, at most M_2-i units are ordered of $M_1 \leqq i \leqq M_2$ and at least M_1-i units are ordered if $i < M_1$. Let $K(i)$ be the set of feasible ordering decisions in state i. We have $K(i) = \{i\}$ for $i > M_2$, $K(i) = \{k|i \leqq k \leqq M_2\}$ for $M_1 \leqq i \leqq M_2$ and $K(i) = \{k|M_1 \leqq k \leqq M_2\}$ for $i < M_1$.

A policy R for controlling the system is a set of functions $\{D_k(h_{t-1},i_t)\}$, $k \in K(i_t)$, satisfying

$$D_k(h_{t-1},i_t) \geqq 0,\; k \in K(i_t) \text{ and } \sum_{k \in K(i_t)} D_k(h_{t-1},i_t) = 1$$

for every "history" $h_{t-1} = (i_1,k_1,\ldots,i_{t-1},k_{t-1})$ and all $i_t \in I$, $t=1,2,\ldots$, where i_n resp. k_n is the observed state resp. the observed decision in period n.

The interpretation being: if at the beginning of period t the history h_{t-1} has been observed and the system is in state i_t, then $k-i_t$ units are ordered with probability $D_k(h_{t-1},i_t)$.

Let $C(M_1,M_2)$ denote the class of all possible policies. A policy R is said to be stationary deterministic if $D_k(h_{t-1},i_t=i) = D_k(i)$, independent of h_{t-1} and t, and if in addition $D_k(i) = 1$, or 0. Suppose that a policy $R \in C(M_1,M_2)$ is followed. Let $\underline{i}_t$ resp. $\underline{k}_t$ be the state resp. the ordering decision in period t. We take as optimality criterion

$$(2.5) \qquad g(i,R) = \liminf_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} \mathcal{E}_R(w(\underline{i}_t,\underline{k}_t)|\, \underline{i}_1=i).$$

We note that the expectations exist. The quantity $g(i,R)$ can be interpreted as the average expected cost per period when the initial state is i and policy R is followed.

Using the fact that $\underline{i}_{t+1} = \underline{k}_t - \underline{\xi}_t$, we have (see also [11])

$$\sum_{t=1}^{n} \mathcal{E}_R(w(\underline{i}_t,\underline{k}_t)|\underline{i}_1=i) = \sum_{t=1}^{n} \mathcal{E}_R\{K\delta(\underline{k}_t-\underline{i}_t)+(\underline{k}_t-\underline{i}_t)c+L(\underline{k}_t)|\underline{i}_1=i\} =$$

$$= \sum_{t=1}^{n} \mathcal{E}_R(K\delta(\underline{k}_t-\underline{i}_t)+L(\underline{k}_t)|\underline{i}_1=i)+n\mu c-ic+c\,\mathcal{E}_R(\underline{i}_{n+1}|\underline{i}_1=i) \;.$$

Since $M_1 - \mu \leq \mathcal{E}_R(\underline{i}_{n+1}|\underline{i}_1=i) \leq \max(i,M_2)$, we have

$$g(i,R) = \liminf_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} \mathcal{E}_R(\hat{w}(\underline{i}_t,\underline{k}_t)|\underline{i}_1=i)+\mu c,$$

where $\hat{w}(i,k) = K\delta(k-i) + L(k)$. We find it convenient to redefine $w(i,k)$ by setting $w(i,k) = K\delta(k-i) + L(k)$. This reduces each $g(i,R)$ with the same finite amount μc.

A policy R^* is called <u>optimal</u> if

$$(2.6) \qquad g(i,R^*) \leq g(i,R) \qquad \text{for all } i\in I, \text{ all } R\in C(M_1,M_2).$$

We formulate now a theorem, which will play a fundamental role in our considerations.

