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THE DYNAMIC INVENTORY MODEL WITH DEMANDS DEPENDING ON THE STOCK LEVEL

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The dynamic inventory model with demands depending on the stock level

by

H.C. Tijms

1. Introduction

The stock level of a single item is reviewed at the beginnings of the periods 1,2,... and only at the beginning of each period an order may be placed. We assume that the delivery of an order is immediate. Let $\phi(k,j)$ be the probability of demand j during any period for which the stock level is k just after any additions to stock in that period. The demand in any period depends only at the stock level at the beginning of that period. Excess demands are backlogged. Hence the stock level may take on negative values. It is assumed that

$$\phi(k,0) < 1 \text{ and } \mu(k) = \sum_{j=0}^{\infty} j\phi(k,j) < \infty \quad \text{for all } k \in I,$$

where I is the set of all integers.

The cost of ordering z units is $K\delta(z) + c(z)$, where $K \geq 0$, $\delta(0) = 0$, and $\delta(z) = 1$ for $z \geq 1$. Let $L(k)$ be the holding and shortage costs in a period, where k is the stock level just after any additions to stock in that period. Finally, there is specified a fixed discount factor α, $0 < \alpha \leq 1$, with the interpretation that a unit cost incurred at the beginning of period t has a value of α^{t-1} at the beginning of period 1.

In section 2 we give some properties of the probabilities $\phi(k,j)$. In section 3 we define for the infinite period inventory model two optimality criteria: the total expected discounted cost ($\alpha < 1$) and the average expected cost per period ($\alpha = 1$). It is shown that for these optimality criteria it is no restriction to assume that the cost of ordering z units is given by $K\delta(z)$. Section 4 is devoted to the (s,S) inventory model. The solution for the total expected cost in the finite period (s,S) model is given. Furthermore, we give both the solution for the total expected discounted cost and the average expected cost per period in the infinite period (s,S) model. Finally, we give both the transient behaviour and the steady-state behaviour of the stock level at the time of review. The results of section 4 are new. Under rather weak conditions on the function L(k) and the probabilities $\phi(k,j)$ we give in section 5 a unified proof for the existence of an optimal (s,S) policy for the infinite period model. This result has been already obtained by Johnson [2] under conditions which differ slightly from our conditions. Our proof is different from Johnson's proof. Johnson shows by an iterative approach, based on Howard's policy improvement method, that an optimal (s,S) policy exists and his proof yields an iteration method for the determination of an optimal (s,S) policy. In our proof, however, it is shown in a direct way that an optimal (s,S) policy exists.

2. Preliminaries

For any $i \in I$ and $j \geq 0$, let

$$\phi^{(n)}(i,j) = \sum_{h=0}^{j} \phi^{(n-1)}(i,h)\,\phi(i-h,j-h), \qquad n \geq 1, \tag{2.1}$$

where

$$\phi^{(0)}(i,j) = \begin{cases} 1 & \text{for } j = 0, \\ 0 & \text{for } j \geq 1. \end{cases} \tag{2.2}$$

Observe that $\phi^{(1)}(i,j) = \phi(i,j)$.

For any $i \in I$ and $j \geq 0$, we have

$$\phi^{(n+m)}(i,j) = \sum_{h=0}^{j} \phi^{(n)}(i,h)\,\phi^{(m)}(i-h,j-h), \qquad m,n \geq 0. \tag{2.3}$$

We prove (2.3) by induction on m, where we fix n. By (2.1) and (2.2) we have that (2.3) is true for m = 0,1. Assuming that (2.3) is true for the integer m, we have

$$\phi^{(n+m+1)}(i,j) = \sum_{h=0}^{j} \phi^{(n+m)}(i,h)\ \phi(i-h,j-h) =$$

$$= \sum_{h=0}^{j} \phi(i-h,j-h) \sum_{k=0}^{h} \phi^{(n)}(i,k)\ \phi^{(m)}(i-k,h-k) =$$

$$= \sum_{k=0}^{j} \phi^{(n)}(i,k) \sum_{h=k}^{j} \phi^{(m)}(i-k,h-k)\ \phi(i-h,j-h) =$$

$$= \sum_{k=0}^{j} \phi^{(n)}(i,k) \sum_{r=0}^{j-k} \phi^{(m)}(i-k,r)\ \phi(i-k-r,j-k-r) =$$

$$= \sum_{k=0}^{j} \phi^{(n)}(i,k)\ \phi^{(m)}(i-k,j-k),$$

which proves (2.3) for m+1. This completes the induction proof.

For any $i \in I$ and $j \geq 0$, let

$$\Phi^{(n)}(i,j) = \sum_{k=0}^{j} \phi^{(n)}(i,k)\ , \qquad n \geq 0\ . \tag{2.4}$$

By (2.3) and (2.4) we have for any $i \in I$ and $j \geq 0$ that

$$\Phi^{(n+m)}(i,j) = \sum_{h=0}^{j} \phi^{(n)}(i,h)\ \Phi^{(m)}(i-h,j-h), \qquad m,n \geq 0\ . \tag{2.5}$$

For any $i \in I$ and $j \geq 0$, let

$$m_\alpha(i,j) = \sum_{n=1}^{\infty} \alpha^n \phi^{(n)}(i,j) \text{ and } M_\alpha(i,j) = \sum_{n=1}^{\infty} \alpha^n \Phi^{(n)}(i,j), \tag{2.6}$$

where $0 < \alpha \leq 1$. Observe that $M_\alpha(i,j) = m_\alpha(i,0)+\ldots+m_\alpha(i,j)$. When $\alpha < 1$, we have for any $i \in I$ and $j \geq 0$ that $M_\alpha(i,j) \leq \alpha/(1-\alpha)$.

Lemma 2.1

The function $M_1(i,j)$, where $i \in I$ and $j \geq 0$, is finite. For any $i \in I$ and $j \geq 0$ holds that $\Phi^{(1)}(i,j)+\ldots+\Phi^{(n)}(i,j)$ converges exponentially fast to $M_1(i,j)$ as $n \to \infty$.

Proof

First we prove that for any $i \in I$ and $j \geq 0$ holds

$$(2.8) \qquad 0 \leq \Phi^{(n+1)}(i,j) \leq \Phi^{(n)}(i,j) \leq 1, \qquad n \geq 0 .$$

Clearly, (2.8) is true for $n = 0$. By (2.5) we have for any $i \in I$ and $j \geq 0$ that

$$(2.9) \qquad \Phi^{(n)}(i,j) = \sum_{h=0}^{j} \phi(i,h)\, \Phi^{(n-1)}(i-h,j-h), \qquad n \geq 1,$$

and from this relation follows now (2.8) by induction.

