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A PRIORITY QUEUEING SYSTEM WITH REMOVABLE SERVER

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ABSTRACT

A non-preemptive priority queueing system is considered in which customers of types 1 and 2 arrive at a service station with a single server. The station is closed down when it becomes empty and the station is reopened when a certain number of customers are present. It is assumed that both the closing-down and the reopening of the station take up time. Two models, A and B, are considered. In model A the closing-down process is interrupted when a new customer arrives, whereas in model B this is not the case. For both models expressions are derived for the average number of customers of type i(=1,2) in the system and the average wait of a customer of type i. A cost structure is imposed on the model and optimization is done. Finally, the models A and B are extended by assuming that after a service completion the server is temporarily not available.

Consider a service station with a single server at which customers of types 1 and 2 arrive in accordance with independent Poisson processes with rates λ_1 and λ_2, respectively. If the server is to select a customer for service, customers of type 1 have priority over customers of type 2. The order in which customers of a given priority class are served is immaterial in our considerations. The priority rule is non-preemptive, i.e., a service of a customer is never interrupted. A customer of type i will be called an i-customer, (i=1,2). Let the service times of different customers be independent random variables with finite first moment μ_i and finite second moment $\mu_i^{(2)}$ for i-customers. Let $\lambda = \lambda_1 + \lambda_2$ and let $\rho_i = \lambda_i \mu_i$, (i=1,2). It is assumed that $\rho_1 + \rho_2 < 1$. The service station will be reopened and closed down from time to time. When the service station is reopened a random time τ_a (the set-up time) will elapse before the server can start servicing. It is assumed that $E\tau_a$ and $E\tau_a^2$ are finite. A decision is taken to close down the service station if, and only if, a service is completed while no customers are awaiting for service. The time needed to finish the closing-down of the station is a random variable τ_b with distribution function G(t) and finite expectation $E\tau_b$. We have to make an assumption regarding the contingency of a customer who arrives while the station is being closed down. We shall consider two alternative models.

In model A it is assumed that on arrival of a new customer the closing-down process is interrupted and the service of this customer commences immediately. Further the time already spent on closing-down in the present attempt is wasted, and so the next attempt will be repeated from the beginning. When the closing-down of the station has been successfully concluded, the station will be reopened at the next epoch at which R customers are at the station, where R is a given positive integer.

In model B it is assumed that the closing-down process is never interrupted and so a customer who arrives while the station is being closed down has to wait at least until the station will be reopened. When the closing-down of the station has been finished, the station will be reopened at the next epoch at which R or more customers are at the station. In model B we also assume that $E\tau_b^2$ is finite. In both models it is assumed that the service times, the set-up times and the close-down times are independent of each other and the arrival processes.

Model A is an extension of a model studied by YADIN and NAOR [10]. These authors assumed one type of customer and derived expressions for the average number of customers in the system and the average wait of a customer.

In this paper we shall derive for the models A and B expressions for the average number of i-customers in the system (queue) and the average amount of time spent by an i-customer in the system (queue). As a by-product we obtain simple and alternative derivations of both COBHAM's formula in non-preemptive priority queueing with two priority classes and the results of YADIN and NAOR. Further, we superimpose a cost structure on the system and optimization will be done. Finally, after we have analysed the models A and B, we incorporate in these models block-times, that is, after completion of a service the server is blocked during a random time before he can commence a new service or close down the station. In references 2 and 8 also models with block-times are studied.

The approach we will follow to analyse the models A and B is quite general and may be applied to a variety of models. This approach, which has been also followed by JEWELL [4] in his proof of the fundamental formula in queueing theory $L = \lambda W$, is based on a simple renewal theoretic argument.

APPROACH

Let us define the amount of time spent by a customer in the queue as the time he awaits for service, and let the time spent by a customer in the system be defined as the time he spends in the queue plus his service time. Correspondingly, the number of customers in the queue and in the system may be defined.

