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SOME RESULTS FOR THE DYNAMIC (s,S) INVENTORY MODEL

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Some results for the dynamic (s,S) inventory model *)

by H.C. Tijms

Summary. The periodic review, single item, stationary (s,S) inventory model is considered. There is a fixed lead time, a linear purchase cost, a fixed set-up cost, a holding and shortage cost function, a discount factor $0 < \alpha \leq 1$ and backlogging of unfilled demand. The solution for the total expected discounted cost for the finite period (s,S) model is found. In addition the time dependent behaviour of the inventory process is found. Further a limit theorem is given, which relates the total expected cost in the finite period (s,S) model with no discounting to the average expected cost per period for the infinite period (s,S) model. As a by-product we obtain known results for the infinite period (s,S) model.

1. *Introduction.*

We consider the dynamic, stationary (s,S) inventory model in which the demands $\underline{\xi}_1, \underline{\xi}_2, \ldots$ for a single item in periods 1,2,... are independent, non-negative, discrete random variables**) with the common probability distribution $\phi(j) = P\{\underline{\xi}_t = j\}$, $(j \geq 0;\ t \geq 1)$.***) It is assumed that $\mu = \mathcal{E}\,\underline{\xi}_t$ is finite and positive.

The stock level is reviewed at the beginning of each period and only then an ordering decision may be made. We shall assume initially that the lead time of an order is zero. If, at review, the stock level is below s, we order up to the level S, i.e., S-i units are ordered. If, at review, the stock level $i \geq s$, then no ordering is done. The numbers s and S are given integers with $s \leq S$. We assume that excess demands are backlogged. Hence the stock level may take on negative values.

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*) Report BW 8/71 of the Operational Research Department of the Mathematical Centre, Amsterdam.

**) Random variables are underlined.

***) The proofs and the results of this paper can be adapted to any general demand distribution.

The cost of ordering z units is $K\delta(z) + cz$, where $K > 0$, $\delta(0) = 0$, and $\delta(z) = 1$ for $z > 0$. Let $L(k)$ be the holding and shortage costs in a period, where k is the amount of stock just after any additions to stock in that period. Finally, there is specified a fixed discount factor α, $0 < \alpha \leq 1$, with the interpretation that a unit of value t periods hence has a present value of α^t.

In the finite period model it is assumed that stock left over at the end of the final period can be salvaged with a return of d per unit. Similarly, any backlogged demand remaining at the end of the final period can be satisfied by a unit cost of d per unit.

The mathematical techniques of this paper are based mainly on renewal theory. Therefore we discuss in section 2 a number of known results in renewal theory. In section 3 the finite period (s,S) model with no discounting ($\alpha = 1$) is treated. Let $f_n(i)$ be the total expected cost over the periods 1,...,n in the n-period (s,S) model with no discounting, where i is the stock just before ordering in period 1. A formula for $f_n(i)$ is found. In addition we find in a simpler way than in [3] the probability distribution of the stock level at the time of review for all periods. In section 4 we determine the Cesarolimit of the sequence $\{f_n(i)-ng\}$, $n \geq 1$, for any i, where g represents the average expected cost per period in the infinite period model. A sufficient condition is given under which the sequence $\{f_n(i)-ng\}$ is convergent for any i. As a by-product we obtain the known stationary probability distribution of the stock level at the time of review [3,4,6,8]. Further, in section 4 a question of IGLEHART [5] is answered partially. Section 5 is devoted to the (s,S) model with discounting ($\alpha < 1$). The solution for the total expected discounted cost for the finite period (s,S) model is found. As a direct corollary we obtain the known solution for the total expected discounted cost for the infinite period (s,S) model [8]. In section 6 we indicate the modifications of the results in the case of a non-zero lead time.

2. *Preliminaries.*

In this section we give a number of known results in renewal theory that will be needed in the analysis that follows.

Let

$$\phi^{(t)}(j) = P\{\underline{\xi}_1+\ldots+\underline{\xi}_t=j\} \quad \text{and} \quad \Phi^{(t)}(j) = P\{\underline{\xi}_1+\ldots+\underline{\xi}_t\leq j\},\ j \geq 0;\ t \geq 1.$$

When $t = 1$, we often drop the superscript. Define for convenience $\phi^{(0)}(0) = 1$, $\phi^{(0)}(j) = 0$ for $j \geq 1$, and $\Phi^{(0)}(j) = 1$ for $j \geq 0$. The convolutionformula

$$\phi^{(t)}(j) = \sum_{k=0}^{j} \phi^{(t-1)}(k)\ \phi(j-k), \qquad j \geq 0;\ t \geq 1 \tag{2.1}$$

is well known [1].

Observe that by the assumption $\mu = \mathcal{E}\underline{\xi}_t > 0$, we have $\phi(0) < 1$. Let

$$m(j) = \sum_{t=1}^{\infty} \phi^{(t)}(j) \quad \text{and} \quad M(j) = \sum_{t=1}^{\infty} \Phi^{(t)}(j), \qquad j \geq 0 \tag{2.2}$$

Clearly, $M(j) = m(0) + \ldots + m(j)$, $j \geq 0$. We note that $M(j)$ can be interpreted as the expected number of periods before the cumulative demand exceeds j. We have from (2.1) and (2.2) that the numbers $m(j)$ can be computed successively from

$$m(j) = \phi(j) + \sum_{k=0}^{j} \phi(j-k)\ m(k), \qquad j \geq 0 \tag{2.3}$$

A direct consequence of the proof of theorem 1 on p. 183 in [2] is the following lemma (see also [8, p.546]).

Lemma 2.1. The renewal function $M(j)$ is finite. For every $j \geq 0$ holds that $\Phi^{(t)}(j)$ and $\Phi^{(1)}(j) + \ldots + \Phi^{(t)}(j)$ converge exponentially fast to 0 and $M(j)$ as $t \to \infty$.

