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THE TRANSIENT BEHAVIOUR AND THE LIMITING DISTRIBUTION OF THE STOCK LEVEL IN A CONTINUOUS TIME (s,S) INVENTORY MODEL

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1. Introduction

Consider the situation that customers arrive at a store and that each customer demands for a single item. The times between arrivals are generated by a renewal process, where the interarrival times have a non-lattice distribution function. The demands of the customers are mutually independent, non-negative and identically distributed random variables with a discrete distribution, and independent of the arrival process. Excess demands are backlogged.

The storekeeper follows an (s,S) policy, i.e. when by the demand of a customer the stock on hand plus on order i falls below s, then S - i units are ordered; otherwise, no ordering is done. There can only be placed an order on the demand epochs. The lead time of an order is a constant $\tau \geq 0$. The numbers s and S are fixed integers with $s \leq S$.

We shall determine for both the stock in hand plus on order and the stock on hand the transient behaviour and the limiting distribution.

2. Preliminaries

Let $\underline{s}_1$, $\underline{s}_2$, ..., be a sequence of random variables such that the interarrival times $\underline{s}_k - \underline{s}_{k-1}$, k = 1, 2, ..., where $\underline{s}_0 = 0$, are mutually independent, positive and identically distributed random variables having the distribution function F(t). It is assumed that F(t) is non-lattice *) with F(0) = 0 and

$$\mu = \int_0^\infty tF(dt) < \infty.$$

Let $\underline{\xi}_1$, $\underline{\xi}_2$, ... be a sequence of mutually independent, non-negative and identically distributed random variables having the discrete distribution $\phi(j) = P\{\underline{\xi}_n = j\}$, $(j \geq 0;\ n \geq 1)$. The sequences $\{\underline{s}_k\}$ and $\{\underline{\xi}_k\}$ are assumed to be independent.

Let the random variable $\underline{s}_n$, $n \geq 1$, denote the epoch on which the n-th customer arrives, and let $\underline{\xi}_n$ denote the size of the demand of the

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*) A distribution function H(t) concentrated on $[0,\infty)$ is said to be lattice if there exist numbers $\alpha,\beta \geq 0$ such that all points of increase of H are among the numbers α, $\alpha+\beta$, $\alpha+2\beta$, ...

n-th customer.

We assume

$$0 < \sum_{j=0}^{\infty} j\phi(j) < \infty.$$

Denote by $F^{(n)}(t)$ and $\phi^{(n)}(j)$, respectively, the n-fold convolution of $F(t)$ and $\phi(j)$, i.e.

$$F^{(n)}(t) = \int_0^t F(t-y)\ F^{(n-1)}(dy) \qquad \text{for } n \geq 1;\ t \geq 0,$$

$$\phi^{(n)}(j) = \sum_{k=0}^{j} \phi(j-k)\phi^{(n-1)}(k) \qquad \text{for } n \geq 1;\ j \geq 0,$$

where $F^{(0)}(t) \equiv 1$ for $t \geq 0$, $\phi^{(0)}(0) = 1$ and $\phi^{(0)}(j) = 0$ for $j \geq 1$. Clearly, $F^{(n)}(t) = P\{\underline{s}_n \leq t\}$ and $\phi^{(n)}(j) = P\{\underline{\xi}_1 + \ldots + \underline{\xi}_n = j\}$.