Next we prove that for every $i \in I$ and $j \geq 0$ there exists an integer $N \geq 1$ such that

$$(2.10) \qquad \Phi^{(n)}(i,j) \leq \Phi^{(N)}(i,j) < 1 \qquad \text{for} \qquad n \geq N.$$

Fix i and j. Suppose to the contrary that $\Phi^{(n)}(i,j) = 1$ for all $n \geq 0$. From (2.9) and the fact that $\phi(i,0) < 1$ it follows that an integer h^* exists such that $1 \leq h^* \leq j$ and $\phi(i,h^*) > 0$. Furthermore, we have by (2.8), (2.9) and the assumption $\Phi^{(n)}(i,j) = 1$, $n \geq 0$, that $\Phi^{(n)}(i-h^*,j-h^*) = 1$ for all $n \geq 0$. Proceeding in this way, we see that there exists an integer $i^* < i$ such that $\Phi^{(n)}(i^*,0) = 1$ for all $n \geq 0$. However, this is a contradiction, since $\Phi^{(n)}(i^*,0) = \{\phi(i^*,0)\}^n < 1$, $n \geq 1$. Thus $\Phi^{(n)}(i,j) < 1$ for at least one $n \geq 0$, and hence by (2.8) we have (2.10).

We are now in a position to prove the lemma. Fix $i_0 \in I$ and $j_0 \geq 0$. From (2.10) follows the existence of a number δ, $0 \leq \delta < 1$, and an integer $N \geq 1$ such that

$$\Phi^{(n)}(i,j) \leq \delta \quad \text{for all } i_0-j_0 \leq i \leq i_0;\ 0 \leq j \leq j_0;\quad n \geq N.$$

For any $k \geq 0$, we have by (2.5) that

$$\Phi^{(kN+N)}(i_0,j_0) = \sum_{h=0}^{j_0} \phi^{(N)}(i_0,h)\ \Phi^{(kN)}(i_0-h,j_0-h) \leq$$

$$\leq \max_{0\leq h\leq j_0} \Phi^{(kN)}(i_0-h,j_0-h) \sum_{h=0}^{j_0} \phi^{(N)}(i_0,h)$$

$$\leq \delta\ \Phi^{(kN)}(i_0',j_0')$$

for some $i_0' \in [i_0-j_0,i_0]$ and $j_0' \in [0,j_0]$, and hence

$$\Phi^{(kN+N)}(i_0,j_0) \leq \delta^{k+1} \qquad k \geq 0. \tag{2.11}$$

The lemma follows now from (2.8) and (2.11).

Corollary

For any $i \in I$ and $j \geq 0$ holds that $\Phi^{(n)}(i,j)$ and $\phi^{(n)}(i,j)$ converge exponentially fast to zero as $n \to \infty$.

Finally, we note that from (2.1) and (2.6) follows

$$m_\alpha(i,j) = \alpha\phi(i,j) + \alpha \sum_{h=0}^{j} \phi(i-h,j-h)\ m_\alpha(i,h), \quad i \in I;\ j \geq 0.$$

Hence the numbers $m_\alpha(i,j)$ can be computed recursively.

3. The optimality criteria for the infinite period model

Let us define the state i of the system in a period as the stock level just before ordering in that period. We take I as the set of all possible states. Let us say that in state i decision k, where $k \geq i$, is made when k-i units are ordered. Hence the ordering decision is identified with the stock level just after ordering. We impose the following mild restriction on the choice of an ordering decision. There are given finite integers m and M, where $m \leq M$, such that no ordering is done if the stock level $i \geq M$, at most M-i units are ordered if $i < M$, and at least m-i units are ordered if $i < m$. Let A(i) denote the set of feasible decisions in state i. Then $A(i) = \{i\}$ for $i \geq M$ and $A(i) = \{k \mid \max(i,m) \leq k \leq M\}$ for $i < M$. The numbers m and M will be specified further in section 5.

A policy R for controlling the inventory system is a set of nonnegative functions $D_k(h_{t-1},i_t)$; $k \in A(i_t)$; $t \geq 1$, satisfying

$$\sum_{k \in A(i_t)} D_k(h_{t-1},i_t) = 1$$

for every "history" $h_{t-1} = (i_1,k_1,\ldots,i_{t-1}, k_{t-1})$ and all $i_t \in I$, $t = 1,2,\ldots$, where i_n and k_n are the observed state and the observed decision in period n.

The interpretation being: if at the beginning of period t the history h_{t-1} has been observed and the system is in state i_t, then $k-i_t$ units are ordered with probability $D_k(h_{t-1},i_t)$.

Let C(m,M) denote the class of all possible policies. A policy R is said to be stationary deterministic if $D_k(h_{t-1},i_t=i) = D_k(i)$ independent of h_{t-1} and t, and if in addition $D_k(i) = 1$, or 0.

Given that the initial state is i and policy R is used, we define the following random variables

$\underline{i}_t$ = the stock level just before ordering in period t, $t \geq 1$,
$\underline{k}_t$ = the stock level just after ordering in period t, $t \geq 1$,
$\underline{\xi}_t$ = the demand in period t, $t \geq 1$.

We have for any $i \in I$ and $R \in C(m,M)$ that

$$\underline{i}_{t+1} = \underline{k}_t - \underline{\xi}_t \qquad t \geq 1. \tag{3.1}$$

For the case $\alpha < 1$, we take as optimality criterion

$$V_\alpha(i;R) = \sum_{t=1}^{\infty} \alpha^{t-1} E_R\{K\delta(\underline{k}_t - \underline{i}_t) + (\underline{k}_t - \underline{i}_t)c + L(\underline{k}_t) | \underline{i}_1 = i\},$$

where E_R denotes the expectation under policy R. We note that the expectations exists and that $V_\alpha(i,R)$ is finite (this is proved by $m \leq \underline{k}_t \leq \max(\underline{i}_1, M)$, (3.1) and $E_R(\underline{\xi}_t | i_1 = i) \leq \max\{\mu(k) | m \leq k \leq \max(i,M)\}$. The quantity $V_\alpha(i;R)$ represents the total expected discounted cost for the infinite period model, when i is the initial state and policy R is followed.

