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THE OPTIMALITY OF (s,S) INVENTORY POLICIES IN THE INFINITE PERIOD MODEL - TOTAL DISCOUNTED COST CRITERION

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The optimality of (s,S) inventory policies in the infinite period model - total discounted cost criterion.

Summary The infinite period stationary inventory model is considered. There is a constant lead time, a nonnegative set-up cost, a linear purchase cost, a holding and shortage function, a discount factor $0 \leq \beta < 1$, and total backlogging of unfilled demand. The optimality criterion is the total expected discounted cost. It is assumed that the negatives of the one period expected holding and shortage costs are unimodal. Under that assumption and a weak assumption*) about the demand distribution, a new proof of the existence of an optimal (s,S) policy is given and further it is shown that any s and S which minimize a quantity depending only on the parameters s and S are optimal for all starting conditions.

1. Introduction

We consider the infinite period stationary inventory model in which demands for a single product in periods 1,2,... are independent, identically distributed random variables. At the beginning of each period an order may be placed for any nonnegative quantity of stock. There is a constant lead time, a fixed set-up cost, a linear purchase cost, a holding and shortage function, a fixed discount factor β, $0 \leq \beta < 1$, and total backlogging of unfilled demand. The optimality criterion is the total expected discounted cost.

Using Scarf's results for the finite period model [5] Iglehart [3] has proved that if the one period holding and shortage costs are convex, then an optimal (s,S) policy exists. Veinott notes in [8] that a modification of Iglehart's proof with the aid of the results of [8] shows that an optimal (s,S) policy also exists under the weaker assumption that the negatives of the one period holding and shortage costs are unimodal. A different proof, based on Howard's policy improvent method, is given in [4]. However that proof seems typically for the discrete demand case.

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*) This assumption will be dropped later (see remark 3.2, p. 9).

Under the assumption that the negatives of the one period holding and shortage costs are unimodal and a weak assumption about the demand distribution we give in this paper a new proof of the existence of an optimal (s,S) policy. The new result of this paper is that any s and S which minimize a quantity depending only on the parameters s and S, are optimal for all starting conditions. Further upper and lower bounds on the optimal values of both s and S are found. By imposing natural bounds on the choice of an ordering decision, our proof will not use any result for the finite period model and it follows immediately from the results of [1,6,7]. We give the proof for the discrete demand case, in which there is a positive probability that the demand in a period equals 1. The proof carries over immediately to the continuous demand case, in which the demand distribution has a positive probability density.

2. Model formulation.

We consider the infinite period stationary model in which demands $\underline{\xi}_1, \underline{\xi}_2, \ldots$ for a single item in periods 1,2,... are independent, nonnegative, discrete random variables with the common probability distribution $p_j = P\{\underline{\xi}_t = j\}$, $(j \geq 0;\ t \geq 1)$. Assume $\mu = \mathcal{E}\underline{\xi}_t < \infty$ and $p_1 > 0$. Only at the beginning of each period the stock on hand plus on order is reviewed. An order may then be placed for any nonnegative, integral quantity of stock. An order placed in period t is delivered at the beginning of period $t+\lambda$, where λ is a known nonnegative integer. The demand takes place at the end of each period. All unsatisfied demand is backlogged and there is no obsolescence of stock.

There is a specified a fixed discount factor β, $0 \leq \beta < 1$, so that a unit cost incurred n periods in future has a present value β^n.

The following costs are considered. In any period the cost of ordering z units is $K\delta(z) + cz$, where $K \geq 0$, $\delta(0) = 0$, and $\delta(z) = 1$ for $z > 0$. Assume that the ordering cost is incurred on the time of delivery of the order. We can always take care that this assumption is satisfied by an appropriate discounting of the ordering cost. Let $g(i)$ be the holding and shortage cost in a period when the amount of stock on hand at the beginning of that period is i just after any additions to stock.