Let

$$U(t) = \sum_{n=1}^{\infty} F^{(n)}(t) \text{ for } t \geq 0, \quad m(j) = \sum_{n=1}^{\infty} \phi^{(n)}(j) \text{ for } j \geq 0.$$

Since $F^{(n)}(t)$ and $\phi^{(n)}(j)$ converge exponentially fast to zero as $n \to \infty$, the functions $U(t)$ and $m(j)$ are finite, and further they satisfy [1]

$$U(t) = F(t) + \int_0^t F(t-y)U(dy), \qquad \text{for } t \geq 0;$$

$$m(j) = \phi(j) + \sum_{k=0}^{j} \phi(j-k)m(k), \qquad \text{for } j \geq 0.$$

Define $\underline{\xi}_0 = 0$. Let

$$\underline{N}(t) = \max\{n \mid \underline{s}_n \leq t\} \qquad \text{for } t \geq 0,$$

$$\underline{m}_k = \max\{n \mid \underline{\xi}_0 + \ldots + \underline{\xi}_n \leq k\} \qquad \text{for } k \geq 0.$$

Note $\underline{N}(t)$ is the number of arrivals in $(0,t]$ and $\underline{m}_k$ is the number of customers before the cumulative demand exceeds k.

It is well-known from renewal theory that [1]

$$\mathcal{E}\underline{N}(t) = U(t) \text{ for } t \geq 0, \quad \mathcal{E}\underline{m}_k = \sum_{j=0}^{k} m(j) \quad \text{for } j \geq 0.$$

For any $t \geq 0$, let

$$\underline{a}(t) = \underline{\xi}_0 + \ldots + \underline{\xi}_{\underline{N}(t)}.$$

Clearly, $\underline{a}(t)$ is the total demand in $(0,t]$. Since $P\{\underline{N}(t) = n\} = P\{\underline{s}_n \leq t < s_{n+1}\} = F^{(n)}(t) - F^{(n+1)}(t)$ for $n \geq 0$, we have

$$(1) \qquad a_k(t) \overset{\text{def}}{=} P\{\underline{a}(t) = k\} = \sum_{n=0}^{\infty} \phi^{(n)}(k)\{F^{(n)}(t) - F^{(n+1)}(t)\}$$

for $k \geq 0$; $t \geq 0$.

For any $k \geq 1$, let

$$(2) \qquad \underline{t}_k = \underline{s}_{\underline{m}_{k-1}+1}.$$

Clearly, $\underline{t}_k$ is the length of the time interval from $t = 0$ up to the epoch on which the cumulative demand exceeds $k - 1$ for the first time. Note that $\underline{t}_k = \underline{s}_k$ if $\phi(1) = 1$. We have by Wald's equation

$$(3) \qquad \mathcal{E}\underline{t}_k = \mathcal{E}\underline{s}_1 . \mathcal{E}(\underline{m}_{k-1}+1) = \mu\{1 + \sum_{j=0}^{k-1} m(j)\}, \qquad \text{for } k \geq 1.$$

Let $\Phi^{(n)}(j) = \phi^{(n)}(0) + \ldots + \phi^{(n)}(j)$ for $j \geq 0$; $n \geq 0$. Since the processes $\{\underline{s}_k\}$ and $\{\underline{\xi}_k\}$ are independent, we have for each $k \geq 0$ that $\Phi^{(n)}(k) - \Phi^{(n+1)}(k)$ is the probability that the cumulative demand will first exceed k on epoch $\underline{s}_{n+1}$, $n \geq 0$, and hence

$$P\{\underline{t}_k \leq t\} = \sum_{n=0}^{\infty} \{\Phi^{(n)}(k-1) - \Phi^{(n+1)}(k-1)\} \; F^{(n+1)}(t),$$

for $t \geq 0$; $k \geq 1$.

Using this formula and using a result from the theory of characteristic functions which characterizes the lattice distributions [1] it is readily verified that for each $k \geq 1$ the distribution function of $\underline{t}_k$ is non-lattice.

We shall need the key renewal theorem. Before we state this theorem, we give the following definition [1]. A function $z(t)$, $t \geq 0$, is said to be directly Riemann integrable if for fixed $h > 0$ the two series $\sum_1^\infty m_n(h)$ and $\sum_1^\infty M_n(h)$ converge absolutely and if

$$\lim_{h\to 0} h \sum_{n=1}^{\infty} m_n(h) = \lim_{h\to 0} h \sum_{n=1}^{\infty} M_n(h),$$

where we denote by $m_n(h)$ and $M_n(h)$, respectively, the largest and the smallest number such that $m_n(h) \leq z(t) \leq M_n(h)$ for $(n-1)h \leq t < nh$.