Let $\{b_n\}$, $n \geq 0$, be a given sequence of finite numbers. Consider the discrete renewal equation

$$u_n = b_n + \sum_{k=0}^{n} u_{n-k}\ \phi(k), \qquad n \geq 0 \tag{2.4}$$

This discrete renewal equation has a unique solution $\{u_n\}$, since the numbers u_n can be computed successively from (2.4). Iterating (2.4) and using (2.1) and the fact that $\phi^{(t)}(j) \to 0$ as $t \to \infty$ for any j, we obtain the known result [2]

$$u_n = b_n + \sum_{k=0}^{n} b_{n-k}\, m(k), \qquad n \geq 0 \tag{2.5}$$

Let

$$\hat{m}(j) = \sum_{t=1}^{\infty} t\phi^{(t)}(j) \quad \text{and} \quad \hat{M}(j) = \sum_{t=1}^{\infty} t\Phi^{(t)}(j), \quad j \geq 0 \tag{2.6}$$

Clearly, $\hat{M}(j) = \hat{m}(0) + \ldots + \hat{m}(j)$, $j \geq 0$. The numbers $\hat{m}(j)$ can be calculated explicitly from

$$\hat{m}(j) = m(j) + \sum_{k=0}^{j} m(j-k)\, m(k), \qquad j \geq 0.$$

This relation can be proved as follows. Using (2.1) and (2.3), we obtain $\hat{m}(j) = m(j) + \{\hat{m}(j)\phi(0) + \ldots + \hat{m}(0)\phi(j)\}$, $j \geq 0$. This equation is a renewal equation as given by (2.4).

For any $i \geq s$, let

$$\rho_i(k) = \begin{cases} 0, & k = 0 \\ \Phi^{(k-1)}(i-s) - \Phi^{(k)}(i-s), & k \geq 1 \end{cases} \tag{2.7}$$

We note that $\rho_i(k)$ can be interpreted as the probability that the cumulative demand will first exceed i-s during the kth period. For any $i \geq s$, we have

$$\sum_{k=0}^{n} \rho_i(k) = 1 - \Phi^{(n)}(i-s)\ , \qquad n \geq 1$$

and (2.8)

$$\sum_{k=0}^{n} k\rho_i(k) = 1 + \sum_{k=1}^{n-1} \Phi^{(k)}(i-s) - n\Phi^{(n)}(i-s), \qquad n \geq 1,$$

where we adopt the convention $\sum_a^b = 0$ if $a > b$. Using lemma 2.1, we have for any $i \geq s$ that

$$\sum_{k=0}^{\infty} \rho_i(k) = 1, \quad \text{and} \quad \sum_{k=0}^{\infty} k\rho_i(k) = 1 + M(i-s) \tag{2.9}$$

Hence we have for any $i \geq s$ that $\{\rho_i(k)\}$, $k \geq 0$, constitutes a probability distribution with a finite, positive first moment.

Put for abbreviation

$$\rho(j) = \rho_{S-s}(j), \qquad j \geq 0 \tag{2.10}$$

Let $\rho^{(1)}(j) = \rho(j)$, $j \geq 0$, and let

$$\rho^{(t)}(j) = \sum_{k=0}^{j} \rho^{(t-1)}(k)\, \rho(j-k), \qquad j \geq 0;\ t \geq 2 \tag{2.11}$$

Define

$$r(j) = \sum_{t=1}^{\infty} \rho^{(t)}(j), \qquad j \geq 0 \tag{2.12}$$

Observe that $r(0) = 0$. The numbers $r(j)$ can be computed successively from

$$r(j) = \rho(j) + \sum_{k=0}^{j} \rho(j-k)\, r(k), \qquad j \geq 0 \tag{2.13}$$

When $\underline{\xi}_t$ has a geometric distribution, then we can evaluate the $m(j)$, $\rho_i(j)$ and $r(j)$ explicitly. Consider now the special case

$$\phi(j) = pq^{j-1}, \qquad j \geq 1,$$

where $0 < p \leq 1$ and $q = 1-p$. It is known that $\underline{\xi}_1 + \ldots + \underline{\xi}_t$ has then a negative binomial distribution [1]. Moreover, we have

$$\Phi^{(k)}(m) - \Phi^{(k+1)}(m) = \binom{m}{k} p^k q^{m-k}, \qquad k \geq 0;\ m \geq 0 \tag{2.14}$$

where we adopt the convention $\binom{m}{k} = 0$ if $k > m$. The relation (2.14) can be proved by the following probabilistic argument. In a sequence of Bernoulli trials with the probability of success p we have that $\phi(j) = pq^{j-1}$ is the probability that the first success occurs at the jth trial. Hence $\Phi^{(k)}(m)$ is the probability that at least k successes occur in m Bernoulli trials. Consequently, $\Phi^{(k)}(m) - \Phi^{(k+1)}(m)$ is the probability that exactly k successes occur in m Bernoulli trials. This interpretation proves (2.14). By (2.14) we have found the $\rho_i(k)$ explicitly. By using the generating function approach we can evaluate the m(j) and the r(j) explicitly. We have

$$m(0) = 0, \quad m(j) = p \quad \text{for } j \geq 1 \tag{2.15}$$

and

$$r(j) = \sum_{k=1}^{j} \binom{k\Delta}{j-k} (p/q)^{j-k} q^{k\Delta}, \qquad j \geq 1, \tag{2.16}$$

where $\Delta = S-s$. We prove only (2.16). The known result (2.15) can be proved in an analogous way. Define the power-series
$V(x) = \rho(1)x + \rho(2)x^2 + \ldots$ and $R(x) = r(1)x + r(2)x^2 + \ldots, |x| < 1$.
We have from (2.7), (2.10) and (2.14) that $V(x) = x(px+q)^{\Delta}$, $|x| < 1$.
We have by (2.13) that $R(x) = V(x) + R(x)V(x)$, $|x| < 1$, and hence

$$R(x) = \frac{V(x)}{1-V(x)} = \sum_{k=1}^{\infty} x^k (px+q)^{k\Delta} = \sum_{k=1}^{\infty} x^k \sum_{m=1}^{\infty} \binom{k\Delta}{m} (px)^m q^{k\Delta-m} .$$

Hence the coefficient of x^j in R(x) is given by (2.16).