For the case $\alpha = 1$, we take as optimality criterion

$$g(i;R) = \liminf_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} E_R\{K\delta(\underline{k}_t - \underline{i}_t) + (\underline{k}_t - \underline{i}_t)c + L(\underline{k}_t) | \underline{i}_1 = i\}.$$

We note that $g(i;R)$ exists and is finite. When the limit exists $g(i;R)$ represents the average expected cost per period when the initial state is i and policy R is used.

Now we shall demonstrate that for these optimality criteria it is no restriction to assume that the ordering cost of z units is given by $K\delta(z)$. For any $i \in I$ and $R \in C(m,M)$, let

$$f_n(i;R;\alpha) = \sum_{t=1}^{n} \alpha^{t-1} E_R\{K\delta(\underline{k}_t - \underline{i}_t) + (\underline{k}_t - \underline{i}_t)c + L(\underline{k}_t) | \underline{i}_1 = i\}, \quad n \geq 1,$$

where $0 < \alpha \leq 1$. Using (3.1), we obtain

$$f_n(i;R;\alpha) = \sum_{t=1}^{n} \alpha^{t-1} E_R\{K\delta(\underline{k}_t - \underline{i}_t) + (1-\alpha)c\underline{k}_t + \alpha c\underline{\xi}_t + L(\underline{k}_t) | \underline{i}_1 = i\} + -ci - c\alpha^n E_R(\underline{i}_{n+1} | \underline{i}_1 = i), \qquad n \geq 1.$$

Since

$$E_R(\underline{\xi}_t | \underline{i}_1 = i) = E_R(\mu(\underline{k}_t) | \underline{i}_1 = i),$$

we have

$$f_n(i;R;\alpha) = \sum_{t=1}^{n} \alpha^{t-1} E_R\{K\delta(\underline{k}_t - \underline{i}_t) + G_\alpha(\underline{k}_t) | \underline{i}_1 = i\} - ci +$$

$$- c\alpha^n E_R(\underline{i}_{n+1} | \underline{i}_1 = i) , \qquad n \geq 1,$$

where

$$G_\alpha(k) = L(k) + (1-\alpha)ck + \alpha c\mu(k), \qquad k \in I.$$

For any $i \in I$ and $R \in C(m,M)$, we have

$$\lim_{n\to\infty} \alpha^n E_R(\underline{i}_{n+1} | \underline{i}_1 = i\} = 0 \qquad \text{for} \quad \alpha < 1,$$

and

$$\lim_{n\to\infty} \frac{1}{n} E_R(\underline{i}_{n+1} | \underline{i}_1 = i) = 0,$$

and hence both $V_\alpha(i;R)$ and $g(i;R)$ are affected identically when we redefine

$$(3.2) \quad V_\alpha(i;R) = \sum_{t=1}^{\infty} \alpha^{t-1} E_R\{K\delta(\underline{k}_t - \underline{i}_t) + G_\alpha(\underline{k}_t) | \underline{i}_1 = i\}$$

and

$$(3.3) \quad g(i;R) = \liminf_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} E_R\{K\delta(\underline{k}_t - \underline{i}_t) + G_1(\underline{k}_t) | \underline{i}_1 = i\}.$$

From now on we assume that the cost of ordering z units is given by $K\delta(z)$ and that $G_\alpha(k)$ is the one period holding and shortage cost.

When $\alpha < 1$ a policy $R^* \in C(m,M)$ is called <u>optimal</u> if

$$V_\alpha(i;R^*) \leq V_\alpha(i;R) \quad \text{for all } i \in I, \text{ all } R \in C(m,M).$$

When $\alpha = 1$ a policy $R^* \in C(m,M)$ is called <u>optimal</u> if

$$g(i;R^*) \leq g(i;R) \qquad \text{for all } i \in I, \text{ all } R \in C(m,M).$$

An easy consequence of theorem 6(f) in [1] is the following theorem (see also section 2 of [5]).

<u>Theorem 3.1</u> (the discounted cost criterion)

Let $\alpha < 1$. If for policy $R^* \in C(m,M)$ holds

$$V_\alpha(i;R^*) = \min_{k \in A(i)} \{K\delta(k-i) + G_\alpha(k) + \alpha \sum_{j=0}^{\infty} V_\alpha(k-j;R^*)\phi(k,j)\}, \; i \in I,$$

then the policy R^* is optimal.

A proof of the next theorem can be found in [4].

<u>Theorem 3.2</u> (the average cost criterion)

Let $\alpha = 1$. Suppose there exists a set of numbers $\{g,v(i)\}$, $i \in I$, such that

$$(3.4) \qquad v(i) = \min_{k \in A(i)} \{K\delta(k-i) + G_1(k) - g + \sum_{j=0}^{\infty} v(k-j)\phi(k,j)\}, \; i \in I$$

and

$$\lim_{n\to\infty} \frac{1}{n} E_R(v(\underline{i}_n) \,|\, \underline{i}_1 = i) = 0 \quad \text{for all } i \in I, \text{ all } R \in C(m,M).$$

Let R^* be any policy which, for each i, prescribes a decision which minimizes the right side of (3.4), then R^* is optimal. Moreover, $g(i;R^*) = g$ for all $i \in I$ and the limit in (3.3) exists for policy R^*.

4. <u>The dynamic (s,S) inventory model</u>

An (s,S) policy, where $s \leq S$ and $s, S \in I$, has the following form: If, at review, the stock level i is below s, order then S-i units; otherwise, do not order.

Fix an (s,S) policy, and define

$$g_n(i;\alpha) = \sum_{t=1}^{n} \alpha^{t-1} E_{(s,S)}\{K\delta(\underline{k}_t - \underline{i}_t) + G_\alpha(\underline{k}_t) | \underline{i}_1 = i\}, \quad i \in I;\ n \geq 1,$$

where $0 < \alpha \leq 1$. Let $g_0(i;\alpha) = 0$, $i \in I$. Clearly, we have

$$g_n(i;\alpha) = K + g_n(S;\alpha), \qquad i < s;\ n \geq 1. \tag{4.1}$$

When the initial state $\underline{x}_1 = i \geq s$ and the (s,S) policy is followed, then the probability of a cumulative demand j in the first k periods is equal to $\phi^{(k)}(i,j)$ for $0 \leq j \leq i-s$, and the probability that the cumulative demand exceeds i-s for the first time during the kth period is $\Phi^{(k-1)}(i,i-s) - \Phi^{(k)}(i,i-s)$. It is now readily seen that

$$(4.2)\quad g_n(i;\alpha) = G_\alpha(i) + \sum_{k=1}^{n-1} \sum_{j=0}^{i-s} \alpha^k G_\alpha(i-j)\phi^{(k)}(i,j) +$$

$$+ \sum_{k=1}^{n-1} \alpha^k \{K + g_{n-k}(S;\alpha)\}\rho_i(k), \quad i \geq s;\ n \geq 1,$$

where

$$\rho_i(k) = \begin{cases} 0, & i \geq s;\ n \geq 1, \\ \Phi^{(k-1)}(i,i-s) - \Phi^{(k)}(i,i-s), & i \geq s;\ n \geq 1. \end{cases}$$

