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A PROBABILISTIC APPROACH TO RENEWAL THEORY

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A probabilistic approach to renewal theory

by

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ABSTRACT

The renewal theorem and some convergence results of Spitzer for lattice random walks will be presented using the methods developed by Ornstein to extend Spitzer's results to the non-lattice case.
Some related results by the speaker on the frequency of visits to infinite sets will also be covered.

KEY WORDS & PHRASES: *Random walk, renewal theory, frequency of visits.*

INTRODUCTION

A *random walk* starting at (a real number) x is a process $\{ x + S_n \}_{n=1}^{\infty}$ with S_n being the cumulative sums of independent and identically distributed random variables.

Renewal theory studied the limiting behaviour of the random walk as it visits the region of the origin, when x is made large. Some of the objects usually of interest are the expected number of visits to some fixed set, the distribution of the first hit to a fixed set, the expected number of visits to some set before some other set is hit.

These questions have been usually dealt with analytically. Probability theory gets involved only in proving, quite trivially, that the probabilities whose limits are sought satisfy the so-called "renewal equation". The rest is an analytic type proof that solutions to such an equation must possess some convergence properties.

We will not even attempt to give a historical account of the work in the field in this short lecture. The interested reader should consult FELLER II or SPITZER.

Spitzer's book presents an excellent exposition of the analytical approach, for lattice random walks, i.e., those for which a positive number d exists such that with probability one dS_1 is integer valued. The book contains many original results, the mayor ones of which were then extended to the non-lattice case by Ornstein, using a constructive, probabilistic approach.

We will present Ornstein's ideas, applied to the lattice case. In this way the simple essence of these ideas will become apparent, once freed from the epsilontic noise that seems to be unavoidable in the treatment of non-lattice random walks.

We will also cover some related work by MEILIJSON.

Throughout this lecture, S_1 will be assumed to be almost surely integer valued. The starting point x will only be given integer values.

An integer valued random variable X is *aperiodic* if the greatest common divisor of the non zero elements of $\{k \mid P(X=k) > 0\}$ is 1. It is *strongly aperiodic* if X + k is aperiodic for every integer k. A random walk is *(strongly) aperiodic* if its increments are (strongly) aperiodic. A random walk is *recurrent* if $P(S_n=0 \text{ i.o.}) = 1$. Recurrence holds for an aperiodic random walk if and only if $P(S_n=k \text{ i.o.}) = 1$ for all integers k. Recurrence holds for a random walk with integrable increments if and only if their expectation is 0. (A good reference for this facts is BREIMAN).

The following lemma is a variant of a construction used in ORNSTEIN I in the proof of theorem 7. Also to be found in MEILIJSON, lemma 3.

LEMMA 1. *Let* F *be the distribution of an integer valued and strongly aperiodic random variable. Then, on some suitable probability space, it is possible to define two processes* $S^{(1)}$ *and* $S^{(2)}$ *and a positive-integers-valued random variable* T *such that* $S^{(1)}$ *and* $S^{(2)}$ *are random walks with* F*-distributed increments and whenever* $n \geq T$, *then* $S^{(2)}_n = S^{(1)}_n + 1$.

PROOF. For every positive integer J denote by F_J the distribution of $(\min(X,J))^+ - (\min(-X,J))^+$ when X is distributed F. Since X is strongly aperiodic, if J is large enough, a variable distributed F_J will also be strongly aperiodic. Fix such a J. Let $X = (X_1,X_2,X_3,\ldots)$ be iid with common distribution F. Let $Y = (Y_1,Y_2,Y_3,\ldots)$ be iid with common distribution equal to the conditional distribution of X_1 given that $|X_1| \leq J$. Let X and Y be independent processes.