<u>Theorem 2.1</u>

Suppose there exists a set of numbers $\{g,v(i)\}$, $i\in I$, such that

$$(2.7) \qquad v(i) = \min_{k\in \mathbf{K}(i)} \{K\delta(k-i) + L(k) - g + \sum_{j=0}^{\infty} v(k-j)p_j\},\ i\in I,$$

and for every $R\in C(M_1,M_2)$

$$(2.8) \qquad \lim_{n\to\infty} \frac{1}{n} \mathcal{E}_R(v(\underline{i}_n)|\underline{i}_1 = i) = 0, \qquad i\in I.$$

Let R^* be a policy which, for each i, prescribes a decision which minimizes the right side of (2.7). The stationary deterministic policy R^* is then optimal and

$$(2.9) \qquad g = \lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} \mathcal{E}_{R^*}(w(\underline{i}_t,\underline{k}_t)|\underline{i}_1 = i) \qquad \text{for all } i\in I.$$

This theorem is a direct consequence of the elegant proof of theorem 1 in [8] (see also [1]).

<u>Remark 1.1</u> The condition (2.7) has its origin in Howard's discrete finite state and finite action Markow programming model with an infinite planning horizon [2]. In that model so-called "relative values" v_i are associated

with every stationary deterministic policy. Under certain conditions it can be shown that the relative values v_i of a stationary deterministic policy can be interpreted as follows. For any two states i and j the difference $v_i - v_j$ is equal to the decrease in total expected cost caused by starting in state i rather than in state j [1,2,6].

3. Some results from renewal theory.

We have defined $p_j^{(n)} = P\{\underline{T}_n = j\}$, where $\underline{T}_0 = 0$ and $\underline{T}_n = \underline{\xi}_1 + \ldots + \underline{\xi}_n$, $n \geq 1$. We write often $p_j^{(1)} = p_j$. We note that $p_0 < 1$, because $\mu = \mathcal{E}\underline{\xi}_t > 0$. The convolutionformula

(3.1) $$p_k^{(n)} = \sum_{j=0}^{k} p_{k-j}\, p_j^{(n-1)}, \qquad k \geq 0;\ n \geq 1$$

is well-known. The renewal quantities m(k) and M(k) are defined by

(3.2) $$m(k) = \sum_{n=1}^{\infty} p_k^{(n)} \text{ and } M(k) = \sum_{j=0}^{k} m(j), \quad k = 0,1,\ldots .$$

The renewal quantity m(k) is the unique bounded solution of [7]

(3.3) $$m(k) = p_k + \sum_{j=0}^{k} p_{k-j}\, m(j), \qquad k = 0,1,\ldots .$$

The renewal function M(k) is finite and furthermore [7]

(3.4) $$\lim_{k\to\infty} M(k)/k = 1/\mu .$$

Suppose b_n (n=0,1,...) are given finite numbers. Consider the discrete renewal equation,

(3.5) $$u_n = b_n + \sum_{j=0}^{n} u_{n-j}\, p_j, \qquad n = 0,1,\ldots .$$

The numbers u_n can be computed successively from (3.5), so that no problem about the existence of a unique solution $\{u_n\}$ arises. Iterating equation (3.5) and using (3.1) and the fact that for k fixed $p_k^{(n)} \to 0$ for $n \to \infty$, yields the well-known result

(3.6) $$u_n = b_n + \sum_{j=0}^{n} b_{n-j}\, m(j), \qquad n = 0,1,\ldots .$$

Define $\underline{N}(k) = \max \{ n \mid \underline{T}_n \leq k \}$, $k = 0,1,\ldots$. Then $\mathcal{E}\underline{N}(k) = M(k)$ [7]. Hence $1+M(k)$ can be interpreted as the expected number of periods needed for a cumulative demand exceeding k. The excess random variable $\underline{\gamma}_k$ is defined by $\underline{\gamma}_k = \underline{T}_{N(k)+1} - k$. Using a standard probabilistic argument it follows [7]

$$(3.7) \qquad P\{\underline{\gamma}_k = j\} = p_{k+j} + \sum_{h=0}^{k} p_{k-h+j}\, m(h), \qquad j = 1,2,\ldots \; .$$

4. The (s,S) policy.

An (s,S) policy, $s,S \in I$ and $s \leq S$, has the following simple form: When the stock on hand plus on order $i < s$, order $S-i$ units; for $i \geq s$, order nothing.