We note that any directly Riemann integrable function is also Riemann integrable over $[0,\infty)$ in the ordinary sense. It is easily seen that a non-negative function $z(t)$, $t \geq 0$, is directly Riemann integrable over $[0,\infty)$ if it is integrable over every finite interval $[0,a)$ and if $\sum_1^\infty M_n(h) < \infty$ for some $h > 0$. Hence a monotone function is directly Riemann integrable if it is Riemann integrable in the ordinary sense.

We now state the key renewal theorem [1]

<u>Key renewal theorem</u>

If the function $z(t)$, $t \geq 0$, is directly Riemann integrable and if $F(t)$ is non-lattice, then

$$\lim_{t\to\infty} \int_0^t z(t-y)\; U(dy) = \frac{1}{\mu} \int_0^\infty z(t)dt.$$

<u>Lemma 1</u>

For each $k \geq 0$ the function $a_k(t)$, $t \geq 0$, is directly Riemann integrable and

$$\int_0^\infty a_k(t)dt = \mu\{\phi^{(0)}(k) + m(k)\}, \qquad \text{for } k \geq 0.$$

Proof

For any $k \geq 0$, let

$$g_k(t) = \sum_{n=0}^{\infty} \phi^{(n)}(k) \{1 - F^{(n+1)}(t)\}, \qquad t \geq 0.$$

By (1), we have for any $k \geq 0$ that $0 \leq a_k(t) \leq g_k(t)$ for $t \geq 0$. It is well-known that if $H(t)$ is a distribution function concentrated on $[0,\infty)$ and having a finite expectation α, then

$$\int_0^{\infty} \{1 - H(t)\}dt = \alpha,$$

and hence

$$(4) \qquad \int_0^{\infty} \{1 - F^{(n+1)}(t)\}dt = (n+1)\mu, \qquad \text{for } n \geq 0.$$

Thus, since $\phi^{(n)}(k)$ converges exponentially fast to zero as $n \to \infty$ for each $k \geq 0$, the non-increasing function $g_k(t)$, $t \geq 0$, is Riemann integrable over $[0,\infty)$ in the ordinary sense. Hence, by a well-known result from analysis, we have for each $k \geq 0$ that $\sum_{n=0}^{\infty} g_k(n) < \infty$. Since $0 \leq a_k(t) \leq g_k(n)$ for $n \leq t < n+1$, it is readily seen that $a_k(t)$ is directly Riemann integrable. Moreover, we have by (1) and (4) that

$$\int_0^{\infty} a_k(t)dt = \sum_{n=0}^{\infty} \phi^{(n)}(k)\mu = \mu\{\phi^{(0)}(k) + m(k)\}, \quad \text{for } k \geq 0.$$

This ends the proof.

Finally let

$$\underline{y}(t) = \underline{s}_{\underline{N}(t)+1} - t, \qquad \text{for } t > 0.$$

The random variable $\underline{y}(t)$ is the length of the time interval between time t and the first demand epoch occurring after t. It is well-known that [1]

$$P\{\underline{y}(t) \leq u\} = F(t+u) - F(t) + \int_0^t \{F(t+u-y) - F(t-y)\}\, U(dy),$$

$$u > 0;\ t \geq 0,$$

and

$$(5) \qquad \lim_{t\to\infty} P\{\underline{y}(t) \leq u\} = \frac{1}{\mu}\int_0^u \{1 - F(x)\}dx, \qquad u > 0.$$

3. The transient behaviour and the limiting distribution of the stock level

The ordering policy followed is an (s,S) policy, i.e. when on a demand epoch the stock on hand plus on order i falls below s, order then S - i units; otherwise, no ordering is done. The lead time of an order is a constant $\tau \geq 0$.