Define, for $n \geq 1$, $X'_n = X_n$ if $|X_n| > J$, $X'_n = Y_n$ otherwise. Define, for $n \geq 1$, $Z_n = X_n - X'_n$. The Z_n are the increments of a recurrent and aperiodic random walk. Hence, the least positive integer T for which $Z_1 + Z_2 + \ldots + Z_T = 1$ is almost surely defined. Define, for $n \geq 1$, $X''_n = X'_n$ if $n \leq T$, $X''_n = X_n$ otherwise. Then $X''_1, X''_2, \ldots$ are iid with common distribution F. Define, for $n \geq 1$, $S^{(1)}_n = X''_1 + X''_2 + \ldots + X''_n$ and $S^{(2)}_n = X_1 + X_2 + \ldots X_n$. Then $S^{(1)}$, $S^{(2)}$ and T possess the desired properties. □

COROLLARY 1 (ORNSTEIN *I, theorem 7*)
For a strongly aperiodic random walk $(S_n)_{n=1}^{\infty}$, $\lim_{n\to\infty} \sum_k |P(S_n=k) - P(x+S_n=k)| = 0$ for all x.

PROOF. It is obviously sufficient to consider the case $x = 1$. In the termonology of lemma 1,

$$P(S_n=k) - P(1+S_n=k) = P(S_n^{(1)}=k) - P(1+S_n^{(1)}=k) =$$

$$= P(S_n^{(1)}=k,\ T>n) - P(S_n^{(2)}=k,\ T>n)$$

So

$$|P(S_n=k) - P(1+S_n=k)| \leq P(S_n^{(1)}=k,\ T>n) + P(S_n^{(2)}=k,\ T>n)$$

and

$$\sum_k |P(S_n=k) - P(1+S_n=k)| \leq 2P(T>n) \to 0 \text{ as } n \to \infty. \quad \square$$

Given a random walk, denote by $g(x;I)$ the expected number of visits to the finite set I, starting at x, divided by the number of points in I. Let $S_0 = 0$ and count number of visits from time 0 on. Denote $g(x) = g(x;\{0\})$. Denote by $p(x)$ the probability of ever visiting 0, starting at x. Unless undefined, the ratio g/p is constant, and is finite for transient random walks.

THEOREM 1. (*The renewal theorem for positive random walks*)
For an aperiodic random walk with $P(S_1 \geq 0) = 1$ *and* $E(S_1) > 0$ (possibly $+\infty$), $\lim_{x \to -\infty} g(x)$ *exists and equals* $1|E(S_1)$.

PROOF. The proof will be divided into four parts, for the case $0 < E(S_1) < \infty$. The case $E(S_1) = \infty$ will be shown at the end to have actually been already covered.

(A) $g(x) - g(x+1) \to 0$ as $x \to -\infty$

(B) $g(x) - g(x;I) \to 0$ as $x \to -\infty$ for every finite set I of integers

(C) $\sup_{x \leq 0} |g(x;[0,M)) - g(0;[0,M))| \to 0$ as $M \to \infty$

(D) $g(0;[0,M)) \to 1|E(S_1)$ as $M \to \infty$.

As the result obviously follows from (A), (B), (C) and (D), and as (B) is a

trivial consequence of (A), we will restrict ourselves to proving (A), (C) and (D).

Proof of (A). For a large enough (negative) x, by aperiodicity, p(x) is positive. By transcience g/p is a positive constant, so it is enough to prove that $p(x) - p(x+1) \to 0$. Since p depends on F only through the conditional F distribution given that the value of the variable is positive, we may assume that $P(S_1=0) > 0$. Coupled with aperiodicity, this implies strong aperiodicity, and lemma 1 may be applied, to express $p(x) - p(x+1) =$ $P(S_n^{(2)}+x$ ever equals $0) - P(S_n^{(1)}+1+x$ ever equals $0) = P(S_n^{(2)}+x$ ever equals 0, and at least one of $S_n^{(2)}+x$ and $S_n^{(1)}+1+x$ reaches $[0,\infty]$ before time T) $-$ $P(S_n^{(1)}+1+x$ ever equals 0, and at least one of $S_n^{(2)}+x$ and $S_n^{(1)}+1+x$ reaches $[0,\infty)$ before time T).