The function

$$(4.1) \qquad a(s,S) = \{L(S) + \sum_{j=0}^{S-s} L(S-j)m(j) + K\}/\{1+M(S-s)\}, \qquad s,S \in I, s \leq S,$$

will play an important part in our considerations. It can be shown that the right side of (4.1), which is well-known from literature [4,6,10,11], represents for each initial state the average expected cost per period when the (S,S) policy is followed. However this result will not be needed in our considerations, only the function $a(s,S)$ itself will be used. Briefly, the right side of (4.1) can be obtained as follows. When an (s,S) policy is followed the stochastic process $(\underline{i}_t, t \geq 1)$ is a denumerable Markov-chain. It can be proved that this Markov-chain has a unique stationary probability distribution, which can be determined explicitly [1,4,6,10,11] . By averaging the one period expected cost $w(i,k)$ with respect to the stationary probability distribution, the right side of (4.1) is obtained.

Lemma 4.1

There exist finite integers s^* and S^*, $s^* \leq S^*$, such that

$$(4.2) \qquad a(s^*,S^*) \leq a(s,S) \qquad \text{for all } s,S \in I,\ s \leq S.$$

5. The optimality of the (s^*,S^*) policy and bounds on s^* and S^*.

In this section we shall define a function $v^*(i)$, $i\in I$ and verify that the set of numbers $\{a^*, v^*(i)\}$, $i\in I$, satisfies the conditions (2.7) and (2.8). This proves the existence of an optimal policy. In addition we shall find the optimality of the (s^*,S^*) policy and bounds on s^* and S^*.

The function $v^*(i)$, $i\in I$, is defined as follows

(5.1) $$v^*(i) = \begin{cases} 0 & \text{for } i < s^*, \\ L(i)-a^*+\sum_{j=0}^{i-s^*} v^*(i-j)p_j & \text{for } i \geq s^*. \end{cases}$$

Remark 5.1. In this remark we motivate the definition of $v^*(i)$. Suppose that $\{g,v(i)\}$, $i\in I$, is a set of numbers satisfying (2.7) and suppose further that the right side of (2.7) is minimized by $k = S^*$ for $i < s^*$ and by $k=i$ for $i \geq s^*$. Then

$$v(i) = \begin{cases} L(i)-g+\sum_{j=0}^{\infty} v(i-j)p_j , & i \geq s^*, \\ K+v(S^*) , & i < s^*. \end{cases}$$

When c is a constant, then the set of numbers $\{g,v(i)+c\}$, $i\in I$, satisfies also (2.7). Hence normalizing $v(i)$ to be zero at $i = s^*-1$, explains definition (5.1).

The function $v^*(i)$, $i\in I$, is uniquely determined by the renewal equation (5.1). Iterating the equation (5.1), yields (c.f.(3.6))

$$(5.2) \qquad v^*(i) = L(i) + \sum_{j=0}^{i-s^*} L(i-j)m(j) - a^*\{1+M(i-s^*)\}, \qquad i \geqq s^*.$$

The functional equation (2.7) suggests to introduce the function

$$(5.3) \qquad J(k) = L(k) - a^* + \sum_{j=0}^{\infty} v^*(k-j)p_j, \qquad k\in I.$$

From (5.1) and (5.3) it follows

$$(5.4) \qquad J(k) = L(k) - a^*, \qquad k < s^*,$$

and

$$(5.5) \qquad J(k) = v^*(k), \qquad k \geqq s^*.$$

Theorem 5.1

(a) $J(k)$ is nonincreasing on $(-\infty, s^*-1]$,

(b) $K + J(S^*) = 0$, $J(s^*-1) \geqq 0$,

(c) $J(k) \geqq J(S^*)$ for all $k\in I$,

(d) $J(k) \leqq 0$ for $s^* \leqq k \leqq S_0$,

(e) $J(k)$ is nonincreasing on $[s^*, S_0]$,

(f) $J(k) - J(i) \geqq L(k) - L(i) - K$ for $k \geqq i \geqq S_0$.

Proof

(a) Since $L(j)$ is nonincreasing on $(-\infty,S_0]$ and by lemma 4.2(b) we have $s^* \leqq S_0$, it follows directly from (5.4) that (a) holds.

(b) Inserting $i = S^*$ in (5.2) and using (5.5), (4.3) and (4.1), yields $J(S^*) = -K$. From (5.4) and lemma 4.2(a) it follows that $J(s^*-1) = L(s^*-1) - a^* \geqq 0$.