The following lemma is well known.

Lemma 2.2. If the sequence $\{a_n\}$, $n \geq 0$, is convergent, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} a_k = \lim_{n\to\infty} a_n .$$

Lemma 2.3. Let $\{a_n\}$, $n \geq 0$, and $\{b_n\}$, $n \geq 0$, be two sequences such that $a_n \geq 0$ and $\sum a_n < \infty$. Suppose b is a finite number. Let the sequence $\{c_n\}$, $n \geq 0$, be defined by $c_n = a_0 b_n + \ldots + a_n b_0$, $n \geq 0$.

(a) If $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} b_k = b$, then $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} c_k = b \sum_{j=0}^{\infty} a_j$.

(b) If $\lim_{n\to\infty} b_n = b$, then $\lim_{n\to\infty} c_n = b \sum_{j=0}^{\infty} a_j$.

Proof. (a) Since the sequence $\{(b_0+\ldots+b_n)/n\}$, $n \geq 1$, has the finite limit b, this sequence is bounded by some positive number N. Let $b_n = 0$ for $n \leq -1$. We can then write

$$\frac{1}{n}\sum_{k=0}^{n} c_k = \frac{1}{n}\sum_{k=0}^{n}\sum_{j=0}^{\infty} a_j\, b_{k-j} = \sum_{j=0}^{\infty} a_j \frac{1}{n}\sum_{k=0}^{n} b_{k-j}, \qquad n \geq 1.$$

Since for any fixed $j \geq 0$ the sequence $\{(b_{-j}+\ldots+b_{n-j})/n\}$, $n \geq 1$, is bounded by N and has limit b, an application of the Lebesque dominated convergence theorem [2, p. 109] yields (a).
(b) The proof of (b) is analogous to that of (a).

We note that this lemma remains valid when we replace the condition $a_n \geq 0$, $\sum a_n < \infty$ by the condition $\sum |a_n| < \infty$.

A proof of the following important theorem can be found in [1, p. 290].

Theorem 2.1. Let $\{a_n\}$, $n \geq 0$, be a sequence such that $a_n \geq 0$, $\sum a_n = 1$, and $0 < \sum na_n < \infty$. Let $\{b_n\}$, $n \geq 0$, be a sequence such that $\sum |b_n| < \infty$.*) Let the sequence $\{u_n\}$, $n \geq 0$, be defined by the recursive relation $u_n = b_n + (a_0 u_n + \ldots + a_n u_0)$, $n \geq 0$.

(a) $$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n} u_k = \sum_{n=0}^{\infty} b_n \Big/ \sum_{n=0}^{\infty} na_n .$$

(b) If the greatest common divisor of the indices n, where $a_n > 0$, is 1, then the sequence $\{u_n\}$, $n \geq 0$, is convergent.

3. *The total expected cost in the n-period model with no discounting.*

In this section the future costs are not discounted, i.e. $\alpha = 1$. A formula will be found for the total expected cost for the finite period (s,S) inventory model.

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*) Actually it is assumed in [1] that $b_n \geq 0$ and $\sum b_n < \infty$. However this condition may be replaced by $\sum |b_n| < \infty$.

Denote by $\underline{x}_t$ and $\underline{y}_t$ the stock level just before ordering and the stock level just after ordering in period t. We note that the stochastic processes $\{\underline{x}_t\}$ and $\{\underline{y}_t\}$ are Markov chains. Clearly $\underline{y}_t = S$ if $\underline{x}_t < s$, and $\underline{y}_t = \underline{x}_t$ if $\underline{x}_t \geq s$. Furthermore, we have

$$\underline{x}_{t+1} = \underline{y}_t - \underline{\xi}_t \ , \qquad t \geq 1 \qquad (3.1)$$

In the n-period (s,S) model the total expected cost $f_n(i)$ is given by

$$f_n(i) = \sum_{t=1}^{n} \mathcal{E}\{K\delta(\underline{y}_t-\underline{x}_t)+(\underline{y}_t-\underline{x}_t)c+L(\underline{y}_t)\,|\,\underline{x}_1=i\}-d\,\mathcal{E}(\underline{x}_{n+1}\,|\,\underline{x}_1=i) \qquad (3.2)$$

Note that by $s \leq \underline{y}_t \leq \max(\underline{x}_1,S)$, $t \geq 1$, and (3.1) the expectations exist and are finite. Using (3.1), we can write (3.2) as (see also [8])

$$f_n(i) = \sum_{t=1}^{n} \mathcal{E}\{K\delta(\underline{y}_t-\underline{x}_t)+L(\underline{y}_t)\,|\,\underline{x}_1=i\}-(d-c)\,\mathcal{E}(\underline{x}_{n+1}\,|\,\underline{x}_1=i)-ci+nc\mu \qquad (3.3)$$

For any i, let

$$f_n^*(i) = \sum_{t=1}^{n} \mathcal{E}\{K\delta(\underline{y}_t-\underline{x}_t) + L(\underline{y}_t)\,|\,\underline{x}_1=i\} \ , \qquad n \geq 1 \qquad (3.4)$$

Define for convenience $f_0^*(i) = 0$ for any i. If $\underline{x}_1 = i < s$, then $\underline{y}_1 = S$, and if $\underline{x}_1 = i \geq s$, then $\underline{y}_1 = i$. Hence

$$f_n(i) = K + (S-i)c + f_n(S), \qquad i < s;\ n \geq 1 \qquad (3.5)$$

and

$$f_n^*(i) = K + f_n^*(S), \qquad i < s;\ n \geq 1 \qquad (3.6)$$

When $\underline{x}_1 = i \geq s$, then the probability that period $t = k+1$ is the first period for which $\underline{x}_t < s$ equals $\rho_i(k)$, where $\rho_i(k)$ is defined by (2.7). Using a standard argument from renewal theory, it follows that

$$f_n^*(i) = L(i)+\sum_{k=1}^{n-1}\sum_{j=0}^{i-s} L(i-j)\phi^{(k)}(j)+\sum_{k=1}^{n-1}\{K+f_{n-k}^*(S)\}\rho_i(k), \quad n\geq 1;\ i\geq s \qquad (3.7)$$

Let

$$g = \{L(S) + \sum_{k=0}^{S-s} L(S-k)m(k) + K\} \; / \; \{1 + M(S-s)\} + c\mu \tag{3.8}$$

It is known [4,5,6,8] that g represents for each initial stock the average expected cost per period in the infinite period model.