So

$$|p(x) - p(x+1)| \le 2P(S_n+x \text{ reaches } [-1,\infty) \text{ before time } T)$$

$$\to 0 \text{ as } x \to \infty.$$

Proof of (D). By considering the constancy of g/p, it is easy to reduce the need of a proof to the case where $P(S_1>0) = 1$. Assume it.

Denote by ψ_M the average number of points in $[0,M)$ visited by the random walk, starting at 0. Since ψ_M is bounded, to prove that $g(0;[0,M))$, which is $E(\psi_M)$, converges to $1|E(S_1)$, it is enough to prove that $1|\psi_M$ converges almost surely to $E(S_1)$. To do this, express $N_M = \max\{n|\ S_n < M\}$ and check that $S_{N_M}|N_M \le 1|\ \psi_M \le (S_{N_M+1}|(N_M+1))((N_M+1)|N_M)$.

Now (D) follows from the strong law of large numbers.

Proof of (C). For $0 < E(S_1) < \infty$, $\Sigma_{j=N}^{\infty} P(S_1 \ge j) \to 0$ as $N \to \infty$. Pick N such that $\Sigma_{j=N}^{\infty} P(S_1 \ge j)$ is small. Now observe that for any $x < 0$, the random walk starting at x will miss the interval $[0,N)$ with probability at most $\Sigma_{j=N}^{\infty} P(S_1 \ge j)$. So, for every $x < 0$,

$$g(0;[0,M-N)).\frac{M-N}{M}.\ (1-\sum_{j=N}^{\infty} P(S_1 \ge j)) \le g(x;[0,M)) \le g(0;[0,M)) \tag{1}$$

and the result follows.

Proof for the case $E(S_1) = +\infty$.

The only point that breaks down in the part of the proof written so far is the left hand side inequality of (1). However, the right hand side holds true, and is sufficient to give the desired result. □

For disjoint finite sets A and B of integers, let $h_x(A,B)$ be the probability that the random walk starting at x will ever visit A, without having visited B before. $H_x(A,B)$ will denote the expected number of visits to A while B hasn't been visited.

For a set A and a number x, $A + x = \{y + x | y \in A\}$ is the x-translate of A.

THEOREM 2. (ORNSTEIN *1, theorem 1*)
For an aperiodic random walk, $\lim_{x\to\infty} h_x(A,B)$ *and* $\lim_{x\to-\infty} h_x(A,B)$ *exist, for any pair* (A,B) *of finite disjoint sets of integers.*

PROOF. Use lemma 1 as in the proof of part (A) of theorem 1 to obtain statement (A):

(A): For every finite interval I of integers and every $\varepsilon > 0$, there is an N such that if $|x| > N$ then $|h_y(A,B) - h_z(A,B)| < \varepsilon$ for y and z in $x + I$.

Denote $s = \limsup_{|x|\to\infty} h_x(A,B)$

Denote, for $\varepsilon > 0$, $0_\varepsilon = \{x \, | h_x(A,B) \geq s - \varepsilon\}$

If $s = 0$ there is nothing to prove, so assume throughout that $s - \varepsilon > 0$.

The crux of the proof is statement (B):

(B): Given A, B and ε, there is an N such that if $|x| > N$ then either $x \in 0_\varepsilon$ or $-x \in 0_\varepsilon$.

Proof of (B). Assume (B) not to hold. Use (A) to state the existence, for an arbitrary finite interval I of integers, of arbitrarily large $|x|$ with $h_y(A,B) < s - \varepsilon$ and $h_{-y}(A,B) < s - \varepsilon$ for every $y \in x + I$. It now follows that for every positive integer M there is a sequence $t_1, t_2, \ldots, t_m$ of integers such that $t_1 = 0$, the sets $(A\cup B) + t_i$ are disjoint and whenever $i \neq j$ and $x \in (A\cup B) + t_i$, $h_x(A+t_j, B+t_j) < s - \varepsilon$. Pick any $M > 4/\varepsilon$, build such a sequence, and denote $E = \cup_{i=1}^m ((A\cup B)+t_i)$.