(c) Since $K \geq 0$, we have by (a) and (b) that $J(k) \geq J(s^*-1) \geq J(S^*)$ for $k < s^*$. Hence it remains to show $J(k) \geq J(S^*)$ for $k \geq s^*$. Suppose there exists an integer $k \geq s^*$, say $k = r$, such that $J(r) < J(S^*)$. From $J(S^*) = -K$ and the formulas (5.2) and (5.5) it follows then

$$a^* > \{L(r) + \sum_{j=0}^{r-s^*} L(r-j)m(j) + K\}/\{1+M(r-s^*)\} .$$

Since by (4.1) the right side of the inequality is $a(r,s^*)$, we have obtained a contradiction. Thus (c) holds.

(d) Since $L(k)$ is nonincreasing on $[s^*,S_0]$, it follows from (5.5), (5.2) and lemma 4.2(a) that $J(k) \leq \{L(s^*)-a^*\}\{1+M(k-s^*)\} \leq 0$ for $s^* \leq k \leq S_0$.

(e) From (5.1) and (5.5) it follows that

$$J(k) = L(k) - a^* + \sum_{j=0}^{k-s^*} J(k-j)p_j , \qquad k \geq s^*. \tag{5.6}$$

By (d) and (5.6) we have for $s^* \leq i \leq k \leq S_0$,

$$J(i) - J(k) \geq L(i) - L(k) + \sum_{j=0}^{i-s^*} \{J(i-j)-J(k-j)\}p_j ,$$

Iterating this inequality, yields for $s^* \leq i \leq k \leq S_0$,

$$J(i) - J(k) \geq L(i) - L(k) + \sum_{j=0}^{i-s^*} \{L(i-j)-L(k-j)\}m(j) ,$$

The assertion (e) follows from this inequality and the fact that $L(k)$ is nonincreasing on $[s^*,S_0]$.

(f) By (b) and (c) we have $J(k) \geq -K$, $k \in I$. From (5.6) it follows now that for $k \geq i \geq s^*$

$$J(k) - J(i) \geq L(k) - L(i) + \sum_{j=0}^{i-s^*} \{J(k-j)-J(i-j)\}p_j +$$
$$- K\{F(k-s^*)-F(i-s^*)\},$$

where $F(n) = p_0 + \ldots + p_n$, $n \geqq 0$. Iterating this inequality, yields for $k \geqq i \geqq s^*$

$$J(k) - J(i) \geqq L(k) - L(i) + \sum_{j=0}^{i-s^*} \{L(k-j) - L(i-j)\}m(j) +$$

$$-K[F(k-s^*) - F(i-s^*) + \sum_{j=0}^{i-s^*} \{F(k-s^*-j) - F(i-s^*-j)\}m(j)].$$

The assertion (f) follows now from the observation that L(k) is non-decreasing on $[S_0,\infty)$ and the fact that the coefficient of -K is the probability that the excess variable $\underline{\gamma}_{i-s^*} \leq$ k-i (see (3.7)) and hence is less than or equal to 1.

Theorem 5.2

(a) The set of numbers $\{a^*, v^*(i)\}$, $i \in I$, satisfies

(5.7) $$v^*(i) = \min_{k \geqq i} \{K\delta(k-i) + L(k) - a^* + \sum_{j=0}^{\infty} v^*(k-j)p_j\}, \quad i \in I.$$

The right side of (5.7) is minimized by $k = S^*$ for $i < s^*$ and by $k = i$ for $i > s^*$.

(b) $$s_1 \leqq s^* \leqq S_0 \leqq S^* \leqq S_1.$$

Proof

(a) By (5.3) we have for each $i \in I$,

$$K\delta(k-i) + L(k) - a^* + \sum_{j=0}^{\infty} v^*(k-j)p_j = K\delta(k-i) + J(k), \quad k \geqq i.$$

Recall $\delta(0) = 0$, and $\delta(j) = 1$ for $j > 0$. Let us consider $K\delta(k-i) + J(k)$ for i fixed and $k \geq i$. We distinguish three cases.