For convenience we assume that at epoch 0 a service has been just completed and no customers are in the system. We define a *cycle* as the time interval between two successive epochs at which for the first time after a reopening of the station no customers are in the system. Observe that for both model A and model B such epochs are regeneration epochs for the queueing process. We shall show that the expected length of a cycle and the expected total amount of time spent by i-customers in the system during one cycle are finite. We now have that the long-run (expected) average number of i-customers in the system equals, with probability one,

$$L^{(i)} = \{\textit{the expected total amount of time spent by i-customers in the system during one cycle}\} \,/\, \{\textit{expected length of a cycle}\}. \qquad (1)$$

This may be seen as follows. Fix i and imagine costs are incurred for i-customers only, where the cost incurred for an i-customer equals the amount of time spent by that customer in the system. Now, by a well known result in renewal theory (see, for instance, reference 7, p.52), the long-run (expected) average cost per unit time equals, with probability one, the quotient of the expected total cost incurred during one cycle and the expected length of a cycle. This gives (1), since the average number of i-customers can be thought of as the average cost per unit time.

The technique which will be used to determine the expected length of a cycle and the expected total amount of time spent by i-customers in the system during one cycle is an adaptation of a technique introduced by TAKÁCS (see reference 9 p.32 and p.61) to determine the distribution of the busy period in the classical single server queue. The determination of the above expectations is based upon the observation that the length of a cycle and the total amount of time spent by i-customers in the system during one cycle do not depend on the order in which customers of a given priority class are served.

When we have determined $L^{(i)}$, it is easy to obtain expressions for $L_q^{(i)}$ (the average number of i-customers in the queue), $W_q^{(i)}$ (the average amount of time spent by i-customers in the queue) and $W = W_q^{(i)} + \mu_i$ (the average amount of time spent by i-customers in the system). Using the results we shall find below, it is easily verified that the assumptions stated in JEWELL's paper [4] are satisfied so that the formulae $L^{(i)} = \lambda_i W^{(i)}$ and $L_q^{(i)} = \lambda_i W_q^{(i)}$ apply.

BASIC MODEL

In order to analyse the models A and B we first consider the simple model in which the set-up time τ_a and the close-down time τ_b are equal to zero with probability one and R = 1. That is, we consider the classical non-preemptive priority model with two priority classes. For this model we introduce the following random variables from which the expectations will be needed in the sequel.

T_{bi} = the time elapsed from the arrival of an i-customer who finds the server idle until the next epoch at which the server becomes idle, (i=1,2).

T_1 = the time elapsed from the arrival of the 1-customer who initiates the busy period T_{b1} until the next epoch at which no 1-customers are in the system.

W_{ik} = the total amount of time spent by i-customers in the system during the busy period T_{bk}, (i,k=1,2).

W_1 = the total amount of time spent by 1-customers in the system during the time T_1.

To determine the expectations of these random variables, we define

S_i =the service time of the i-customer who initiates the busy period T_{bi}, (i=1,2).

N_{1i} = the number of 1-customers who arrived during the time S_i, (i=1,2).

U_2 = the time elapsed from the completion of the service of the 2-customer who initiates T_{b2} until the next epoch at which no 1-customers are in the system.

M_{2i} = the number of 2-customers in the system just after the first epoch in the busy period T_{bi} at which a service is completed while no 1-customers are in the system.

Observe that the distributions of the above random variables do not depend on the order in which customers of a given class are served. Further, we shall frequently use the following property of the Poisson process. Given that n events of a Poisson process have occurred during (0,s), then the n epochs at which events occur are distributed independently and uniformly on (0,s).