Let $\underline{T}_0 = 0$ and let $\underline{T}_n = \underline{\xi}_1+\ldots+\underline{\xi}_n$, $n \geqq 1$. Define $p_j^{(n)} = P\{\underline{T}_n=j\}$, $(j \geqq 0;\ n \geqq 0)$. Assume that for each integer k

$$(2.1) \qquad L(k) = \sum_{j=0}^{\infty} g(k-j)\ p_j^{(\lambda)} \quad \text{and} \quad G_\beta(k) = L(k) + (1-\beta)c\,k$$

exist and are finite. The function $L(k)$ represents the expected holding and shortage cost in period $t+\lambda$ when k is the stock on hand plus on order just after ordering in period t. The following conditions are imposed on the function $G_\beta(k)$:

(i) There exists a finite integer S_0, such that $G_\beta(i) \leqq G_\beta(j)$ for $j \leqq i \leqq S_0$ and $G_\beta(i) \geqq G_\beta(j)$ for $i \geqq j \geqq S_0$.

(ii) $\lim_{|k|\to\infty} G_\beta(k) > L(S_0) + K$. .

Because of (ii) we may assume that S_0 is the largest integer for which (i) holds. Let s_1 be the smallest integer for which

$$(2.2) \qquad G_\beta(s_1) \leqq G\ (S_0) + (1-\beta p_0)K$$

and let S_1 be the largest integer for which

$$(2.3) \qquad G_\beta(S_1) \leqq G_\beta(S_0) + \beta K\ .$$

Let us define the state of the system in a period as the stock on hand plus on order just before ordering in that period. We take the set I of all integers as the set of all possible states. Every ordering decision is based on the stock on hand plus on order. We say that in state i decision k ($k \geqq i$) is made when k-i units are ordered. We impose the following mild restrictions on the choice of an ordering decision. There are finite integers $M_1 \leqq s_1$ and $M_2 \geqq S_1$ such that nothing is ordered if the stock on hand plus on order $i \geqq M_2$, at most M_2-i units are ordered if $i < M_2$, and at least M_1-i units are ordered if $i < M_1$. Let K(i) be the set of feasible decisions in state i. We have K(i) = $= \{k|\max(i,M_1) \leqq k \leqq M_2\}$ for $i < M_2$, $K(i) = \{i\}$ for $i \geqq M_2$.

Let $C(M_1,M_2)$ be the class of all possible policies for controlling the inventory system considered (see [6] for a precise description).

Given a policy $R \in C(M_1,M_2)$ and an initial state $i \in I$, define $\underline{i}_t$ and $\underline{k}_t$ as the state and the decision in period t ($t \geq 1$). We take as optimality criterion

$$V_\beta(i;R) = \sum_{t=1}^{\infty} \beta^{t-1} \mathcal{E}_R\{K\delta(\underline{k}_t-\underline{i}_t)+(\underline{k}_t-\underline{i}_t)c+L(\underline{k}_t)\,|\,\underline{i}_1=i\} ,$$

where $\mathcal{E}_R$ denotes the expectation under policy R. We note that the expectations and the infinite summations exist, since the cost function $K\delta(k-i) + (k-i)c + L(k)$ is bounded from below. The quantity $V_\beta(i;R)$ represents the total expected discounted costs over the periods $\lambda+1,\lambda+2,\ldots$, all discounted to the beginning of period $\lambda+1$, when the state in period 1 is i and policy R is followed. Observe that the expected discounted cost over the first λ periods is not taken into account. However this is no restriction, since that cost cannot be influenced by any policy.

Using the fact that $\underline{i}_{t+1} = \underline{k}_t-\underline{\xi}_t$, we have (see also [7])

$$\sum_{t=1}^{n} \beta^{t-1}\mathcal{E}_R\{K\delta(\underline{k}_t-\underline{i}_t)+(\underline{k}_t-\underline{i}_t)c+L(\underline{k}_t)\,|\,\underline{i}_1=i\} =$$

$$= \sum_{t=1}^{n} \beta^{t-1} \mathcal{E}_R\{K\delta(\underline{k}_t-\underline{i}_t)+G_\beta(\underline{k}_t)\,|\,\underline{i}_1=1\}-ci+ \beta\mu c \sum_{t=0}^{n-1} \beta^t + \beta^n c\,\mathcal{E}_R(\underline{i}_{n+1}|\underline{i}_1=i).$$