Case 1 $i < s^*$. By theorem 5.1(a), 5.1(b) and 5.1(c) we have

$$J(i) \geqq J(s^*-1) \geqq K + J(S^*) = \min_{k>i} \{K\delta(k-i) + J(k)\}.$$

Hence the right side of (5.7) is minimized by $k = S^*$ for $i < s^*$. By theorem 5.1 (b) and (5.1) we have $K + J(S^*) = 0 = v^*(i)$, $i < s^*$. This proves assertion (a) for $i < s^*$.

Case 2 $s^* \leq i \leq S_0$. By theorem 5.1 (c), 5.1 (b), 5.1 (d) and (5.5) we have $K + J(k) \geq K + J(S^*) = 0 \geq J(i) = v^*(i)$ for $k > i$. This proves (a) for $s^* \leq i \leq S_0$.

Case 3 $i > S_0$. Since $L(k)$ is nondecreasing on $[S_0,\infty)$, it follows from theorem 5.1 (f) and (5.5) that $K + J(k) \geq J(i) = v^*(i)$ for $k > i$. This proves (a) for $i > S_0$.

(b) In lemma 4.2 (b) we have already shown that $s_1 \leq s^* \leq S_0$. Since $S^* \geq s^*$ it follows from theorem 5.1 (c) and 5.1 (e) that $S^* \geq S_0$. By theorem 5.1 (c) and 5.1 (f) we have $0 \geq J(S^*) - J(S_0) \geq L(S^*) - L(S_0) - K$. From the definition (2.3) of S_1 it follows now $S^* \leq S_1$.

Theorem 5.3

The (s^*,S^*) policy is optimal among the class $C(M_1,M_2)$ of policies.

Proof

Since $M_1 \leq s_1$ and $M_2 \geq S_1$ we have by theorem 5.2 (b) that $M_1 \leq s^*$ and $M_2 \geq S^*$. It follows now from theorem 5.2 (a) that

$$(5.8) \qquad v^*(i) = \min_{k \in K(i)} \{K\delta(k-i) + L(k) - a^* + \sum_{j=0}^{\infty} v^*(k-j)p_j\}, \qquad i \in I,$$

and that the right side of (5.8) is minimized by $k = S^*$ for $i < s^*$ and by $k = i$ for $i \geq s^*$. Since $v^*(i) = 0$ for $i < s^*$, we have for every $R \in C(M_1,M_2)$,

$$|\mathcal{E}_R(v(\underline{i}_n) | \underline{i}_1 = i)| \leq \sum_{j=s^*}^{\max(i,M_2)} |v^*(j)|, \qquad i \in I.$$

Since by the finiteness of the function $v^*(j)$ we have $v^*(j)/n \to 0$ for $n \to \infty$, $j \in I$, above inequality implies that condition (2.8) from theorem 2.1 is satisfied. Theorem 5.3 follows by applying theorem 2.1.

Remark 5.2. The theorems 2.1 and 5.3 have as an additional consequence that for each initial state $a^* = a(s^*,S^*)$ represents the average expected cost per period when the (s^*,S^*) policy is followed.

Remark 5.3. By the introduction of the natural bounds M_1 and M_2 on the inventory level, it was possible to treat the infinite period stationary model independent of the finite one. Assume now $c \geqq 0$ and consider the class of policies consisting of that policies R from the class $C(-\infty,+\infty)$ for which $\mathcal{E}_R(\underline{i}_{n+1}|\underline{i}_1=i)/n \to 0$ as $n\to\infty$ for each $i \in I$ (roughly speaking, only that policies are considered for which in the long run the average expected order quantity per period equals the expected demand per period). Among the policies from that sub-class there is an optimal (s,S) and every optimal s and S satisfy the bounds given in theorem 5.2. This can be proved by combining theorem 5.2 with the discrete version of the results from the section 2 in [12] and the section 5 in [4].