We will need ET_1 and ET_1^2. Since 1-customers have priority and any 1-customer arriving in S_1 creates a busy period of type T_1, we have $E(T_1|S_1=s,N_{11}=n) = s + nET_1$ and $E(T_1^2|S_1=s,N_{11}=n) = E(s+T_n)^2$, where T_n is distributed as the sum of n independent random variables which are distributed as T_1. From this and the fact that the conditional distribution of

N_{11} given that $S_1 = s$ is Poisson with mean $\lambda_1 s$, we find the well known result [6,9]

$$ET_1 = \mu_1 / (1-\rho_1) \quad \text{and} \quad ET_1^2 = \mu_1^{(2)} / (1-\rho_1)^3. \tag{2}$$

Similarly, by $E(W_1|S_1=s, N_{11}=n) = s + ns/2 + nEW_1 + (1/2)\, n(n-1)\, ET_1$, we have $E(W_1|S_1=s) = s + \lambda_1 s^2/2 + \lambda_1 sEW_1 + \lambda_1^2 s^2 ET_1/2$, from which we get

$$EW_1 = \mu_1 / (1-\rho_1) + \lambda_1\mu_1^{(2)} / 2(1-\rho_1)^2. \tag{3}$$

Since $ET_{b1} = ET_1 + \lambda_2 ET_1 ET_{b2}$, $E(T_{b2}|S_2=s, N_{12}=n) = s + nET_1 + \lambda_2\,(s+ +nET_1)\, ET_{b2}$ and the conditional distribution of N_{12} given that $S_2 = s$ is Poisson with mean $\lambda_1 s$, we find after some algebra the well known result [6]

$$ET_{bi} = \mu_i / (1-\rho_1-\rho_2) \qquad \text{for } i = 1,2. \tag{4}$$

To determine EW_{11} and EW_{12}, we observe that

$$EW_{11} = EW_1 + \lambda_2 ET_1 EW_{12} \quad \text{and} \quad E(W_{12}|S_2=s, N_{12}=n, M_{22}=k) = ns/2 + nEW_1 + + (1/2)\, n(n-1)\, ET_1 + kEW_{12}. \tag{5}$$

Since $E(M_{22}|S_2=s, N_{12}=n) = \lambda_2(s+nET_1)$ and the conditional distribution of N_{12} given that $S_2 = s$ is Poisson with mean $\lambda_1 s$, we find after some algebra

$$EW_{11} = \{1/(1-\rho_1-\rho_2)\}\, \{\mu_1^{(2)}\lambda_1(1-\rho_2)/2(1-\rho_1) + \mu_2^{(2)}\lambda_2\rho_1/2(1-\rho_1) + \mu_1(1-\rho_2)\}, \tag{6}$$

and

$$EW_{12} = \{1/(1-\rho_1-\rho_2)\}\, \{\mu_1^{(2)}\lambda_1^2\mu_2/2(1-\rho_1) + \mu_2^{(2)}\lambda_1/2 + \rho_1\mu_2\}. \tag{7}$$

Since $E(W_{21}|T_1=t, M_{21}=k) = kt/2 + kEW_{22} + (1/2)\, k(k-1)\, ET_{b2}$ and the conditional distribution of M_{21} given that $T_1 = t$ is Poisson with mean $\lambda_2 t$, we find

$$EW_{21} = \lambda_2 ET_1^2/2 + \lambda_2 ET_1 EW_{22} + \lambda_2^2 ET_1^2 ET_{b2}/2. \tag{8}$$

From $E(W_{22}|S_2=s,N_{12}=n,U_2=t,M_{22}=k) = s + k(s+t)/2 + kEW_{22} + (1/2)\, k(k-1)\, ET_{b2}$ and the fact that the conditional distribution of M_{22}, given that $S_2 = s$ and $U_2 = t$, is Poisson with mean $\lambda_2(s+t)$, we get

$$E(W_{22}|S_2=s,N_{12}=n,U_2=t) = s + \lambda_2(s+t)^2/2 + \lambda_2(s+t)EW_{22} + \lambda_2^2(s+t)^2 ET_{b2}/2.$$