Clearly, we have for any $i \geq s$ and $n \geq 1$ that

$$(4.3)\quad \sum_{k=0}^{n} \rho_i(k) = 1 - \Phi^{(n)}(i,i-s);\ \sum_{k=0}^{n} k\rho_i(k) = 1 + \sum_{k=1}^{n-1} \Phi^{(k)}(i,i-s) - n\Phi^{(n)}(i,i-s),$$

and hence by the corollary of lemma 2.1 we have

$$(4.4)\qquad \sum_{k=0}^{\infty} \rho_i(k) = 1 \quad \text{and} \quad \sum_{k=0}^{\infty} k\rho_i(k) = 1 + M_1(i,i-s), \quad i \geq s.$$

Thus we have for any $i \geq s$ that $\{\rho_i(k)\}$, $k \geq 0$, constitutes a probability distribution with a finite first moment.

We can write (4.2) as

$$(4.5) \quad g_n(i;\alpha) = b_n(i;\alpha) + \sum_{k=0}^{n} g_{n-k}(S;\alpha)\alpha^k \rho_i(k), \quad i \geq s;\ n \geq 0,$$

where

$$b_n(i;\alpha) = \begin{cases} 0, & i \geq s;\ n = 0, \\ G_\alpha(i) + \sum_{k=1}^{n-1} \sum_{j=0}^{i-s} \alpha^k G_\alpha(i-j)\phi^{(k)}(i,j) + K\sum_{k=1}^{n-1} \alpha^k \rho_i(k), & i \geq s;\ n > 0. \end{cases}$$

The equation (4.5) constitutes for $i = S$ a renewal equation.

Let

$$\rho(k;\alpha) = \alpha^k \rho_S(k), \qquad k \geq 0,$$

Let $\rho^{(1)}(k;\alpha) = \rho(k;\alpha)$, $k \geq 0$, and let

$$\rho^{(t)}(j;\alpha) = \sum_{k=0}^{j} \rho^{(t-1)}(k;\alpha)\, \rho(j-k;\alpha), \quad j \geq 0;\ t \geq 2.$$

Define

$$r(j;\alpha) = \sum_{t=1}^{\infty} \rho^{(t)}(j;\alpha), \qquad j \geq 0.$$

We note that $r(j;\alpha) = \rho(j;\alpha) + \{\rho(0,\alpha)\, r(j;\alpha) + \ldots + \rho(j;\alpha)\, r(0;\alpha)\}$, $j \geq 0$. Observe that $\rho(0;\alpha) = 0$, and hence $r(0;\alpha) = 0$.

In the same way as in section 5 of [6] the following results can be proved.

Theorem 5.1.

$$g_n(i;\alpha) = b_n(S;\alpha) + \sum_{k=0}^{n} b_{n-k}(S;\alpha)\ r(k;\alpha) + K\ , \quad i < s;\ n \geq 1,$$

and

$$g_n(i;\alpha) = b_n(i;\alpha) + \sum_{k=0}^{n} \{b_{n-k}(S;\alpha) + \sum_{j=0}^{n-k} b_{n-k-j}(S;\alpha)\ r(j;\alpha)\}\alpha^k \rho_i(k),$$

$$i \geq s,\ n \geq 1.$$

Theorem 5.2.

Let $\alpha < 1$, then

$$V_\alpha(i;(s,S)) = \lim_{n\to\infty} g_n(i;\alpha) = \frac{a_\alpha(s,S)}{1-\alpha}\ , \quad i < s,$$

and

$$V_\alpha(i;(s,S)) = \lim_{n\to\infty} g_n(i;\alpha) = G_\alpha(i) + \sum_{j=0}^{i-s} G_\alpha(i,j)\ m_\alpha(i,j) +$$

$$+ \frac{a_\alpha(s,S)}{1-\alpha}\{\alpha-(1-\alpha)\ M_\alpha(i,i-s)\},\quad i \geq s,$$

where

$$a_\alpha(s,S) = \{G_\alpha(S) + \sum_{k=0}^{S-s} G_\alpha(S-k)\ m_\alpha(S,k) + K\}/\{1+M_\alpha(S,S-s)\}.$$

Observe that

$$\lim_{\alpha\uparrow 1} (1-\alpha)\ V_\alpha(i;(s,S)) = a_1(s,S) \quad \text{for all } i \in I,$$

where

$$a_1(s,S) \overset{\text{def}}{=} \{G_1(S) + \sum_{k=0}^{S-s} G_1(S-k)\ m_1(S,k) + K\}/\{1+M_1(S,S-s)\}.$$

Next consider the case $\alpha = 1$. Let

$$g_n^*(i) = g_n(i;1) - na_1(s,S)\ , \quad i \in I;\ n \geq 0.$$

From (4.1) and (4.5) it follows that

$$(4.6)\qquad g_n^*(i) = b_n^*(i) + \sum_{k=0}^{n} g_{n-k}^*(S)\,\rho_i(k), \quad i \geq s;\ n \geq 0,$$

where

$$b_n^*(i) = \begin{cases} 0, & i \geq s;\ n = 0 \\ G_1(i) + \sum_{k=1}^{n-1}\sum_{j=0}^{i-s} G_1(i-j)\phi^{(k)}(i,j) - a_1(s,S)\{1+\sum_{k=1}^{n-1}\Phi^{(k)}(i,i-s)\} + & \\ \quad + K\{1-\Phi^{(n-1)}(i,i-s)\}, & i \geq s;\ n \geq 1 \end{cases}$$

The equation (4.6) constitutes for i = S a renewal equation. It is now readily seen that the following theorem holds (c.f. theorem 3.1 in section 3 of [6]).