Since $M_1-\mu \leq \mathcal{E}_R(\underline{i}_{n+1}|\underline{i}_1=i) \leq \max(i,M_2)$ and $\beta^n \to 0$ as $n \to \infty$, we see that

$$V_\beta(i;R) = \sum_{t=1}^{\infty} \beta^{t-1} \mathcal{E}_R\{K\delta(\underline{k}_t-\underline{i}_t) + G_\beta(\underline{k}_t)\,|\,\underline{i}_1=i\} - ci + \beta\mu c/(1-\beta) .$$

Since the term $-ci + \beta\mu c/(1-\beta)$ is not affected by the choice of the policy R, we find it convenient to redefine $V_\beta(i;R)$ by setting

$$(2.4) \qquad V_\beta(i;R) = \sum_{t=1}^{\infty} \beta^{t-1} \mathcal{E}_R\{K\delta(\underline{k}_t-\underline{i}_t) + G_\beta(\underline{k}_t)\,|\,\underline{i}_1=i\} .$$

A strategy $R^* \in C(M_1,M_2)$ is called <u>optimal</u> if

$$V_\beta(i;R^*) \leq V_\beta(i;R) \qquad \text{for all } i\in I, \text{ all } R\in C(M_1,M_2) .$$

Theorem 2.1 (Blackwell)

Let $R^* \in C(M_1,M_2)$ and suppose that

$$(2.5) \qquad V_\beta(i;R^*) = \min_{k\in K(i)} \{K\delta(k-i)+G_\beta(k)+\beta \sum_{j=0}^{\infty} V_\beta(k-j;R^*)p_j\} \ , \quad i\in I.$$

Then the policy R^* is optimal among the policies from $C(M_1,M_2)$.

Proof

Fix some integer $i_0 \in I$. Let $M = \max(i_0,M_2)$. Consider the following decision model. We have a system with $I(M) = \{i \mid i\in I, i \leq M\}$ as the set of possible states. At discrete times $t = 1,2,\ldots$ we observe the current state of the system and then one of a number of possible decisions is made. Let $K(i)$ be the set of feasible decisions in state i. If the system is in state i at time t and decision k is made, then two things occur (1) we incur an immediate cost $K\delta(k-i) + G_\beta(k)$ (2) the system moves at time t+1 to state j, $j \leq k$, with probability p_{k-j}. Finally there is specified a discount factor β, $0 \leq \beta \leq 1$. Obviously we have for this model that the total expected discounted cost over $t = 1,2,\ldots$, is given by (2.4), when the initial state is $i (\in I(M))$ and policy $R(\in C(M_1,M_2))$ is followed. The equation (2.5) holds for each $i\in I(M)$. Since $K\delta(k-i) + G_\beta(k)$, $k\in K(i)$, $i\in I(M)$, uniformly bounded in k and i, we can now apply theorem 6(f) in [1] (see also [2]). This theorem tells us that $V_\beta(i;R^*) \leq V_\beta(i;R)$ for all $i\in I(M)$, all $R\in C(M_1,M_2)$. Hence in particular we have found $V_\beta(i_0;R^*) \leq V_\beta(i_0;R)$ for all $R\in C(M_1,M_2)$. This proofs the theorem, since i_0 was chosen arbitrarily.

3. The optimality of an (s,S) policy and bounds on the optimal s and S.

Define

$$(3.1) \qquad m_\beta(j) = \sum_{n=1}^{\infty} \beta^n p_j^{(n)} \quad \text{and} \quad M_\beta(j) = \sum_{k=0}^{j} m_\beta(k) \ , \quad j = 0,1,\ldots \ .$$

The function $M_\beta(j)$ is the renewal function of the defective probability distribution $\{\beta p_j, j \geqq 0\}$ with defect $1-\beta$. Obviously we have $M_\beta(j) \leqq \beta/(1-\beta)$, $j \geqq 0$, so $m_\beta(j) \to 0$ as $j \to \infty$. By $p_j^{(n)} = p_0^{(n-1)} p_j + \ldots + p_j^{(n-1)} p_0$ ($j \geqq 0$; $n \geqq 2$), we have

$$(3.2) \qquad m_\beta(j) = \beta p_j + \beta \sum_{k=0}^{j} m_\beta(k) p_{j-k}, \qquad j \geqq 0.$$