Given that $N_{12} = n$, the random variable U_2 has the same distribution as the sum of n independent random variables which are distributed as T_1, so

$$E(W_{22}|S_2=s,N_{12}=n) = s + (1/2)\, \{\lambda_2+\lambda_2^2 ET_{b2}\}\, \{s^2+2snET_1 + n(n-1)(ET_1)^2 + nET_1^2\} + \\ + \lambda_2(s+nET_1)EW_{22}.$$

Using that the conditional distribution of N_{12} given that $S_2 = s$ is Poisson with mean $\lambda_1 s$, we find after some simple manipulations that

$$EW_{22} = \{1/(1-\rho_1-\rho_2)\}\, \{\mu_1^{(2)}\lambda_1\rho_2/2(1-\rho_1)(1-\rho_1-\rho_2) + \mu_2^{(2)}\lambda_2/2(1-\rho_1-\rho_2) + \\ + (1-\rho_1)\mu_2\}. \qquad (9)$$

From (2), (4), (8) and (9),

$$EW_{21} = \{1/(1-\rho_1-\rho_2)\}\, \{\mu_1^{(2)}\lambda_2(1-\rho_2)/2(1-\rho_1)(1-\rho_1-\rho_2) + \\ + \mu_2^{(2)}\lambda_2^2\mu_1/2(1-\rho_1)(1-\rho_1-\rho_2) + \rho_2\mu_1\}. \qquad (10)$$

Remark. Let us define for the above model a cycle as the time interval between two successive epochs at which the server becomes idle. Using that λ_i/λ represents the probability that an arbitrary customer is an i-customer, it follows that the expected length of a cycle equals $1/\lambda + (\lambda_1/\lambda)ET_{b1} + (\lambda_2/\lambda)ET_{b2}$ and that the expected total amount of time spent by i-customers in the system during one cycle equals $(\lambda_1/\lambda)EW_{i1} + (\lambda_2/\lambda)EW_{i2}$. Since the long-run (expected) average number of i-customers in the system equals, with probability one, the quotient of the expected total amount of time spent by i-customers in the system during one cycle and the expected length of a cycle, we find that this average is given by

$$\rho_i + \{\lambda_i(\lambda_1\mu_1^{(2)} + \lambda_2\mu_2^{(2)})\} \,/\, 2(1-\rho_1)(1-\rho_1-\rho_2-\rho_i^*) \;,$$

where $\rho_1^* = \rho_1 + \rho_2$ and $\rho_2^* = 0$. This formula is well known [3, 5, 6].

MODEL A: CLOSING-DOWN PROCESS WITH INTERRUPTIONS.

We have defined a cycle as the time interval between two successive epochs at which for the first time after a reopening of the station no customers are in the system. Denote by the random variable T_c the length of a cycle and denote by $W_c(i)$ the total amount of time spent by i-customers in the system during one cycle. To determine ET_c and $EW_c(i)$, let

$$\pi = \int_0^\infty e^{-\lambda t}\, G(dt).$$

Then $1 - \pi$ represents the probability that an attempt to close down the station is interrupted by the arrival of a new customer. The number of unsuccessful attempts within one cycle to complete the closing-down of the station is a geometrically distributed random variable N with mean

$$\beta = (1-\pi)/\pi.$$

The gross close-down time per cycle is defined as the sum of the N partial, interrupted close-down times and the final successful close-down time. The expected gross close-down time equals β/λ [1,10]. Further the expected amount of time elapsed from the arrival of a customer who interrupts the closing-down process until the next epoch at which no customers are in the system equals $(\lambda_1/\lambda)ET_{b1} + (\lambda_2/\lambda)ET_{b2}$. Hence the expected amount of time elapsed from the first attempt in a cycle to close down the station until the next epoch at which the station is reopened equals

$$\beta/\lambda+\beta\{(\lambda_1/\lambda)ET_{b1} + (\lambda_2/\lambda)ET_{b2}\}+ R/\lambda = \beta/\lambda(1-\rho_1-\rho_2) + R/\lambda. \qquad (11)$$