Theorem 4.2.

$$(4.7)\quad g_n(i;1) = na_1(s,S) + b_n^*(S) + \sum_{k=0}^{n} b_{n-k}^*(S)\, r(k;1) + K, \quad i < s;\ n > 0,$$

and

$$g_n(i;1) = na_1(s,S) + b_n^*(i) + \sum_{k=0}^{n}\{b_{n-k}^*(S) + \sum_{j=0}^{n-k} b_{n-k-j}^*(S)\, r(j;1)\}\,\rho_i(k),$$

$$(i \geq s;\ n > 0).$$

Theorem 4.3.

$$g(i;(s,S)) = \lim_{n\to\infty} \frac{g_n(i)}{n} = a_1(s,S) \quad \text{for all } i \in I.$$

Proof.

From lemma 2.1 and the definition of $a_1(s,S)$ it follows that $b_n^*(S)$ converges exponentially fast to zero as $n \to \infty$, and hence $\Sigma|b_n^*(S)| < \infty$. From renewal theory (see, for instance, [3]) we have that the renewal quantity $r(k;1)$, $k \geq 0$, is bounded. From (4.7) it follows now that $g_n(i;1)/n \to a_1(s,S)$ as $n \to \infty$ for $i < s$. Moreover, by (4.7) and the relation $g_n^*(S) = g_n^*(i) - K$ for $i < s$ it is readily seen that $g_n^*(S)$, $n \geq 0$, is bounded. Since for any $i \geq s$ the sequence $b_n^*(i)$ has a finite limit and using the fact that $\{\rho_i(k)\}$, $k \geq 0$, constitutes a probability distribution for any $i \geq s$, we have by (4.6) that $g_n^*(i)/n \to 0$ as $n \to \infty$ for any $i \geq s$. This ends the proof

Observe that the average expected cost per period for the infinite period (s,S) model does not depend on the initial stock. Finally, it is interesting to note that in the same way as in the sections 3 and 4 of [6] for the (s,S) policy the probability distribution of $\underline{k}_n$, $n \geq 1$, and the stationary probability distribution of the Markov chain $\{\underline{k}_n\}$ can be determined. Let

$$p_{ij}^{(n)} = P_{(s,S)}\{\underline{k}_{n+1}=j \mid \underline{i}_1=i\}, \quad i,j \in I;\ n \geq 0.$$

We have the following theorem.

<u>Theorem 4.4.</u>

(a) $$p_{ij}^{(n)} = \phi^{(n)}(i,i-j) + \sum_{k=0}^{n} p_{Sj}^{(n-k)} \rho_i(k), \quad s \leq j \leq \max(i,S);\ i \geq s;\ n \geq 0,$$

and

$$p_{S_j}^{(n)} = p_{ij}^{(n)} = \phi^{(n)}(S,S-j) + \sum_{k=0}^{n} \phi^{(n-k)}(S,S-j)\, r(k;1),$$

$$(s \leq j \leq S;\ i < s;\ n \geq 0),$$

where $\phi^{(n)}(i,j) = 0$ for $j < 0$.

(b) $$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} p_{ij}^{(k)} = q_j \qquad \text{for all } i,j \in I,$$

where

$$q_j = \begin{cases} [\phi^{(0)}(S,S-j) + m_1(S,S-j)]/[1+M_1(S,S-s)], & s \leq j \leq S, \\ 0, & \text{otherwise} \end{cases} .$$

If the greatest common divisor of the indices n, where $\rho(n;1) > 0$, is 1, then the sequence $\{p_{ij}^{(n)}\}$ is convergent for any i,j.

5. The optimality of an (s,S) policy for the infinite period model.

For the existence proof of an optimal (s,S) policy we impose the following conditions on the function $G_\alpha(k)$.

(i) There exists a finite integer S_0 such that $G_\alpha(i) \leq G_\alpha(j)$ for $j \leq i \leq S_0$ and $G_\alpha(i) \geq G_\alpha(j)$ for $i \geq j \geq S_0$.

(ii) $\lim_{|k|\to\infty} G_\alpha(k) > G_\alpha(S_0) + K.$

We assume that S_0 is the smallest integer for which (i) holds. The function $G_\alpha(k)$ takes on its absolute minimum at $k = S_0$. Let S_0' be the largest integer at which $G_\alpha(k)$ takes on its absolute minimum.

Let s_1 be the smallest integer for which

$$G_\alpha(s_1) \leq G_\alpha(S_0) + (1-\alpha\phi(S_0,0))K$$

and let S_1 be the largest integer for which

$$G_\alpha(S_1) \leq G_\alpha(S_0) + \alpha K.$$

Observe that $s_1 \leq S_0 \leq S_0' \leq S_1$.

We impose on the probabilities $\phi(i,j)$ the condition

$$(5.1) \qquad \sum_{k=i-h}^{\infty} \phi(i,k) \geq \sum_{k=j-h}^{\infty} \phi(j,k) \quad \text{for all } S_0 \leq i < j \text{ and } 0 \leq h \leq i.$$

First, we shall give some properties of the function $a_\alpha(s,S)$, where $s \leq S$ and $s,S \in I$. For any α, $0 < \alpha \leq 1$, let

$$a_\alpha^* = \min_{s \leq S;\, s,S \in I} a_\alpha(s,S).$$

Lemma 5.1.

Let $0 < \alpha \leq 1$ and let s^* and S^* be integers such that $s^* \leq S^*$ and $a_\alpha(s^*,S^*) = a_\alpha^*$.

(a) If $m_\alpha(S^*,S^*-s^*+1) > 0$, then $G_\alpha(s^*-1) \geq a_\alpha^*$.

(b) If $s^* = S^*$, then $G_\alpha(s^*) \leq a_\alpha^*$.

(c) If $s^* < S^*$ and if $m_\alpha(S^*,S^*-s^*) > 0$, then $G_\alpha(s^*) \leq a_\alpha^*$.

(d) If $\phi(k,1) > 0$ for $k \leq S^*$, then $G_\alpha(s^*-1) \geq a_\alpha^* \geq G_\alpha(s^*)$.

(e) If $G_\alpha(s^*-1) \geq a_\alpha^* \geq G_\alpha(s^*)$, then $s_1 \leq s^* \leq S_0'$ when $K = 0$, and $s_1 \leq s^* \leq S_0$ when $K > 0$.

Proof.

The proof is identical to the proof of lemma 4.2 of [5].

Lemma 5.2.