Remark 5.4: In this remark we shall show that an optimal (s,S) policy also exists under the following weaker assumptions about L(k): (i) there exists a finite integer S_0 for which L(k) assume: its absolute minimum; (ii) there exist finite integers $v_1 \leqq S_0$ and $V_1 \geqq S_0$ such that L(k) is nonincreasing on $[v_1,S_0]$, L(k) is nondecreasing on $[S_0,V_1]$ and both $z_1 = \inf_{k<v_1} L(k)$ and $Z_1 = \inf_{k>V_1} L(k)$ is larger than $L(S_0)+K$. We may assume that S_0 is the largest integer for which (i) holds. Clearly, $s_1 \geqq v_1$ and $S_1 \leqq V_1$, where s_1 and S_1 are as in section 2. Write for convenience $L_1(k) = L(k)$. Define $L_2(k)$ as follows: $L_2(k) = \min(z_1,L_1(k))$ for $k \leqq S_0$ and $L_2(k) = \min(Z_1,L_1(k))$ for $k \geqq S_0$. Denote g(i,R) and a(s,S) (see (2.6) and (4.1)) by $g_1(i,R)$ and $a_1(s,S)$ resp. by $g_2(i,R)$ and $a_2(s,S)$ when $L(k) = L_1(k)$ resp. $L(k) = L_2(k)$. Obviously we have $g_1(i,R) \geqq g_2(i,R)$ for all i,R. The function $L_2(k)$ satisfies the conditions (i) and (ii) from section 2. Let s^* and S^* be any integers for which $a_2(s,S)$ assumes its absolute minimum. By theorem 5.2 we have $g_2(i,R) \geqq a_2(s^*,S^*)$ for all i,R and further $s_1 \leqq s^* \leqq S^* \leqq S_1$. Since $L_1(k) = L_2(k)$ on $[s_1,S_1]$, $s_1 \leqq s^*$ and $S_1 \geqq S^*$, we have by the structure of formula (4.1) that $g_1(i,R) \geqq g_2(i,R) \geqq a_2(s^*,S^*) = a_1(s^*,S^*)$ for all i,R. For the inventory model with $L(k) = L_1(k)$ we have for each initial state that $a_1(s^*,S^*)$

represents the average expected cost per period when the policy (s^*,S^*) is followed (strictly speaking, we have not proved this; for a proof see [1]). Hence under the above weaker assumptions about $L(k)$ there exists also an optimal (s,S) policy. Further, it follows from $\min a_1(s,S) = \min a_2(s,S)$ that every optimal s and S satisfy the bounds given in theorem 5.2.

Remark 5.5. The continuous demand case, in which the distribution function $F(\xi)$ of the random variables $\underline{\xi}_t (t \geq 1)$ has a density $f(\xi)$, can be treated in a quite similar way. The following conditions are now imposed on $L(y)$ (which is defined as the expected holding and shortage cost in period $t+\lambda$ when y is the stock on hand plus on order just after ordering in period t): (i) there exists a finite number S_0 such that $L(y)$ is non-increasing for $y \leq S_0$ and nondecreasing for $y \geq S_0$, (ii) $L(y) > L(S_0)+K$ for $|y|$ sufficient large, (iii) $L(y)$ is defferentiable. We may assume that S_0 is the largest number for which (i) holds. Let s_1 be the smallest number for which $L(s_1) = L(S_0)+K$ and let S_1 be the largest number for which $L(S_1) = L(S_0)+K$.

Define $F^{(n)}(\xi) = P\{\underline{\xi}_1+\ldots+\underline{\xi}_n \leq \xi\}$, $n \geq 1$, and let $M(\xi) = \sum_{n=1}^{\infty} F^{(n)}(\xi)$. Denote the derivate of the renewal function $M(\xi)$ by $m(\xi)$.

Analogous to the discrete demand case the following results can be obtained. There exist finite numbers s^* and S^*, $s^* \leq S^*$, for which

$$a(s,S) = \{L(S) + \int_0^{S-s} L(S-\xi)m(\xi)d\xi+K\}/\{1+M(S-s)\}, \qquad s \leq S,$$

assumes its absolute minimum. The (s^*,S^*) policy is optimal and s^* and S^* satisfy $s_1 \leq s^* \leq S_0 \leq S^* \leq S_1$.