Let the positive integer N be so large that $E \subset [-N, N]$ and whenever $|x| > N$, $|h_x(A+t_i, B+t_i) - h_x(A+t_j, B+t_j)| < \varepsilon/4$ for every $1 \le i \le M$, $1 \le j \le M$. We will arrive at a contradiction by showing that whenever $|x| > N$, $h_x(A,B) < s - \frac{\varepsilon}{2}$. Let x be such that $|x| > N$. For some $1 \le i \le M$, $h_x((A\cup B) + t_i, E - ((A\cup B) + t_i)) < \frac{1}{M} < \frac{\varepsilon}{4}$. Fix such an i.

$$h_x(A,B) = h_x(A+t_i, B+t_i) + (h_x(A,B) - h_x(A+t_i, B+t_i)) <$$

$$<(1 - \frac{1}{M})(s-\varepsilon) + \frac{1}{M} + \frac{\varepsilon}{4} < s - \varepsilon + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = s - \frac{\varepsilon}{2} .$$

We will now finish the proof of the theorem, differentiating between the recurrent and transient cases. For the recurrent case, let $\bar{s} = \limsup_{|x|\to\infty} h_x(B,A) = 1 - \liminf_{|x|\to\infty} h_x(A,B) \ge 1 - s$. If $\bar{s} + s = 1$, there is nothing to prove. If $\bar{s} + s > 1$, apply statement (B) to both h(A,B) and h(B,A) to obtain that $h_x(A,B)$ has exactly two limit points, s and $1 - \bar{s}$, as $|x| \to \infty$. Now use the slow variation claimed by statement (A) to obtain than convergence must hold when making $|x|$ large keeping the sign of x fixed. For the transient case, $h_x(A,B)$ and $h_x(B,A)$ need not add up to 1, so the above proof fails. However, the same ideas can be applied, only that instead of using statement (B) twice, use statements (B) and (C). Let $p_x(I)$ be the probability of ever visiting the set I and $g_x(I)$ be the expected number of visits to I, starting at x. ($p_x(I) = h_x(I,\emptyset)$, $g_x(I) = H_x(I,\emptyset)$).

(C): Assume the random walk in transient. Given a finite set I of integers and $\varepsilon > 0$, there is an N such that if $|x| > N$, $\min(p_x(I), p_{-x}(I)) < \varepsilon$.

<u>Proof of (C)</u>: As a first step, we prove that $\liminf_{|x|\to\infty} g_x(I) = 0$. Since $g_0(I)$ is finite, $E(g_{S_n}(I))$, the expected number of visits to I from time n on starting at 0, must converge to zero as $n \to \infty$. Since for every N, $P(|S_n| \le N) \to 0$ as $n \to \infty$, the result follows. As a second step, observe that $\liminf_{|x|\to\infty} p_x(I) \le \liminf_{|x|\to\infty} g_x(I) = 0$. Finally, assume the negation of (C). For arbitrarily large $|x|$, $p_x(I) > \varepsilon$ and $p_{-x}(I) > \varepsilon$. Pick an arbitrary y. $p_y(I) \ge p_y(I+x) \cdot \min_{\zeta\in I+x} p_\zeta(I) = p_{y-x}(I) \cdot \min_{\zeta\in I+x} p_\zeta(I)$. Since $p_{y-x}(I)$

becomes arbitrarily close to $p_{-x}(I)$ and $\min_{\zeta \in I+x} p_\zeta(I)$ becomes arbitrarily close to $p_x(I)$ as $|x| \to \infty$, we obtain that $p_y(I) \geq \varepsilon^2$, contradicting the second step. □

COROLLARY 2. *For an aperiodic random walk,* $\lim_{x\to\infty} H_x(A,B)$ *and* $\lim_{x\to-\infty} H_x(A,B)$ *exist, for any pair* (A,B) *of finite disjoint sets of integers.*

PROOF.

$$H_x(A,B) = \sum_{y\in A} H_x(\{y\}, B) = \sum_{y\in A} h_x(\{y\}, B)\, H_y(\{y\}, B). \quad \square$$

As we have seen in these proofs, renewal-type properties of random walks stem from the uniform way in which a random walk spreads itself. Corollary 1 is a crisp statement of this uniformity.