Let $0 < \alpha \leq 1$. There exist integer s^* and S^*, $s^* \leq S^*$, such that $a_\alpha(s^*,S^*) = a_\alpha^*$ and $G_\alpha(s^*-1) \geq a_\alpha^* \geq G_\alpha(s^*)$. If $K = 0$, then $s^* = S_0$ and $S^* = S_0$ satisfy these conditions.

Proof.

The proof is identical to the proof of lemma 4.3 of [5].

Let $0 < \alpha \leq 1$. From now on s^* and S^* are two fixed integers for which $a_\alpha(s^*,S^*) = a_\alpha^*$ and $G_\alpha(s^*-1) \geq a_\alpha^* \geq G_\alpha(s^*)$, where we choose $s^* = S^* = S_0$ when $K = 0$.
The function $v_\alpha^*(i)$, $i \in I$, is defined as follows

$$(5.2)\qquad v_\alpha^*(i) = \begin{cases} 0\ , & i < s^*\ , \\ G_\alpha(i) - a_\alpha^* + \alpha \sum_{j=0}^{i-s^*} v_\alpha^*(i-j)\ \phi(i,j), & i \geq s^*\ . \end{cases}$$

The function $v_\alpha^*(i)$ is uniquely determined by (5.2). Iterating (5.2) and using (2.1), yields

$$v_\alpha^*(i) = G_\alpha^*(i) - a_\alpha^* + \sum_{k=1}^{n} \sum_{j=0}^{i-s^*} \{G_\alpha(i-j)-a_\alpha^*\}\alpha^k\phi^{(k)}(i,j) +$$

$$+ \sum_{j=0}^{i-s^*} v_\alpha^*(i-j)\alpha^{n+1}\phi^{(n+1)}(i,j),\ i \geq s^*;\ n \geq 0,$$

and hence by using the corollary of lemma 2.1 we obtain

$$(5.3)\qquad v_\alpha^*(i) = \begin{cases} 0, & i < s^*, \\ G_\alpha^*(i) + \sum_{j=0}^{i-s^*} G_\alpha(i-j)\ m_\alpha(i,j) - a_\alpha^*\{1+M_\alpha(i,i-s^*)\}, & i \geq s^*. \end{cases}$$

For convenience we define the function

$$(5.4)\qquad J_\alpha(k) = G_\alpha(k) - a_\alpha^* + \alpha \sum_{j=0}^{\infty} v_\alpha^*(k-j)\ \phi(k,j), \qquad k \in I.$$

By (5.2) and (5.4) we have

$$(5.5)\qquad J_\alpha(k) = \begin{cases} G_\alpha(k) - a_\alpha^*\ , & k < s^*\ , \\ v_\alpha^*(k)\ , & k \geq s^* \end{cases}$$

Theorem 5.1

(a) $J_\alpha(k)$ is nonincreasing on $(-\infty, s^*-1]$.

(b) $K+J_\alpha(S^*) = 0$; $J_\alpha(s^*-1) \geq 0$.

(c) $J_\alpha(k) \geq J_\alpha(S^*)$ for all $k \in I$.

(d) $K+J_\alpha(k) > 0$ for $k > S_1$.

(e) $J_\alpha(k) \leq 0$ for $s^* \leq k \leq S_0$.

(f) $K+J_\alpha(k) \geq J_\alpha(i)$ for $k \geq i \geq s^*$.

Proof

(a), (b) and (c). The proof of (a), (b) and (c) is identical to the proof of theorem 5.1(a), 5.1(b) and 5.1(c) of [5].

(d) By (5.2), (5.4) and (5.5) we have

$$J_\alpha(k) = G_\alpha(k) - a_\alpha^* + \alpha \sum_{j=0}^{k-s^*} J_\alpha(k-j)\, \phi(k,j), \qquad k \geq s^* . \tag{5.6}$$

We have by (b) and (c) that $J_\alpha(k) \geq -K$, $k \in I$, and by the definition of S_1 we have $G_\alpha(k) > G_\alpha(S_0) + \alpha K$ for $k > S_1$. Further, $a_\alpha^* \leq a_\alpha(S_0, S_0) = G_\alpha(S_0) + K/\{1+M_\alpha(S_0,0)\} \leq G_\alpha(S_0) + K$. Hence

$$K+J_\alpha(k) > K+G_\alpha(S_0) + \alpha K - G_\alpha(S_0) - K - \alpha K \sum_{j=0}^{k-s^*} \phi(k,j) \geq 0, \quad k > S_1 .$$

(e) The proof of (e) is identical to the proof of theorem 5.1(d) of [5].

(f) By (b), (c) and (e) we have $K+J_\alpha(k) \geq 0 \geq J_\alpha(i)$ for $k \geq i$ and $s^* \leq i \leq S_0$. The proof that $K+J_\alpha(k) \geq J_\alpha(i)$ for $k \geq i > S_0$ is a slight modification of a proof in [2, pp. 84-85]. Assuming that $i-1 \geq S_0$ is an integer such that $K+J_\alpha(k) \geq J_\alpha(h)$ for $k \geq h$ and $S_0 \leq h \leq i-1$, we shall demonstrate that $K+J_\alpha(k) \geq J_\alpha(i)$ for $k \geq i$.

To prove this induction step we need the following lemma [2].

Lemma 5.3. If a_j and b_j, $j = 0,1,\ldots,N$, are nonnegative real numbers such that

$$\sum_{j=0}^{N} a_j = \sum_{j=0}^{N} b_j$$

$$\sum_{j=0}^{H} a_j \geq \sum_{j=0}^{H} b_j \quad \text{for all } H = 0,\ldots,N-1,$$

and if $f(.)$ and $g(.)$ are functions on the integers $0,1,\ldots,N$ such that $f(h) \leq g(j)$ whenever $h \leq j$, then

$$\sum_{j=0}^{N} a_j\, f(j) \leq \sum_{j=0}^{N} b_j\, g(j).$$

Clearly, this lemma remains true when we replace the condition $f(h) \leq g(j)$ whenever $h \leq j$ by the weaker condition $f(h) \leq g(j)$ for any pair (h,j) with $h \leq j$ and $a_h\, b_j > 0$.

By (5.5) we have that $J_\alpha(h) = v_\alpha^*(h)$ for $h \geq s^*$. To prove that $K+v_\alpha^*(k) \geq v_\alpha^*(i)$ for $k \geq i$, we fix k and we distinguish between $\phi(i,0) \geq \phi(k,0)$ and $\phi(i,0) \leq \phi(k,0)$.