Since the probability that k customers of type 1 (and so R - k customers of type 2) are at the station when it is reopened equals $\binom{R}{k}(\lambda_1/\lambda)^k(\lambda_2/\lambda)^{R-k}$, we find that the expected amount of time elapsed from a reopening of the station until the next epoch at which no customers are in the system is given by

$$\sum_{k=0}^{R} \binom{R}{k}(\lambda_1/\lambda)^k(\lambda_2/\lambda)^{R-k} [E\tau_a + (k+\lambda_1 E\tau_a)ET_1 + \{R - k +$$

$$+ \lambda_2(E\tau_a+(k+\lambda_1 E\tau_a)ET_1)\} ET_{b2}] = \{R(\rho_1+\rho_2) + \lambda E\tau_a\} / \lambda(1-\rho_1-\rho_2). \qquad (12)$$

From (11) and (12),

$$ET_c = (R+\lambda E\tau_a+\beta) / \lambda(1-\rho_1-\rho_2). \qquad (13)$$

To determine $EW_c(i)$, let us first observe that the expected total amount of time spent by i-customers in the system during the time elapsed from the first attempt in a cycle to close down the station until the next epoch at which the station is reopened equals

$$\beta\{(\lambda_1/\lambda)EW_{i1} + (\lambda_2/\lambda)EW_{i2}\} + (\lambda_i/\lambda)(1/2)\ R(R-1)\ /\ \lambda. \qquad (14)$$

Denote by the random variable U_{ik} the total amount of time spent by i-customers in the system during the time elapsed from a reopening of the station until the next epoch at which no customers are in the system, given that k customers of type 1 are at the station when it is reopened. Let ν_1 be the number of 1-customers arriving during the set-up time τ_a. Then

$$E(U_{1k}|\tau_a=s,\nu_1=n) = ks + ns\ /\ 2 + (k+n)\ EW_1 + (1/2)(k+n)(k+n-1)\ ET_1 +$$
$$+ [R - k + \lambda_2\ \{s+(k+n)ET_1\}]\ EW_{12},$$

from which we get

$$EU_{1k} = kE\tau_a + \lambda_1 E\tau_a^2\ /\ 2 + (k+\lambda_1 E\tau_a)\ EW_1 + (1/2)\ \{\lambda_1^2 E\tau_a^2 + 2k\lambda_1 E\tau_a +$$
$$+ k(k-1)\}\ ET_1 + [R - k + \lambda_2\ \{E\tau_a+(k+\lambda_1 E\tau_a)\ ET_1\}]\ EW_{12},$$

since the conditional distribution of ν_1 given that $\tau_a = s$ is Poisson with mean $\lambda_1 s$. Using (2), (3) and the first part of (5), we find after some algebra that the expected total amount of time spent by 1 - customers in the system during the time elapsed from a reopening of the station until the next epoch at which no customers are in the system is given by

$$\sum_{k=0}^{R}\binom{R}{k}(\lambda_1/\lambda)^k(\lambda_2/\lambda)^{R-k}\ EU_{1k} = (R+\lambda E\tau_a)\ \{(\lambda_1/\lambda)\ EW_{11} + (\lambda_2/\lambda)\ EW_{12}\} +$$
$$+ \{\lambda_1/(1-\rho_1)\}\ \{R(R-1)\rho_1\ /\ 2\lambda^2 + RE\tau_a\ /\ \lambda + E\tau_a^2\ /\ 2\}. \qquad (15)$$

The determination of EU_{2k} is very similar to that of EW_{22}. To determine EU_{2k}, denote by the random variable τ the time elapsed from the start of the first service after a reopening of the station until the next epoch at which no 1-customers are in the system. Let the random variable η be the number of 2-customers who arrived during the time elapsed from a reopening of the station until the next epoch at which a service commences while no 1-customers are in the system. Then

$$E(U_{2k} | \tau_a = s, \nu_1 = n, \tau = t, \eta = m) = (R-k)(s+t) + m(s+t) / 2 + (R-k+m) EW_{22} + + (1/2)(R-k+m)(R-k+m-1) ET_{b2}.$$