Let

$$g^* = g - c\mu, \tag{3.9}$$

and for any i, let

$$g_n^*(i) = f_n^*(i) - ng^*, \qquad n \geq 0 \tag{3.10}$$

From (3.10), (3.8), (3.7) and (2.8) it follows after some straightforward calculations that

$$g_n^*(i) = b_n^*(i) + \sum_{k=0}^{n} g_{n-k}^*(S)\, \rho_i(k), \qquad i \geq s;\ n \geq 1, \tag{3.11}$$

where

$$b_n^*(i) = L(i)+\sum_{k=1}^{n-1}\sum_{j=0}^{i-s} L(i-j)\phi^{(k)}(j)-g^*\{1+\sum_{k=1}^{n-1} \Phi^{(k)}(i-s)\}+K\{1-\Phi^{(n-1)}(i-s)\} \tag{3.12}$$

$$(i \geq s;\ n \geq 1)$$

Define for convenience $b_0^*(i) = 0$, $i \geq s$. We have by (3.11) that

$$g_n^*(S) = b_n^*(S) + \sum_{k=0}^{n} g_{n-k}^*(S)\, \rho(k), \qquad n \geq 0 \tag{3.13}$$

This renewal equation has the unique solution (c.f. section 2)

$$g_n^*(S) = b_n^*(S) + \sum_{k=0}^{n} b_{n-k}^*(S)\, r(k), \qquad n \geq 0 \tag{3.14}$$

The relations (3.6), (3.10), (3.11), and (3.14) in conjunction yield a formula for $f_n^*(i)$. From (3.3) and (3.4) it follows that the solution for $f_n(i)$ is obtained by determining $E(\underline{x}_{n+1}|\underline{x}_1=i)$. From (3.1) we have for any i that

$$\mathcal{E}(\underline{x}_{n+1}|\underline{x}_1=i) = \mathcal{E}(\underline{y}_n|\underline{x}_1=i) - c\mu\ , \qquad n \geq 1 \qquad (3.15)$$

For any i, j, let

$$p_{ij}^{(n)} = P\{\underline{y}_{n+1} = j|\underline{x}_1=i\}\ , \qquad n \geq 0.$$

For any i, we have

$$p_{ij}^{(n)} = 0, \qquad j\notin[s,\max(i,S)];\ n \geq 0 \qquad (3.16)$$

Furthermore, we have

$$p_{ij}^{(n)} = p_{Sj}^{(n)} \qquad i < s;\ n \geq 0 \qquad (3.17)$$

Using a standard argument from renewal theory, we have for $n \geq 0$ that

$$p_{ij}^{(n)} = \phi^{(n)}(i-j) + \sum_{k=0}^{n} p_{Sj}^{(n-k)}\rho_i(k), s \leq j \leq \max(i,S);\ i \geq s, \qquad (3.18)$$

where $\phi^{(n)}(k) = 0$ for $k \leq -1$; $n \geq 0$. We have in particular

$$p_{Sj}^{(n)} = \phi^{(n)}(S-j) + \sum_{k=0}^{n} p_{Sj}^{(n-k)}\rho(k), \qquad s \leq j \leq S;\ n \geq 0 \qquad (3.19)$$

For any $j\epsilon[s,S]$ the equation (3.19) constitutes a renewal equation, and hence (c.f. section 2)

$$p_{Sj}^{(n)} = \phi^{(n)}(S-j) + \sum_{k=0}^{n} \phi^{(n-k)}(S-j)r(k), \qquad s \leq j \leq S;\ n \geq 0 \qquad (3.20)$$

The relations (3.16), (3.17), (3.18), and (3.20) in conjunction yield the probability distribution of $\underline{y}_{n+1}$. Observe that by (3.1) the probability distribution of $\underline{x}_{n+1}$ follows from that of $\underline{y}_n$.

We note that in [3] the probability distribution of $\underline{x}_n$ has been found by another approach, which is less simple than our approach.

It is interesting to note that if $s > 0$, then $p_{ij}^{(t)}$ represents also in the lost sales model with zero lead time the probability that just after ordering in period t+1 the stock level is j, where i ($i \geq 0$) is the initial stock.

Combining (3.3), (3.4), (3.9), (3.10), (3.11), and (3.14), we obtain the following theorem.

Theorem 3.1. For any $n \geq 1$ holds

$$f_n(i) = ng + b_n^*(S) + \sum_{k=0}^{n} b_{n-k}^*(S)r(k) + K - (d-c)\left(\sum_{j=s}^{S} jp_{Sj}^{(n-1)} - \mu\right) - ci, \quad i < s$$

and

$$f_n(i) = ng + b_n^*(i) + \sum_{k=0}^{n} \{b_{n-k}^*(S) + \sum_{j=0}^{n-k} b_{n-k-j}^*(S)r(j)\}\, \rho_i(k) +$$

$$- (d-c)\left(\sum_{j=s}^{\max(i,S)} jp_{ij}^{(n-1)} - \mu\right) - ci, \qquad i \geq s.$$