Denote by $\underline{z}_t$ the stock on hand plus on order at time t, where on the demand epochs, $\underline{z}_t$ is measured just after ordering (if any). *)

For any $t \geq 0$, let

$$p_{ij}(t) = P\{\underline{z}_t = j \mid \underline{z}_0 = i\}, \qquad \text{for } i,\ j \geq s.$$

For any $i \geq s$, denote by $G_i(t)$ the distribution function of the random variable $\underline{t}_{i-s+1}$ (cf. formula (2)). Using a standard argument from renewal theory, we have for any $t \geq 0$ that

$$(6) \qquad p_{ij}(t) = a_{i-j}(t) + \int_0^t p_{Sj}(t-u)G_i(du), \quad \text{for all } i,\ j \geq s,$$

where $a_k(t) = 0$ for $k < 0$; $t \geq 0$. In particular we have

$$(7) \qquad p_{Sj}(t) = a_{S-j}(t) + \int_0^t p_{Sj}(t-u)G_S(du), \quad \text{for } s \leq j \leq S;\ t \geq 0.$$

Clearly

$$(8) \qquad p_{Sj}(t) = 0 \qquad \text{for } j > S;\ t \geq 0.$$

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*) It is interesting to note that the stock on hand plus on order on the demand epochs just after ordering behaves exactly as the stock on hand plus on order just after ordering in the classical, periodic review (s,S) inventory model.

For each $j \in [s,S]$ the equation (7) is a renewal equation. Let $G_S^{(n)}(t)$ be the n-fold convolution of $G_S(t)$, and let the renewal function $V(t)$ be defined by

$$V(t) = \sum_{n=1}^{\infty} G_S^{(n)}(t), \qquad t \geq 0.$$

For each $k \geq 0$ the function $a_k(t)$, $t \geq 0$, satisfies $0 \leq a_k(t) \leq 1$ (cf. (1)). Iterating (7) and using the fact that $G_S^{(n)}(u) \to 0$ as $n \to \infty$ for each u, yields

$$(9) \qquad p_{Sj}(t) = a_{S-j}(t) + \int_0^t a_{S-j}(t-u)V(du), \quad \text{for } s \leq j \leq S;\ t \geq 0.$$

The formulas (6), (8) and (9) in conjunction yield the time dependent solution of the distribution of the stock on hand plus on order.

We shall next determine the limiting distribution of the stock on hand plus on order. Therefore we note that the distribution function $G_S(t)$ is non-lattice, as already proved in section 2. Further we have for each $k \geq 0$ that $a_k(t) \to 0$ as $t \to \infty$. From the key renewal theorem, lemma 1, (3) and (9), it follows now that

$$(10) \qquad \lim_{t\to\infty} p_{Sj}(t) = \frac{1}{\mathcal{E}\underline{t}_{S-s+1}} \int_0^\infty a_{S-j}(u)du = \frac{\phi^{(0)}(S-j)+m(S-j)}{1+M(S-s)}, \qquad s \leq j \leq S,$$

where

$$M(k) = m(0) + \ldots + m(k), \qquad k \geq 0.$$

From (6), (8) and (10), it follows that

$$(11) \quad \lim_{t\to\infty} p_{ij}(t) = \begin{cases} \{\phi^{(0)}(S-j)+m(S-j)\}/\{1+M(S-s)\}, & \text{for all } i \geq s;\ s \leq j \leq S \\ 0 & \text{, for all } i \geq s;\ j > S. \end{cases}$$