Lemma 1 is one of the strongest technical tools in the field, while at the same time being probably the best intuitive way to approach it. We will conclude this lecture by another application of lemma 1. So far we have studied the behavior of a long-travelled random walk in the ephemeral moments it approaches a certain region. Now we will study one aspect of the overall behavior of the random walk along time.

Let K be a set of integers, and let V_n denote the proportion of times from 1 to n spent by the random walk (that started at zero) in the set K. Let $d(m,n,K)$, for m an integer and n a positive integer, be the proportion of integers between m and $m + n - 1$ that belong to the set K. Let $d(K) = \limsup_{n\to\infty} d(0,n,K)$ and $\bar{d}(K) = \limsup_{n\to\infty} (\sup_m d(m,n,K))$.
MEILIJSON has shown that for an aperiodic random walk with a finite positive mean, $V_n - d(0,[nE(S_1)], K)$ converges to zero a.s. In particular, $d(K) = 0$ implies that $\limsup_{n\to\infty} V_n = 0$ a.s. . However, this last fact fails to hold when the assumption of finiteness of the mean is lifted. There may exist random walks that visit with almost surely positive frequency a set of density zero. Theorem 3 will provide examples of this phenomenon. A second result in MEILIJSON states that if $\bar{d}(K) = 0$ then for every random walk (other than $S_1 \equiv 0$), $\limsup_{n\to\infty} V_n = 0$ a.s., and if $\bar{d}(K) > 0$, there exists a distribution F such that the random walk with F distributed increments has

$\limsup_{n\to\infty} V_n > 0$ a.s. . Theorem 4 will prove the first half of this statement and theorem 3 will prove partially the second half, for sets K with $\bar{d}(K) = 1$.

THEOREM 3. *Assume* $\bar{d}(K) = 1$. *Then for some random walk,* $\limsup_{n\to\infty} V_n > 0$ a.s.

PROOF. By the Hewitt-Savage 0-1 theorem (see BREIMAN) $\limsup_{n\to\infty} V_n$ is a.s. constant. By Fatou's lemma, $E(\limsup_{n\to\infty} V_n) \geq \limsup_{n\to\infty} E(V_n)$. So it is enough to prove that $\limsup_{n\to\infty} E(V_n) > 0$. $\bar{d}(K) = 1$ means that for every n there is an m such that the interval $[m, m+n)$ is entirely contained in K. Obviously, if $\bar{d}(K) = 1$ then either $\bar{d}(K\cap(0,\infty)) = 1$ or $\bar{d}(K\cap(-\infty,0)) = 1$, so assume, without loss of generality, that $K \subset (0,\infty)$.

Build a distribution F for S_1 in the following manner: $P(S_1=a_i) = 1/(i(i+1))$, with $a_1 = 1$ and a_{i+1} being such that (1) $a_{i+1} > a_i$ and (2) the interval $[a_{i+1}, a_{i+1} + ia_i]$ is entirely contained in K.

To see that this does it, let $X_1, X_2, X_3, \ldots$ be independent and identically F-distributed random variables, denote $X_1 + X_2 + \ldots + X_n = S_n$ and let B_n be the event: "Exactly one of the variables $X_1, X_2, \ldots, X_n$ is bigger or equal a_n and all the variables $X_{n+1}, X_{n+2}, \ldots, X_{2n}$ are less than a_n". Since $\Sigma_{i=n}^{\infty} 1/(i(i+1)) = 1/n$, the probability of B_n is $n\cdot(1/n)\cdot(1-(1/n))^{2n-1}$, and so $P(B_n) \to e^{-2}$ as $n \to \infty$. On the event B_n, the random walk spends all times from n + 1 to 2n in the set K, so $E(V_n|B_n) \geq \frac{1}{2}$ and $\liminf_{n\to\infty} E(V_n) \geq \liminf_{n\to\infty} P(B_n)\, E(V_n|B_n) \geq 1/(2e^2) > 0$. □

THEOREM 4. (*part a) of* MEILIJSON *theorem 2*)
If $\bar{d}(K) = 0$ *and* $P(S_1=0) < 1$, *then* $V_n \to 0$ a.s.