The conditional distribution of η, given that $\tau_a = s$ and $\tau = t$, is Poisson with mean $\lambda_2(s+t)$. Given that $\nu_1 = n$, the random variable τ has the same distribution as the sum of $k + n$ independent random variables which are distributed as T_1. Finally, the conditional distribution of ν_1 given that $\tau_a = s$ is Poisson with mean $\lambda_1 s$. Now, by taking expectations succesively on η, τ, ν_1 and τ_a and using (2), we find after some simple manipulations that

$$EU_{2k} = \{(R-k)(1-\rho_1) / (1-\rho_1-\rho_2) + \lambda_2 EW_{22}\} \{E\tau_a / (1-\rho_1) + kET_1\} + + (1/2) \{\lambda_2 + \lambda_2^2 ET_{b2}\} \{E\tau_a^2 / (1-\rho_1)^2 + k(k-1)(ET_1)^2 + (k+\lambda_1 E\tau_a) ET_1^2 + + 2\mu_1 E\tau_a k / (1-\rho_1)^2\} + (R-k) EW_{22} + (R-k)(R-k-1) ET_{b2} / 2.$$

Using (2), (4) and (8), we find after some algebra that the expected total amount of time spent by 2-customers in the system during the time elapsed from a reopening of the station until the next epoch at which no customers are in the system is given by

$$\sum_{k=0}^{R} \binom{R}{k}(\lambda_1/\lambda)^k(\lambda_2/\lambda)^{R-k}\ EU_{2k} = \{R + \lambda E\tau_a\}\ \{(\lambda_1/\lambda)\ EW_{21} + (\lambda_2/\lambda)\ EW_{22}\} +$$

$$+ [\lambda_2/(1-\rho_1)(1-\rho_1-\rho_2)]\ [R(R-1)\ \{1-(1-\rho_1)(1-\rho_1-\rho_2)\}\ /\ 2\lambda^2 + RE\tau_a/\lambda + E\tau_a^2/2]. \tag{16}$$

From (14), (15) and (16),

$$EW_c(i) = \{R + \lambda E\tau_a + \beta\}\ \{(\lambda_1/\lambda)\ EW_{i1} + (\lambda_2/\lambda)\ EW_{i2}\} +$$

$$+ \{\lambda_i/(1-\rho_1)(1-\rho_1-\rho_2-\rho_i^*)\}\ \{R(R-1)/2\lambda^2 + RE\tau_a/\lambda + E\tau_a^2/2\},$$

where

$$\rho_1^* = \rho_1 + \rho_2 \quad \text{and} \quad \rho_2^* = 0.$$

Now, by (1), the long-run (expected) average number of i-customers in the system is equal to $L^{(i)} = EW_c(i)\ /\ ET_c$, (i=1,2). Using (6), (7), (9) and (10), we find that

$$L^{(i)} = \rho_i + \{\lambda_i(\lambda_1\mu_1^{(2)} + \lambda_2\mu_2^{(2)})\}\ /\ 2(1-\rho_1)(1-\rho_1-\rho_2-\rho_i^*) +$$

$$+ \{\lambda\lambda_i(1-\rho_i^*)\ /\ (1-\rho_1)(R+\lambda E\tau_a+\beta)\}\ \{R(R-1)\ /\ 2\lambda^2 + RE\tau_a\ /\ \lambda + E\tau_a^2\ /\ 2\}.$$

If we put $\lambda_2 = 0(\lambda_1=0)$ in the expression for $L^{(1)}(L^{(2)})$ we obtain formula (26) in YADIN and NAOR [10]. Finally, it easily follows from JEWELL's paper [4] that the formulae $L^{(i)} = \lambda_i W^{(i)}$ and $L_q^{(i)} = \lambda_i W_q^{(i)}$ apply. Hence, by $W^{(i)} = W_q^{(i)} + \mu_i$, the long-run (expected) average number of i-customers in the queue equals $L_q^{(i)} = L^{(i)} - \rho_i$, (i=1,2).