Corollary. Consider the special case $s = S = \bar{x}$. We note that such an (s,S) policy is frequently used when the set-up cost K is zero. We have then $g = L(\bar{x}) + K(1-\phi(0)) + c\mu$, $\rho(k) = \{\phi(0)\}^{k-1}\{1-\phi(0)\}$, $k \geq 1$, and $b_k^*(\bar{x}) = -K\rho(k)$, $k \geq 1$. Since $\{\rho(k)\}$ is a geometric probability distribution, we have $r(j) = 1 - \phi(0)$, $j \geq 1$ (c.f. (2.15)). Furthermore, we have for any $n \geq 0$ that $p_{ij}^{(n)} = 1$ for $j = \bar{x}$, $i \leq \bar{x}$, and $p_{ij}^{(n)} = \phi^{(n)}(i-j)$ for $\bar{x} < j \leq i$, $i > \bar{x}$. It is now straightforward to verify that

$$f_n(i) = n\{L(\bar{x}) + K(1-\phi(0)) + c\mu\} + K\phi(0) - (d-c)(\bar{x}-\mu) - ci, \qquad i < \bar{x}$$

and

$$f_n(i) = n\{L(\bar{x}) + K(1-\phi(0)) + c\mu\} + L(i) + \sum_{k=1}^{n-1} \sum_{j=0}^{i-\bar{x}} L(i-j)\phi^{(k)}(j) +$$

$$- \{L(\bar{x}) + K(1-\phi(0))\}\{1 + \sum_{k=1}^{n-1} \Phi^{(k)}(i-\bar{x})\} + K\phi(0)\{1 - \Phi^{(n-1)}(i-\bar{x})\} +$$

$$- (d-c)\left(\sum_{j=\bar{x}+1}^{i} j\phi^{(n-1)}(i-j) + \bar{x}\,\{1 - \sum_{j=\bar{x}+1}^{i} \phi^{(n-1)}(i-j)\} - \mu\right) - ci,$$

$$i \geq \bar{x}.$$

A direct consequence of theorem 3.1 is the known result that $f_n(i)/n \to g$ as $n \to \infty$ for any i, i.e. the average expected cost per period in the infinite period model is g irrespective of the initial stock.

4. *Cesarolimit of $f_n(i) - ng$.*

In this section we find the Cesarolimit of the sequence $\{f_n(i) - ng\}$, $n \geq 1$, for any i. A sufficient condition will be given under which the sequence $\{f_n(i) - ng\}$ is convergent for any i. As a by-product we find the known stationary probability distribution of the Markov chain $\{\underline{y}_t\}$. Further a question of IGLEHART will be answered partially.

From (3.12) and lemma 2.1, we obtain

$$\lim_{n\to\infty} b_n^*(i) = v^*(i), \qquad i \geq s, \quad (4.1)$$

where

$$v^*(i) = L(i) + \sum_{j=0}^{i-s} L(i-j)m(j) + K - g^*\{1+M(i-s)\}, \qquad i \geq s \quad (4.2)$$

From (3.8), (3.9) and (4.2) we have that $v^*(S) = 0$. Furthermore, we have by lemma 2.1 that the sequence $\{b_n^*(S)\}$ converges exponentially fast to $v^*(S) = 0$, and hence

$$\sum_{n=0}^{\infty} |b_n^*(S)| < \infty \quad (4.3)$$

Using lemma 2.1, we obtain after some straightforward calculations

$$\sum_{n=0}^{\infty} b_n^*(S) = \lim_{N\to\infty} \sum_{n=0}^{N} b_n^*(S) = \\ = - \sum_{j=0}^{S-s} \{L(S-j)-g^*\}\hat{m}(j) - K\{1+M(S-s)\}, \quad (4.4)$$

where $\hat{m}(j)$ is defined by (2.6).

From (4.3), (4.4), (3.13), theorem 2.1(a) and (2.9) it follows that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \{f_k^*(S)-kg^*\} = \sum_{n=0}^{\infty} b_n^*(S) \;/\; \sum_{n=0}^{\infty} n\rho(n) =$$
$$= -[\sum_{j=0}^{S-s} \{L(S-j) - g^*\}\, \hat{m}(j)] \;/\; [1 + M(S-s)] - K \tag{4.5}$$

Lemma 4.1.

(a) $$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \{f_k^*(i) - kg^*\} = K + \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \{f_k^*(S) - kg^*\}, \qquad i < s$$

and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \{f_k^*(i) - kg^*\} = v^*(i) + \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \{f_k^*(S) - kg^*\}, \qquad i \geq s.$$

(b) If the greatest common divisor of the indices n, where $\rho(n) > 0$, is 1, then the sequence $\{f_n^*(i) - ng^*\}$, $n \geq 0$, is convergent for any i.

Proof. (a) From (3.6) it follows trivially that assertion (a) holds for $i < s$. From (4.5), (4.1), (3.11), lemma 2.3(a), lemma 2.2 and (2.9) it follows that assertion (a) holds for $i \geq s$.
(b) If g.c.d. $\{n|\rho(n)>0\} = 1$, then by (4.3), (3.13), theorem 2.1(b) and (2.9) we have that the sequence $\{f_n^*(S) - ng^*\}$ is convergent. Next it follows from (4.1), (3.11), (3.6), lemma 2.3(b) and (2.9) that the sequence $\{f_n^*(i) - ng^*\}$ is convergent for any i.

Lemma 4.2.

(a) For any i,j holds that $(p_{ij}^{(0)} + \ldots + p_{ij}^{(n)})/n \to q_j$ as $n \to \infty$, where

$$q_j = \begin{cases} [\phi^{(0)}(S-j) + m(S-j)]/[1 + M(S-s)], & s \leq j \leq S, \\ 0, & \text{otherwise.} \end{cases}$$

(b) If the greatest common divisor of the indices n, where $\rho(n) > 0$, is 1, then $\{p_{ij}^{(n)}\}$ is convergent for any i,j.