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Errata and addenda to report BW 2/70.

page	line	
3	12	for ... S_0 is the largest ... read ... S_0 is the smallest ...
9	5	for ... minimum. Let ... read ... minimum. If $K = 0$, we take $s^* = S^* = S_0$. Note that for the case $K = 0$ we have that $\min a(s,S) = a(S_0,S_0) = L(S_0) < L(S_0-1)$. Let ...
	15	for This inequality leads ... read By $p_1 > 0$ we have that $m(k) > 0$, $k > 0$. The inequality $b(s^*-1,\Delta^*+1) \geq b(s^*,\Delta^*)$ together with $m(\Delta^*+1) > 0$ lead ...
	19	for leads ... read together with $m(\Delta^*) > 0$ lead ...
	22	replace the lines 22, 23 and 24 by: and $K > 0$. If $K = 0$ we have by the choice $s^* = S_0$ that $s^* \leq S_0$. Consider...
12	19	for ..., $k \in I$. From ... read ..., $k \in I$. By (d) we have that $J(k) \leq 0$ for $s^* \leq k \leq S_0$. From ...
	20	for $k \geq i \geq s^*$ read $k \geq i \geq S_0$
	21	for $\sum_{j=0}^{i-s^*}$ read $\sum_{j=0}^{i-S_0}$
	22	for $F(i-s^*)$ read $F(i-S_0)$
13	2	for $k \geq i \geq s^*$ read $k \geq i \geq S_0$
	3,4	for $\sum_{j=0}^{i-s^*}$ read $\sum_{j=0}^{i-S_0}$
	4	for $F(i-s^*)$ read $F(i-S_0)$ for $F(i-s^*-j)$ read $F(i-S_0-j)$
	7	for $\gamma_{i-s^*} \leq k-i$ read $\gamma_{i-S_0} \leq k-i+S_0-s^*$
14	10	replace line 10, 11, 12 and 13 by: (b) By lemma 4.2(b) and the choice $s^* = S_0$ when $K = 0$ we have that $s_1 \leq s^* \leq S_0$. Assume to the contrary that $S^* < S_0$. Then $L(S_0) < L(S^*)$. By using theorem 5.1(d), 5.1(e) and (5.6) it is now easy to verify that $S^* < S_0$ implies $J(S_0) < J(S^*)$. This contradicts theorem 5.1(c). Thus $S^* \geq S_0$. By theorem 5.1(f) and 5.1(c) we have next that $L(S^*) - L(S_0) - K \leq 0$. Thus $S^* \leq S_1$.

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21 *for* S_0 is the largest ... *read* S_0 is the smallest ...

29 *for* By theorem 5.2 ... *read* If $K = 0$ we take $s^* = S^* = S_0$. By theorem 5.2 ...

16 15 *for* S_0 is the largest ... *read* S_0 is the smallest ...

23 *for* optimal and s^* ... *read* optimal and, if $K > 0$, then s^* ...

Add to page 16:

Remark 5.6. The condition $p_1 > 0$, used in the proof of lemma 4.2(a), is now dropped. We shall prove the following lemma:

Lemma. There exist integers s and S such that $a(s,S) = a^*$ and $L(s-1) \geq a^* \geq L(s)$.

Proof. By lemma 4.1 there exist integers s' and S' such that $a(s',S') = a^*$. When $m(S'-s'+1) = 0$, we have by (4.1) that also $a(s'-1,S') = a^*$. By the proof of lemma 4.1 we have that $a(s,S') > a^*$ for s sufficient small. This proves now that there exist integers s and S such that $a(s,S) = a^*$ and $m(S-s+1) > 0$. By the proof of lemma 4.2(a) we have now shown that the set $T = \{(s,S) | a(s,S) = a^* \leq L(s-1)\}$ is non-empty. Let (s^*,S^*) be a policy from T such that $S^*-s^* \leq S-s$ for all $(s,S) \in T$. We shall show that $a^* \geq L(s^*)$. When $s^* = S^*$ we have trivially $a^* \geq L(s^*)$. Consider now the case $s^* < S^*$. Suppose to the contrary that $L(s^*) > a^*$. By the proof of lemma 4.2(a) we have then $m(S^*-s^*) = 0$. Hence by (4.1) we have that $a(s^*+1,S^*) = a^*$. By $L(s^*) > a^*$ we have now the contradiction $(s^*+1,S^*) \in T$. This ends the proof.

Lemma 4.2 and the theorems 5.1, 5.2 and 5.3 are valid when we choose s^* and S^* as follows. If $K = 0$, we take $s^* = S^* = S_0$. If $K > 0$, we take s^* and S^* such that $a(s^*,S^*) = a^*$ and $L(s^*-1) \geq a^* \geq L(s^*)$.