First, consider the case $\phi(i,0) \geq \phi(k,0)$. By (5.2) we have

$$(5.7) \qquad v_\alpha^*(i) = G_\alpha(i) - a_\alpha^* + \alpha\phi(k,0)\, v_\alpha^*(i) + \alpha \sum_{j=0}^{i-s^*+1} v_\alpha^*(i-j)\, \hat{\phi}(i,j) \ ,$$

where

$$\hat{\phi}(i,j) = \begin{cases} \phi(i,0) - \phi(k,0), & j = 0 \ , \\ \phi(i,j) \ , & j = 1,\ldots,i-s^*, \\ \sum_{h=i-s^*+1}^{\infty} \phi(i,h), & j = i-s^*+1 \ . \end{cases}$$

Further, we have

$$(5.8)\quad v_\alpha^*(k) = G_\alpha(k) - a_\alpha^* + \alpha\phi(k,0)\, v_\alpha^*(k) + \sum_{j=1}^{k-s^*+1} v_\alpha^*(k-j)\, \hat{\phi}(k,j)\ ,$$

where

$$\hat{\phi}(k,j) = \begin{cases} \phi(k,j)\ , & j = 1,\ldots,k-s^*, \\ \sum_{h=k-s^*+1}^{\infty} \phi(k,h)\ , & j = k-s^*+1\ . \end{cases}$$

Since $G_\alpha(\cdot)$ is nondecreasing on $[S_0,\infty]$, we have

$$(5.9)\quad G_\alpha(i) - G_\alpha(k) \le 0 \le K - \alpha K \sum_{j=0}^{\infty} \phi(k,j).$$

From (5.7), (5.8) and (5.9) it follows

$$(5.10)\quad \{v_\alpha^*(i)-v_\alpha^*(k)\}\{1-\alpha\phi(k,0)\} \le K\{1-\alpha\phi(k,0)\}+ \alpha \sum_{j=0}^{i-s^*+1} v_\alpha^*(i-j)\, \hat{\phi}(i,j) +$$

$$- \alpha \sum_{j=1}^{k-s^*+1} \{K+v_\alpha^*(k-j)\}\, \hat{\phi}(k,j).$$

Let

$$a_j = \begin{cases} \hat{\phi}(i,i-s^*+1-j)\ , & \text{for } 0 \le j \le i-s^*+1\ , \\ 0\ , & \text{for } i-s^*+1 < j \le k-s^*, \end{cases}$$

and

$$b_j = \hat{\phi}(k,k-s^*+1-j), \qquad \text{for } 0 \le j \le k-s^*.$$

Define

$$f(j) = v_\alpha^*(s^*-1+j) \text{ and } g(j) = K+v_\alpha^*(s^*-1+j),\quad 0 \le j \le k-s^*,$$

then

$$\sum_{j=0}^{k-s^*} a_j f(j) = \sum_{j=0}^{i-s^*+1} v_\alpha^*(i-j)\hat{\phi}(i,j);\ \sum_{j=0}^{k-s^*} b_j g(j) = \sum_{j=1}^{k-s^*+1} \{K+v_\alpha^*(k-j)\}\hat{\phi}(k,j).$$

Using condition (5.1), the relation $J_\alpha(h) = v_\alpha^*(h)$ for $h \geq s^*$, the induction hypothesis and the relation $v_\alpha^*(s^*-1) = 0 \leq K+v_\alpha^*(h)$, $h \geq s^*$ (cf. (b) and (c)), it is straightforward to verify that lemma 5.3 can be applied, and hence $v_\alpha^*(i) \leq K+v_\alpha^*(k)$.

The case $\phi(i,0) \leq \phi(k,0)$ can be treated in an analogous way. In a similar way as (5.10), we obtain the inequality

$$\{v_\alpha^*(i)-v_\alpha^*(k)\}\{1-\alpha\phi(i,0)\} \leq K\{1-\alpha\phi(i,0)\} + \alpha \sum_{j=1}^{i-s^*+1} v_\alpha^*(i-j)\hat{\phi}(i,j) +$$

$$- \alpha \sum_{j=0}^{k-s^*+1} \{K+v_\alpha^*(k-j)\} \hat{\phi}(k,j) \ ,$$

where $\hat{\phi}(i,j)$ and $\hat{\phi}(k,j)$ are the same as before except $\hat{\phi}(k,0) = \phi(k,0) - \phi(i,0)$. From this inequality and lemma 5.3 it follows easily that $v_\alpha^*(i) \leq K+v_\alpha^*(k)$. This ends the proof.

Theorem 5.2.

(a) The set of numbers $\{a_\alpha^*, v_\alpha^*(i)\}$, $i \in I$, satisfies

$$(5.11) \quad v_\alpha^*(i) = \min_{k \geq i}\{K\delta(k-i)+G_\alpha(k) - a_\alpha^* + \alpha \sum_{j=0}^{\infty} v_\alpha^*(k-j)\phi(k,j)\}, \ i \in I,$$

where the right side of (5.11) is minimized by $k = S^*$ for $i < s^*$ and by $k = i$ for $i \geq s^*$.

(b) $\quad s_1 \leq s^* \leq S_0$ and $s^* \leq S^* \leq S_1$.

Proof.

(a) By (5.4) we have for any $i \in I$ that

$$K\delta(k-i) + G_\alpha(k) - a_\alpha^* + \alpha \sum_{j=0}^{\infty} v_\alpha^*(k-j)\phi(k,j) = K\delta(k-i) + J_\alpha(k), \quad k \geq i.$$

Consider $K\delta(k-i) + J_\alpha(k)$ for i fixed and $k \geq i$. Distinguish the cases $i < s^*$ and $i \geq s^*$.

1. $i < s^*$. By theorem 5.1(a), 5.1(b) and 5.1(c) we have that

$$J_\alpha(i) \geq J_\alpha(s^*-1) \geq K+J_\alpha(S^*) = \min_{k>i}\{K+J_\alpha(k)\}.$$

Hence the right side of (5.11) is minimized by $k = S^*$ for $i < s^*$. By theorem 5.1(b) and (5.2) we have $K+J_\alpha(S^*) = v_\alpha^*(i)$ for $i < s^*$. This proves (a) for $i < s^*$.

2. $i \geq s^*$. By theorem 5.1(f) and (5.5) we have $K+J_\alpha(k) \geq J_\alpha(i) = v_\alpha^*(i)$ for $k \geq i \geq s^*$. This proves (a) for $i \geq s^*$.