Remark. It is easily verified that for the case where interruptions of the closing-down process involve no loss of close-down time, the average number of i-customers in the system is given by the above expression for $L^{(i)}$ with β replaced by $\lambda E\tau_b$.

MODEL B: CLOSING-DOWN PROCESS WITHOUT INTERRUPTIONS.

For this model, let the random variable T_c^* be the length of a cycle and let the random variable $W_c^*(i)$ be the total amount of time spent by i-customers in the system during one cycle, where a cycle is the time between two successive epochs at which a service completion occurs while no customers are awaiting for service. Let

$$p_n = \int_0^\infty e^{-\lambda t}(1/n!)(\lambda t)^n \, G(dt) \qquad (n=0,1,\ldots).$$

Then p_n represents the probability that n customers will arrive during the close-down time τ_b. Let $p_n^* = p_n$ for $n > R$, and let $p_R^* = \sum_{k=0}^R p_k$. We have by (13) that the expected amount of time elapsed from a reopening of the station until the next epoch at which no customers are in the system equals $(n+\lambda E\tau_a) / \lambda(1-\rho_1-\rho_2) - n/\lambda$ given that n customers are at the station when it is reopened. Now it is easily seen that

$$ET_c^* = E\tau_b + \sum_{k=0}^R p_k\,(R-k)/\lambda + \sum_{n=R}^\infty p_n^*\,\{(n+\lambda E\tau_a)/\lambda(1-\rho_1-\rho_2) - n/\lambda\}.$$

Using $\sum_{n=R}^\infty np_n^* = \lambda E\tau_b + \sum_{k=0}^R (R-k)p_k$, we find that

$$ET_c^* = \{\lambda E\tau_a + \lambda E\tau_b + \sum_{k=0}^R (R-k)p_k\} / \lambda(1-\rho_1-\rho_2).$$

To determine $EW_c^*(i)$, denote by ϕ_{1n} and ϕ_{2n} the right-hand side of (15) and (16), respectively, with R replaced by n. Then ϕ_{in} represents the expected total amount of time spent by i-customers in the system during the time elapsed from a reopening of the station until the next epoch at which no customers are in the system given that n customers are at the station when it is reopened. Further, we observe that if k customers have been arrived during the close-down time τ_b then the expected number of i-customers who arrived during τ_b is equal to $k\lambda_i / \lambda$. Now it is readily seen that

$$EW_c^*(i) = \lambda_i E\tau_b^2 / 2 + \sum_{k=0}^{R} p_k \{(k\lambda_i/\lambda)(R-k)/\lambda + (\lambda_i/\lambda)(1/2)(R-k)(R-k-1)/\lambda\} + \sum_{n=R}^{\infty} \phi_{in} p_n^*.$$

Now, by (1), the long-run average number of i-customers in the system equals $L^{(i)} = EW_c^*(i) / ET_c^*$. Using $\sum_{n=R}^{\infty} n(n-1)p_n^* = \lambda^2 E\tau_b^2 + \sum_{k=0}^{R}\{R(R-1)-k(k-1)\}p_k$, we find after some algebra that

$$L^{(i)} = \rho_i + \{\lambda_i(\lambda_1\mu_1^{(2)} + \lambda_2\mu_2^{(2)})\} / 2(1-\rho_1)(1-\rho_1-\rho_2-\rho_i^*) +$$

$$+ [\lambda\lambda_i(1-\rho_i^*) / (1-\rho_1)(\lambda E\tau_a+\lambda E\tau_b+\sum_{k=0}^{R}(R-k)p_k)][(E\tau_a^2+E\tau_b^2) / 2 +$$