Let $q_j = \lim_{t\to\infty} p_{ij}(t)$ for i, j $\geq$ s. We note that if $\phi(1) = 1$, then

$$q_j = 1/(S-s+1), \qquad \text{for } s \leq j \leq S.$$

This result has been obtained in [3] for the case that the arrival process is a Poisson proces, although the derivation in [3] needs an additional argument. If $\phi(j)$ has a geometric distribution, i.e. $\phi(j) = p(1-p)^{j-1}$ for $j \geq 1$, where $0 < p \leq 1$, then $m(j) = p$ for $j \geq 1$, and hence

$$q_j = \begin{cases} 1/\{1 + (S-s)p\}, & \text{for } j = S, \\ p/\{1 + (S-s)p\}, & \text{for } s \leq j < S. \end{cases}$$

This result has been obtained in [5] for the case the arrival process is a Poisson process. Moreover, we note that in [2,4] a number of results for the distribution of the stock level are obtained for the case $\phi(1) = 1$ and the lead time is random.

We shall next determine the distribution of the stock on hand. For any $t \geq 0$, let

$$b_k(t,\tau) = P\{\underline{a}(t+\tau) - \underline{a}(t) = k\}, \qquad \text{for } k \geq 0,$$

i.e. $b_k(t,\tau)$ is the probability that the total demand in $(t,t+\tau]$ will be k. Note that $b_k(t,\tau) = a_k(\tau)$ if the renewal process $\{\underline{s}_k\}$ is a Poisson process.

For any $t \geq \tau$, let $r_{ij}(t)$ be the probability that the stock on hand at time t will be j, given that $\underline{z}_0 = i$. Since anything on order at time t will have arrived by time t + τ and since anything ordered after time t will arrive after time t + τ, we have for any $t \geq 0$ that

$$(12) \qquad r_{ij}(t+\tau) = \sum_{k=s}^{\max(i,S)} p_{ik}(t) b_{k-j}(t,\tau), \quad \text{for } i \geq s;\ j \in I,$$

where I is the set of all integers and $b_k(t,\tau) = 0$ for $k < 0$.

Using the definition of the random variable $\underline{y}(t)$ (cf. section 2), it is readily seen that

$$(13) \qquad b_0(t,\tau) = P\{\underline{y}(t) > \tau\} + \int_0^\tau \{\phi(0) + a_0(\tau-u)\}\, dP\{\underline{y}(t) \leq u\}, \qquad t \geq 0,$$

and

$$(14) \qquad b_k(t,\tau) = \sum_{j=0}^{k} \int_0^\tau \{\phi(j) + a_{k-j}(\tau-u)\}\, dP\{y(t) \leq u\}, \qquad k \geq 1;\ t \geq 0.$$

By (12), (13) and (14) the transient behaviour of the stock on hand is determined.

Finally, we determine the limiting distribution of the stock on hand. From (5), it follows that

$$(15) \qquad b_0(\tau) \overset{\text{def}}{=} \lim_{t\to\infty} b_0(t,\tau) =$$

$$= \frac{1}{\mu}\int_\tau^\infty \{1 - F(u)\}du + \frac{1}{\mu}\int_0^\tau \{\phi(0) + a_0(\tau-u)\}\{1 - F(u)\}du,$$

and

$$(16) \qquad b_k(\tau) \overset{\text{def}}{=} \lim_{t\to\infty} b_k(t,\tau) =$$

$$= \frac{1}{\mu}\sum_{j=0}^{k}\int_0^\tau \{\phi(j) + a_{k-j}(\tau-u)\}(1-F(u))du, \quad k \geq 1.$$

From (11), (12), (15) and (16), it follows

$$\lim_{t\to\infty} r_{ij}(t) = \sum_{k=s}^{S} q_k b_{k-j}(\tau), \qquad i \geq s;\ j \in I,$$

where $b_k(\tau) = 0$ for $k < 0$.

Remark

The results of this paper can be adapted to any general demand distribution. Further, the case that F has a lattice distribution can be solved in a similar way (in [6] the case is treated that F is degenerate).

By imposing a cost structure on the inventory model, the average expected cost per unit time for an (s,S) policy can be determined.

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