Proof. (a) From (3.16) it follows trivially that if $j \notin [s,S]$, then assertion (a) holds for any i. By (3.19), theorem 2.1(a) and (2.9) we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} p_{Sj}^{(k)} = \sum_{n=0}^{\infty} \phi^{(n)}(S-j) / \sum_{n=0}^{\infty} n\rho(n) = q_j, \qquad s \le j \le S.$$

Using the fact that $\phi^{(n)}(j)$ converges to zero as $n \to \infty$ for any j, it follows from (3.18), lemma 2.3(a), lemma 2.2 and (2.9) that assertion (a) holds for any $i \geq s$. Finally it follows from (3.17) that assertion (a) holds for any i,j.

(b) From (3.16) it follows trivially that if $j \notin [s,S]$, then assertion (b) holds for any i. If g.c.d.$\{n | \rho(n)>0\} = 1$, then it follows from (3.19), theorem 2.1(b) and (2.9) that $\{p_{Sj}^{(n)}\}$ is convergent for any $j \in [s,S]$. Next it follows from (3.18), (3.17), lemma 2.3(b) and (2.9) that assertion (b) holds for any i,j.

Corollary.

(i) For any i holds $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \mathcal{E}(\underline{x}_{k+1} | \underline{x}_1 = i) = \sum_{j=s}^{S} j q_j - \mu.$

(ii) If the greatest common divisor of the indices n, where $\rho(n) > 0$, is 1, then the sequence $\{\mathcal{E}(\underline{x}_{n+1} | \underline{x}_1 = i\}$ is convergent for any i.

It is interesting to note that from Markov chain theory it follows that $\{q_j\}$ is the unique stationary probability distribution of the Markov chain $\{\underline{y}_t\}$. Using lemma 4.2, (3.1) and (2.3) we obtain the unique stationary probability distribution $\{a_j\}$ of the Markov chain $\{\underline{x}_t\}$. We have [3,4,6,8] that $a_j = \{\phi(S-j)+(\phi(S-j)m(0)+\ldots+\phi(s)m(S-s))\}/\{1+M(S-s)\}$ for $j < s$, $a_j = m(S-j)/\{1+M(S-s)\}$ for $s \le j \le S$, and $a_j = 0$ for $j > S$.

A direct consequence of (4.5), lemma 4.1 and the corollary of lemma 4.2 is the following theorem.

Theorem 4.1.

(a) $$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \{f_k(i)-kg\} = -[\sum_{j=0}^{S-s} \{L(S-j)-g^*\}\hat{m}(j)]/[1+M(S-s)] +$$

$$- (d-c)(\sum_{j=s}^{S} jq_j-\mu) - ci, \qquad i < s$$

and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \{f_k(i)-kg\} = L(i) + \sum_{j=0}^{i-s} L(i-j)m(j) - g^*\{1+M(i-s)\} +$$

$$-[\sum_{j=s}^{S} \{L(S-j)-g^*\}\hat{m}(j)]/[1+M(S-s)] - (d-c)(\sum_{j=s}^{S} jq_j-\mu) - ci, \qquad i \geq s.$$

(b) If the greatest common divisor of the indices n, where $\rho(n) > 0$, is 1, then the sequence $\{f_n(i) - ng\}$ is convergent for any i.

Corollary. Consider the special case $s = S = \bar{x}$. Since $\phi(0) < 1$, we have that $\rho(1) = 1 - \phi(0) > 0$. Hence g.c.d.$\{n|\rho(n)>0\} = 1$. This shows that $\{f_n(i) - ng\}$ is convergent for any i. It is straightforward to verify that

$$\lim_{n\to\infty} [f_n(i)-n\{L(\bar{x})+K(1-\phi(0))+c\mu\}] = K\phi(0) - (d-c)(\bar{x}-\mu) - ci, \quad i < \bar{x}$$

and

$$\lim_{n\to\infty} [f_n(i)-n\{L(\bar{x})+K(1-\phi(0))+c\mu\}] = L(i) + \sum_{j=0}^{i-\bar{x}} L(i-j)m(j) +$$

$$- \{L(\bar{x})+K(1-\phi(0))\}\{1+M(i-\bar{x})\} + K\phi(0) - (d-c)(\bar{x}-\mu) - ci, \qquad i \geq \bar{x}.$$

This limit relation has been derived in a quite different way in [7] for the case K = 0 and d = c.

Remark. In this remark we present an example, where $\{f_n(i) - ng\}$ oscillates for any i. Moreover, this example answers partially a question of IGLEHART [5, p. 31]. Suppose $P\{\underline{\xi}_t=1\} = 1$, c = d = 0 and K = 1. Let

$L(1) = L(2) = 0$, $L(j) > 1$ for $j \neq 1,2$, $L(j)$ is increasing for $j \geq 2$, and $L(j)$ is decreasing for $j \leq 1$. Consider the (s,S) policy with $s = 1$ and $S = 2$. Clearly $f_n(0) = (n+1)/2$ if n odd, and $f_n(0) = n/2$ if n even. Hence the sequence $\{f_n(0) - ng\}$ is not convergent.

It is not difficult to show that the (1,2) policy minimizes both the total expected cost in the finite period models and the average expected cost per period in the infinite period model. So this example shows that IGLEHART's conjecture from [5] may not be valid when the demand variable $\underline{\xi}_t$ is bounded.

5. The (s,S) inventory model with discounting.

Suppose that future costs are discounted by a fixed factor α, $0 < \alpha < 1$. Denote by $f_n(i;\alpha)$ the total expected discounted cost for the n-period (s,S) model, where i is the initial stock. Using (3.1), we have (see also [8])

$$f_n(i;\alpha) = \sum_{t=1}^{n} \alpha^{t-1} \mathcal{E}\{K\delta(\underline{y}_t-\underline{x}_t)+(\underline{y}_t-\underline{x}_t)c+L(\underline{y}_t)\,|\,\underline{x}_1=i\} - \alpha^n d\, \mathcal{E}(\underline{x}_{n+1}\,|\,\underline{x}_1=i) =$$

$$= \sum_{t=1}^{n} \alpha^{t-1} \mathcal{E}\{K\delta(\underline{y}_t-\underline{x}_t)+G_\alpha(\underline{y}_t)\,|\,\underline{x}_1=i\} - \alpha^n(d-c)\, \mathcal{E}(\underline{x}_{n+1}\,|\,\underline{x}_1=i) +$$

$$+ \alpha c\mu \sum_{t=0}^{n-1} \alpha^t - ci,$$

where

$$G_\alpha(k) = L(k) + c(1-\alpha)k.$$

For any i, let $f_0^*(i;\alpha) = 0$ and let

$$f_n^*(i;\alpha) = \sum_{t=1}^{n} \alpha^{t-1} \mathcal{E}\{K\delta(\underline{y}_t-\underline{x}_t)+G_\alpha(\underline{y}_t)\,|\,\underline{x}_1=i\}, \qquad n \geq 1.$$