(b) The relation $s_1 \leq s^* \leq S_0$ follows from lemma 5.1(e). The relation $S^* \leq S_1$ follows from theorem 5.1(b) and 5.1(d).

Suppose that the numbers m and M, which determine the class C(m,M) of policies, satisfy $m \leq s_1$ and $M \geq S_1$.

A direct consequence of theorem 3.2 and theorem 5.2 is the following theorem (see also theorem 5.3 of [5]).

<u>Theorem 5.3</u> (average cost criterion)

Let $\alpha = 1$, then

$$\min_{R\in C(m,M)} g(i;R) = a_1^* \qquad \text{for all } i \in I.$$

If $K = 0$, then the (S_0,S_0) policy is optimal. If $K > 0$, then any (s,S) policy such that $a_1(s,S) = a_1^*$ and $G_1(s-1) \geq a_1 \geq G_1^*(s)$ is optimal and has the property $s_1 \leq s \leq S_0$ and $S \leq S_1$. If $\phi(k,1) > 0$ for $k \leq S$, then $a_1(s,S) = a_1^*$ implies $G_1(s-1) \geq a_1^* \geq G_1(s)$.

Next consider the case $\alpha < 1$. From theorem 4.1 and (5.2) we have

$$v_\alpha^*(i) = V_\alpha(i;(s^*,S^*)) - \frac{a_\alpha^*}{1-\alpha}, \qquad i \in I,$$

and hence by theorem 5.2(a) we have

$$V_\alpha(i;(s^*,S^*)) = \min_{k\geq i}\{K\delta(k-i)+G_\alpha(k) + \alpha \sum_{j=0}^{\infty} V_\alpha(k-j;(s^*,S^*))\phi(k,j)\},\ i \in I,$$

where the right side of this equality is minimized by $k = S^*$ for $i < s^*$. A direct consequence of this relation and theorem 3.1 is the following theorem (see also theorem 5.4 of [5]).

Theorem 5.4. (the total discounted cost criterion)

Let $\alpha < 1$, then

$$\min_{R\in C(m,M)} V_\alpha(i;R) = V_\alpha(i;(s^*,S^*)), \qquad i \in I.$$

If $K = 0$, then the (S_0,S_0) policy is optimal. If $K > 0$, then any (s,S) policy such that $a_\alpha(s,S) = a_\alpha^*$ and $G_\alpha(s-1) \geq a_\alpha^* \geq G_\alpha(s)$ is optimal *) and has the property $s_1 \leq s^* \leq S_0$ and $S^* \leq S_1$. If $\phi(k,1) > 0$ for $k \leq S$, then $a_\alpha(s,S) = a_\alpha^*$ implies $G_\alpha(s-1) \geq a_\alpha^* \geq G_\alpha(s)$.

Remark.

When we require that condition (5.1) holds for all $s_1 \leq i < j$ and $0 \leq h \leq i$, then

$$S_0 \leq S^* .$$

We indicate the proof briefly. First, it can be proved that $J_\alpha(k)$ is nonincreasing on $(-\infty,S_0]$. Using this property of $J_\alpha(k)$ it can be shown with the aid of lemma 5.3 and theorem 5.1(b),(c) that $S_0 \leq S^*$. The proof that $J_\alpha(k)$ is nonincreasing on $(-\infty,S_0]$ runs as follows. By theorem 5.1(a), 5.1(b) and 5.1(e) we have that $J_\alpha(k)$ is nonincreasing on $(-\infty,s^*]$. Consider the case $s^* < S_0$. Assuming that $J_\alpha(k)$ is nonincreasing on $(-\infty,i]$ for some integer i with $s^* \leq i < S_0$, we shall demonstrate that $J_\alpha(i+1) \leq J_\alpha(i)$.

*) This optimality criterion has been already found by Johnson [2, p.89], and hence it is stated wrongly in [5] that this result is new.

The proof is a modification of a proof in [2, pp. 83-84]. Distinguish between $\Phi^{(1)}(i,i-s^*) \leq \phi(i+1,0)$ and $\Phi^{(1)}(i,i-s^*) \geq \phi(i+1,0)$. First consider the case $\Phi^{(1)}(i,i-s^*) \leq \phi(i+1,0)$. Since $G_\alpha(.)$ is nonincreasing on $[s^*,S_0]$ we have by (5.6), theorem 5.1(e) and the induction hypothesis that

$$J_\alpha(i) - J_\alpha(i+1) \geq \alpha J_\alpha(i)\ \Phi^{(1)}(i,i-s^*) - \alpha J_\alpha(i+1)\ \phi(i+1,0) \geq$$

$$\geq \alpha\phi(i+1,0)\ \{J_\alpha(i) - J_\alpha(i+1)\},$$

and hence $J_\alpha(i) \geq J_\alpha(i+1)$. Next consider the case $\Phi^{(1)}(i,i-s^*) \geq \phi(i+1,0)$. Using (5.1), (5.6), theorem 5.1(b) and 5.1(e) it is readily seen that

$$J_\alpha(i) - J_\alpha(i+1) \geq \alpha \sum_{j=0}^{i-s^*+1} J_\alpha(i-j)\ \hat{\phi}(i,j) + \alpha\phi(i+1,0)\ J_\alpha(i) +$$

$$- \alpha \sum_{j=1}^{i-s^*+2} J_\alpha(i+1-j)\ \hat{\phi}(i+1,j) - \alpha\phi(i+1,0)\ J_\alpha(i+1),$$

where $\hat{\phi}(i,0) = \Phi^{(1)}(i,i-s^*) - \phi(i+1,0)$, $\hat{\phi}(i,j) = \phi(i,j)$ for $1 \leq j \leq i-s^*$, $\hat{\phi}(i,i-s^*+1) = 1 - \Phi^{(1)}(i,i-s^*)$, $\hat{\phi}(i+1,j) = \phi(i+1,j)$ for $1 \leq j \leq i-s^*+1$, and $\hat{\phi}(i+1,i-s^*+2) = 1 - \Phi^{(1)}(i+1,i-s^*+1)$.
From above inequality and lemma 5.3 it can be easily deduced that $J_\alpha(i) \geq J_\alpha(i+1)$.

Remark

The conditions imposed on $G_\alpha(k)$ can be weakened in the same way as in remark 5.2 in section 5 of [5]. Furthermore we note that under Johnson's conditions for the case $\alpha < 1$ only the existence of an (s,S) policy can be shown for which the total expected discounted cost is minimal for each initial stock $i \leq S^0$, where S^0 is defined in [2, p.81]

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