Using only the fact that $G_\beta(k)$ is bounded from below, it is shown in [7] that for an (s,S) policy (order S-i units, when the stock on hand plus on order $i < s$; order nothing, when $i \geqq s$) we have that

$$(2.3) \qquad V_\beta(i;(s,S)) = \begin{cases} a_\beta(s,S)/(1-\beta), & i < s \\ G_\beta(i) + \sum_{j=0}^{i-s} G_\beta(i-j)\, m_\beta(j) + \\ \quad + \{a_\beta(s,S)/(1-\beta)\}\{\beta-(1-\beta)M_\beta(i-s)\}, & i \geqq s, \end{cases}$$

where

$$(3.4) \qquad a_\beta(s,S) = \{G_\beta(S) + \sum_{j=0}^{S-s} G_\beta(S-j)\, m_\beta(j) + K\}/\{1+M_\beta(S-s)\}.$$

Consider now the function $a_\beta(s,S)$, $s \leqq S$, $s,S \in I$. The function $a_1(s,S)$ has been extensively examined in [6]. It is not difficult to verify that the function $a_\beta(s,S)$, where β fixed and $0 \leqq \beta < 1$, can be treated in a quite similar way. The lemmas 4.1 and 4.2 in [6] remain true when we replace $L(j)$, $a(s,S)$ and a^* by $G_\beta(j)$, $a_\beta(s,S)$ and $\min a_\beta(s,S)$ respectively (use in the proof of lemma 4.1 that $m_\beta(j) \to 0$ as $j \to \infty$).

From now on s^* and S^* are fixed and such that $a_\beta(s,S)$ assumes its absolute minimum for $s = s^*$ and $S = S^*$.
Define (see (5.1) in [6])

$$(3.5) \qquad v_\beta^*(i) = \begin{cases} 0 & i < s^*, \\ G_\beta(i) - a_\beta(s^*,S^*) + \beta \sum_{j=0}^{i-s^*} v_\beta^*(i-j)\, p_j, & i \geqq s^*. \end{cases}$$

The function $v_\beta^*(i)$, $i \in I$, is uniquely determined by (3.5). Iterating the renewal equation (3.5) and using $p_j^{(n)} = \sum_k p_k^{(n-1)} p_{j-k}$ $(j \geq 0; n \geq 1)$, yields

$$(3.6) \quad v_\beta^*(i) = \begin{cases} 0 \,, & i < s^*, \\ G_\beta(i) + \sum_{j=0}^{i-s^*} G_\beta(i-j)m_\beta(j) - a_\beta(s^*,S^*)\{1+M(i-s^*)\}, & i \geq s^*. \end{cases}$$

It is not difficult to verify that the theorems 5.1 and 5.2 in [6] remain true when we replace $L(j)$, p_j and $v^*(j)$ by $G_\beta(j)$, βp_j and $v_\beta^*(j)$ respectively. Only the inequality $S^* \leq S_1$, where S_1 is defined by (2.3), needs some comment. From the proof of theorem 5.1(f) in [6] it follows immediately that for the discounted model considered theorem 5.1(f) can be sharpened to $J(k) - J(i) \geq G_\beta(k) - G_\beta(i) - \beta K$ for $k \geq i \geq S_0$. The proof of theorem 5.2(b) in [6] implies now directly that $S^* \leq S_1$. Hence by theorem 5.2 in [6] we have the following theorem.

Theorem 3.1

$$(3.7) \quad v_\beta^*(i) = \min_{k \geq i}\{K\delta(k-i)+G_\beta(k)-a_\beta(s^*,S^*)+\beta\sum_{j=0}^{\infty} v_\beta^*(k-j)p_j\}\,, \qquad i \in I.$$

The right side of (3.7) is minimized by $k = S^*$ for $i < s^*$ and by $k = i$ for $i \geq s^*$. Further s^* and S^* satisfy

$$(3.8) \qquad s_1 \leq s^* \leq S_0 \leq S^* \leq S_1 \,.$$

From (3.3) and (3.6) it follows immediately that

$$(3.9) \qquad v_\beta^*(i) = V_\beta(i;(s^*,S^*)) - a_\beta(s^*,S^*)/(1-\beta) \qquad \text{for all } i \in I.$$