Clearly,

$$f_n^*(i;\alpha) = K + f_n^*(S;\alpha), \qquad i < s;\ n \geq 1 \tag{5.1}$$

Using a standard argument from renewal theory, we have

$$f_n^*(i;\alpha) = G_\alpha(i) + \sum_{k=1}^{n-1}\sum_{j=0}^{i-s}\alpha^k G_\alpha(i-j)\phi^{(k)}(j) + \sum_{k=1}^{n-1}\alpha^k\{K+f_{n-k}^*(S;\alpha)\}\rho_i(k) =$$

$$= b_n^*(i;\alpha) + \sum_{k=0}^{n} f_{n-k}^*(S;\alpha)\alpha^k\rho_i(k), \qquad i \geq s;\ n \geq 1, \quad (5.2)$$

where

$$b_n^*(i;\alpha) = G_\alpha(i) + \sum_{k=1}^{n-1}\sum_{j=0}^{i-s} G_\alpha(i-j)\alpha^k\phi^{(k)}(j) + K\sum_{k=1}^{n-1}\alpha^k\rho_i(k), \quad i \geq s;\ n \geq 1.$$

If we define $b_0^*(i;\alpha) = 0$, $i \geq s$, then (5.2) is also valid for $n = 0$. We have in particular

$$f_n^*(S;\alpha) = b_n^*(S;\alpha) + \sum_{k=0}^{n} f_{n-k}^*(S;\alpha)\rho(k;\alpha), \qquad n \geq 0, \quad (5.3)$$

where

$$\rho(k;\alpha) = \alpha^k\rho(k), \qquad k \geq 0.$$

Let $\rho^{(1)}(j;\alpha) = \rho(j;\alpha)$, $j \geq 0$, and let

$$\rho^{(t)}(j;\alpha) = \sum_{k=0}^{j}\rho^{(t-1)}(k;\alpha)\,\rho(j-k;\alpha), \qquad j \geq 0;\ t \geq 2. \quad (5.4)$$

Define

$$u(j;\alpha) = \sum_{t=1}^{\infty}\rho^{(t)}(j;\alpha), \qquad j \geq 0.$$

We note that $u(j;\alpha) = \rho(j;\alpha) + \{\rho(0;\alpha)u(j;\alpha)+\ldots+\rho(j;\alpha)u(0;\alpha)\}$, $j \geq 0$.

Iterating (5.3) and using the fact that $\rho^{(t)}(j;\alpha) \to 0$ as $t \to \infty$ for any j, we obtain

$$f_n^*(S;\alpha) = b_n^*(S;\alpha) + \sum_{k=0}^{n} b_{n-k}^*(S;\alpha)u(k;\alpha) \qquad n \geq 0 \quad (5.5)$$

The relations (5.1), (5.2), and (5.5) in conjunction yield a formula for $f_n^*(i;\alpha)$. Since the solution for $E(\underline{x}_{n+1}|\underline{x}_1=i)$ has been already determined in section 3, we have found a formula for $f_n(i;\alpha)$.

Theorem 5.1. For any $n \geq 1$ holds

$$f_n(i;\alpha) = b_n^*(S;\alpha) + \sum_{k=0}^{n} b_{n-k}^*(S;\alpha)u(k;\alpha) + K - \alpha^n(d-c)\left(\sum_{j=s}^{S} jp_{Sj}^{(n-1)} - \mu\right) +$$

$$+ \alpha c\mu \sum_{t=0}^{n-1} \alpha^t - ci, \qquad i < s$$

and

$$f_n(i;\alpha) = b_n^*(i;\alpha) + \sum_{k=0}^{n} \{b_{n-k}^*(S;\alpha) + \sum_{j=0}^{n-k} b_{n-k-j}^*(S;\alpha)u(j;\alpha)\}\ \alpha^k\rho_i(k) +$$

$$- \alpha^n(d-c)\left(\sum_{j=s}^{\max(i,S)} jp_{ij}^{(n-1)} - \mu\right) + \alpha c\mu \sum_{t=0}^{n-1} \alpha^t - ci, \quad i \geq s.$$

We note that the formula for $f_n(i;\alpha)$ can be simplified in the special case $s = S$. We omit details.

Next we shall determine the limit of the sequence $\{f_n(i;\alpha)\}$ for any i. Let

$$m(j;\alpha) = \sum_{t=1}^{\infty} \alpha^t\phi^{(t)}(j) \quad \text{and} \quad M(j;\alpha) = \sum_{t=1}^{\infty} \alpha^t\Phi^{(t)}(j), \qquad j \geq 0.$$

Clearly, $M(j;\alpha) = m(0;\alpha) + \ldots + m(j;\alpha)$, $j \geq 0$. The numbers $m(j;\alpha)$ can be computed from $m(j;\alpha) = \alpha\phi(j) + \alpha\{\phi(0)m(j;\alpha)+\ldots+\phi(j)m(0;\alpha)\}$, $j \geq 0$.