$$+ E\tau_a\{\lambda E\tau_b + \sum_{k=0}^{R}(R-k)p_k\} / \lambda + (1/2\lambda^2) \sum_{k=0}^{R}\{R(R-1) - k(k-1)\}p_k],$$

where $\rho_1^* = \rho_1 + \rho_2$ and $\rho_2^* = 0$. Finally, it easily follows from reference 4 that the formulae $L^{(i)} = \lambda_i W_i$ and $L_q^{(i)} = \lambda_i W_q^{(i)}$ apply. Hence, in particular $L_q^{(i)} = L^{(i)} - \rho_i$.

COST OPTIMIZATION

We consider the following cost structure. There are a holding cost of $h_i > 0$ per unit time per i-customer in the system and a fixed cost of $K > 0$

per cycle for reopening and closing-down the station. The long-run average cost per unit time can be easily found. For convenience we only consider model A. By the elementary renewal theorem, the long-run (expected) number of cycles per unit time equals ET_c^{-1} with probability one. Hence the long-run (expected) average cost per unit time equals, with probability one, $\phi(R) = h_1 L^{(1)} + h_2 L^{(2)} + K / ET_c$, where ET_c and $L^{(i)}$ are given by (13) and (16). To determine the value of R for which $\phi(R)$ is minimal, let us treat R as a positive continuous variable. Straightforward calculations show that $\phi''(R) > 0$ for $R > 0$, and so $\phi(R)$ is strictly convex for $R > 0$. Putting $\phi'(R) = 0$, we may find the optimal value of R. Since the expression for the optimal R is very complicated it will be omitted. For the special case where the set-up time and the close-down time are zero, we obtain from $\phi'(R) = 0$ that

$$R^* = [\{2K\lambda^2(1-\rho_1)(1-\rho_1-\rho_2)\} \ / \ \{h_1\lambda_1(1-\rho_1-\rho_2) + h_2\lambda_2\}]^{1/2}.$$

Since $\phi(R)$ is convex, the optimal positive integer value of R is one of the integers $[R^*]$ and $[R^*] + 1$.

BLOCK-TIMES

Let us extend the model A and B as follows. We now suppose that after completion of the service of an i-customer the server is blocked during a random time B_i before he can commence a new service or close-down the station (see reference 2 for examples). When the block-time has been passed, the server commences a new service when customers are at the station, otherwise he decides to close-down the station. We suppose that the first two moments of B_i are finite, $(i=1,2)$. Let $\rho_1' = \lambda_i(\mu_i+Eb_i)$, $(i=1,2)$. It is assumed that $\rho_1' + \rho_2' < 1$. Further we suppose that the block-times, the service times, the set-up times and the close-down time are independent of each other and of the arrival process. The long-run averages $L^{(i)}$, $L_q^{(i)}$, $W^{(i)}$ and $W_q^{(i)}$ are easily found for this model. To do this, we observe that with respect to the number of customers in the queue the model with block-times is equivalent to the model with no block-times and a service time $S_i + B_i$ for an i-customer, where S_i is the service time for an i-customer in the original models A and B. Hence for the models A and B with block times we obtain expressions for $L_q^{(i)}$ when we replace μ_i, $\mu_i^{(2)}$ and ρ_i by $\mu_i + EB_i$, $\mu_i^{(2)} + 2\mu_i EB_i + EB_i^2$ and ρ_i', respectively, in the expressions for $L_q^{(i)}$ which we have found for the original models A and B. Expressions for $L^{(i)}$, $W^{(i)}$ and $W_q^{(i)}$ next follow from $L^{(i)} = \lambda_i W^{(i)}$, $L_q^{(i)} = \lambda_i W_q^{(i)}$ and $W^{(i)} = W_q^{(i)} + \mu_i$.

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