Substituting (3.9) in (3.7) yields

$$(3.10) \quad V_\beta(i;(s^*,S^*)) = \min_{k \geq i}\{K\delta(k-i)+G_\beta(k)+\beta\sum_{j=0}^{\infty} V_\beta(k-j;(s^*,S^*))p_j\}, \quad i \in I,$$

where the right side of (3.10) is minimized by $k = S^*$ for $i < s^*$ and by $k = i$ for $i \geq s^*$. Since $S^* \leq S_1 \leq M_2$ and $s^* \geq s_1 \geq M_1$, we have that $V_\beta(i;(s^*,S^*))$, $i \in I$, also satisfies the optimality equation (2.5). Hence by theorem 2.1 the (s^*,S^*) policy is optimal. So we have proved that any s and S for which $a_\beta(s,S)$ assumes its absolute minimum, are optimal for all initial states and satisfy (3.8)

Remark 3.1. Suppose now that the demand distribution $F(\xi)$ of the demand variables $\{\underline{\xi}_t, t \geq 1\}$ has a positive probability density $f(\xi)$. Define now $F^{(n)}(\xi) = P\{\underline{\xi}_1 + \ldots + \underline{\xi}_n \leq \xi\}$, $n \geq 1$ and (cf. (2.1))

$$G_\beta(y) = \begin{cases} g(y) + (1-\beta)\, cy & \text{if } \lambda = 0, \\ \int_0^\infty g(y-\xi)\, f^{(\lambda)}(\xi)\, d\xi + (1-\beta)\, cy & \text{if } \lambda \geq 1, \end{cases}$$

where $f^{(\lambda)}(\xi)$ is the density of $F^{(\lambda)}(\xi)$. The following assumptions are made about the function $G_\beta(y)$: (i) there exists a finite number S_0 such that $G_\beta(y)$ is nonincreasing for $y \leq S_0$ and nondecreasing for $y \geq S_0$, (ii) $G_\beta(y) > G_\beta(S_0) + K$ for $|y|$ sufficient large, (iii) $G_\beta(y)$ is differentiable. Assume that S_0 is the largest number for which (i) holds. Let s_1 be the smallest number for which $G_\beta(s_1) \leq G_\beta(S_0) + K$ and let S_1 be the largest number for which $G_\beta(S_1) \leq G_\beta(S_0) + \beta K$. Define $M_\beta(\xi) = \sum_{n=1}^{\infty} \beta^n F^{(n)}(\xi)$. The derivate $m_\beta(\xi)$ of $M_\beta(\xi)$ satisfies

$$m_\beta(\xi) = \beta f(\xi) + \beta \int_0^\xi f(\xi-\eta)\, m(\eta)\, d\eta\ , \qquad \xi \geq 0,$$

Analogous to the discrete demand case the following result can be obtained. Any numbers s^* and S^* for which

$$a_\beta(s,S) = \{G_\beta(S) + \int_0^{S-s} G_\beta(S-\xi) m_\beta(\xi) + K\}/\{1+M_\beta(S-s)\}\ , \qquad s \geq S,$$

assumes its absolute minimum, satisfy $s_1 \leq s^* \leq S_0 \leq S^* \leq S_1$ and the (s^*,S^*) policy is optimal for all initial states.

Remark 3.2. When $p_1 = 0$ it is not difficult to show that integers s^{**} and S^{**} exist for which $a_\beta(s,S)$ attains its absolute minimum and which satisfy $G_\beta(s^{**}-1) \geq \min a_\beta(s,S) \geq G_\beta(s^{**})$. Lemma 4.2 and the theorems 5.1 and 5.2 in [6] are valid when $s^* = s^{**}$ and $S^* = S^{**}$. Consequently the (s^{**},S^{**}) policy is optimal for all starting conditions and s^{**} and S^{**} satisfy $s_1 \leq s^{**} \leq S_0 \leq S^{**} \leq S_1$. Such a policy is found by choosing from $\{(s^*,S^*) | G_\beta(s^*-1) \geq a_\beta(s^*,S^*) = \min a_\beta(s,S)\}$ a policy for which $S^* - s^*$ is minimal. (see remark 3.6 in [6]).

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