For any $i \geq s$, we have

$$\sum_{k=0}^{\infty} \alpha^k\rho_i(k) = \sum_{k=1}^{\infty} \alpha^k\{\Phi^{(k-1)}(i-s)-\Phi^{(k)}(i-s)\} = \alpha - (1-\alpha)M(i-s;\alpha) \qquad (5.6)$$

For any $t \geq 1$, we have $\rho^{(t)}(0;\alpha) + \rho^{(t)}(1;\alpha) + \ldots = \{\alpha-(1-\alpha)M(S-s;\alpha)\}^t$, as can be easily proved from (5.4) by induction. Thus

$$\sum_{k=0}^{\infty} u(k;\alpha) = \{\alpha - (1-\alpha)M(S-s;\alpha)\}/(1-\alpha)\{1+M(S-s;\alpha)\} \qquad (5.7)$$

Using (5.6), we have

$$\lim_{n\to\infty} b_n^*(i;\alpha) = v^*(i;\alpha), \qquad i \geq s, \quad (5.8)$$

where

$$v^*(i;\alpha) = G_\alpha(i) + \sum_{j=0}^{i-s} G_\alpha(i-j)m(j;\alpha) + K\{\alpha-(1-\alpha)M(i-s;\alpha)\}, \qquad i \geq s.$$

From (5.8), (5.7), (5.5) and lemma 2.3(b), we have

$$\lim_{n\to\infty} f_n^*(S;\alpha) = v^*(S;\alpha)/(1-\alpha)\{1+M(S-s;\alpha)\} \tag{5.9}$$

Using (5.9), (5.8), (5.6), (5.2) and lemma 2.3(b), we obtain

$$\lim_{n\to\infty} f_n^*(i;\alpha) = v^*(i;\alpha)+\{\alpha-(1-\alpha)M(i-s;\alpha)\} \lim_{n\to\infty} f_n^*(S;\alpha), \quad i \geq s \tag{5.10}$$

The relations (5.1), (5.9), and (5.10) in conjunction yield the solution for $\lim f_n^*(i;\alpha)$. Since $\lim f_n(i;\alpha) = \lim f_n^*(i;\alpha)+\alpha c\mu/(1-\alpha)-ci$, we obtain after some calculations the following known result [8]

$$\lim_{n\to\infty} f_n(i;\alpha) = g_\alpha^*/(1-\alpha) + \alpha c\mu/(1-\alpha) - ci, \qquad i < s$$

and

$$\lim_{n\to\infty} f_n(i;\alpha) = G_\alpha(i) + \sum_{j=0}^{i-s} G_\alpha(i-j)m(j;\alpha) - \{g_\alpha^*/(1-\alpha)\}\{\alpha-(1-\alpha)M(i-s;\alpha)\} +$$

$$+ \alpha c\mu/(1-\alpha) - ci, \qquad i \geq s,$$

where

$$g_\alpha^* = \{G_\alpha(S) + \sum_{k=0}^{S-s} G_\alpha(S-k)m(k;\alpha)+K\}/\{1+M(S-s;\alpha)\}.$$

6. The (s,S) inventory model with a fixed lead time.

Suppose that an order placed in period t (= 1,2,...) is delivered at the beginning of period $t+\lambda$, where λ is a fixed positive integer. There is a fixed discount factor α with $0 < \alpha \leq 1$. In this section we consider the cases $\alpha = 1$ and $\alpha < 1$ simultaneously. We assume that the ordering costs are incurred at the time of delivery of the order. We shall demonstrate that the results of the sections 3, 4 and 5 carry over with a slight modification.

The (s,S) policy is now based on the stock on hand plus on order. Denote now by $\underline{x}_t$ and $\underline{y}_t$ the stock on hand plus on order just before ordering and the stock on hand plus on order just after ordering in period t. Since excess demands are backlogged, the stochastic processes $\{\underline{x}_t\}$ and $\{\underline{y}_t\}$ behave exactly as they done in the (s,S) model with zero lead time. Since everything on order in period t will have arrived by period $t+\lambda$, we have that $\underline{y}_t - (\underline{\xi}_t+\ldots+\underline{\xi}_{t+\lambda-1})$ is the stock on hand at the beginning of period $t+\lambda$ just after any additions to stock. Suppose

$$L_\lambda(k) = \sum_{j=0}^{\infty} L(k-j)\phi^{(\lambda)}(j)$$

exists and is finite for any k. Clearly, $L_\lambda(k)$ represents the expected holding and shortage costs in period $t+\lambda$, given that $\underline{y}_t = k$.

In the n-period model there are made only ordering decisions in the periods $1,\ldots,n$ and we denote by $f_n(i;\alpha)$ the total expected (discounted) cost over the periods $\lambda+1,\ldots,\lambda+n$ all discounted to the beginning of period $\lambda+1$, when $\underline{x}_1=i$. Using (3.1), we have (see also [8])

$$f_n(i;\alpha) = \sum_{t=1}^{n} \alpha^{t-1} \mathcal{E}\{K\delta(\underline{y}_t-\underline{x}_t)+(\underline{y}_t-\underline{x}_t)c+L_\lambda(\underline{y}_t)\,|\,\underline{x}_1=i\} +$$

$$- \alpha^n d\,\mathcal{E}(\underline{x}_{n+1}-\sum_{t=n+1}^{n+\lambda} \underline{\xi}_t\,|\,\underline{x}_1=i) =$$

$$= \sum_{t=1}^{n} \alpha^{t-1} \mathcal{E}\{K\delta(\underline{y}_t-\underline{x}_t)+L_\lambda(\underline{y}_t)+c(1-\alpha)\underline{y}_t\,|\,\underline{x}_1=i\} - \alpha^n(d-c)\,\mathcal{E}(\underline{x}_{n+1}\,|\,\underline{x}_1=i) +$$

$$+ \alpha c\mu \sum_{t=0}^{n-1} \alpha^t + \alpha^n d\lambda\mu - ci.$$

It will now be clear that the theorems 3.1, 4.1 and 5.1 remain valid provided that we replace L(k) by $L_\lambda(k)$, replace $-ci$ by $d\lambda\mu-ci$ in theorem 4.1 and replace $-ci$ by $\alpha^n d\lambda\mu-ci$ in theorem 5.1.

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