PROOF. Without loss of generality it may be assumed that the random walk is aperiodic, since otherwise we may work on its lattice. Denote by $V_n(x)$ the proportion of times between 1 and n the random walk starting at x spends in the set K. (So, $V_n=V_n(0)$). As a first step, we will prove
(A): $\sup_x E(V_n(x)) \to 0$ as $n \to \infty$.

Proof of (A): Fix $\varepsilon > 0$. Let n_0 be such that every interval of length at least n_0 contains a proportion at most $\varepsilon/4$ of points in K. Without loss of generality, assume strong aperiodicity. (Otherwise, mix the distribution of S_1 with a small mass at 0. The statement holds for the walk if and only if it holds for the retarded walk, as is easy to check).

Use lemma 1 repeatedly to build an n_0 "sweep", i.e., processes $S^{(1)}, S^{(2)}, \ldots, S^{(n_0)}$ and a random variable T such that each $S^{(j)}$ is a random walk with the given distribution and whenever $n \geq T$, $S_n^{(j+1)} = S_n^{(j)} + 1$ for each $1 \leq j < n_0$. The idea is that from time T on, the sweep forms an interval of length n_0, so at any moment, a proportion less than $\varepsilon/4$ of the random walks $S^{(j)}$ is in K. This says about each one of them that it spends on the average a small part of the time in K.

Formally, let the positive integers a and ℓ satisfy $P(T>a) \leq \varepsilon/2$ and $4a/\varepsilon \leq \ell$. For an arbitrary x, on the event $\{T \leq a\}$ the number of pairs (i,j) with $1 \leq i \leq \ell$ and $1 \leq j \leq n_0$ for which $x + S_i^{(j)} \in K$ is at most $an_0 + (\varepsilon/4)\, n_0\, \ell \leq (\varepsilon/2)\, n_0\, \ell$. On the set $\{T > a\}$, the number of those pairs is of course at most $n_0\, \ell$, so the expected number of those pairs is at most $(\varepsilon/2)\, n_0\, \ell\, P(T\leq a) + n_0\, \ell\, P(T>a) \leq \varepsilon\, n_0\, \ell$. Since $S^{(1)}, S^{(2)}, \ldots, S^{(n_0)}$ are identically distributed, we obtain finally that for every $\varepsilon > 0$ there exists an ℓ_0 such that if $n \geq \ell_0$, $E(V_n(x)) < \varepsilon$ for every x. This proves (A). Fix an ℓ_0 as above.

Let $Y_0 \equiv 0$ and let Y_n, for $n \geq 1$, denote the proportion of times between $(n-1)\ell_0 + 1$ and $n\,\ell_0$ spent in K, and let $Z_n = Y_n - E(Y_n | Y_0, Y_1, \ldots, Y_{n-1})$. Since $(Z_n)_{n=1}^{\infty}$ are the increments of a martingale with mean zero and uniformly bounded increments, they satisfy Lévy's strong law of large numbers. (see LÉVY section 69, p.250 or NEVEU p.146).
By our construction, $0 \leq E(Y_n | Y_0, Y_1, \ldots Y_{n-1}) < \varepsilon$ a.s.
Hence,

$$\limsup_{n\to\infty} V_n = \limsup_{n\to\infty} V_{n\ell_0} = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} Y_i =$$

$$= \limsup_{n\to\infty} \left(\frac{1}{n} \sum_{i=1}^{n} Z_i + \frac{1}{n} \sum_{i=1}^{n} E(Y_i | Y_0, Y_1, \ldots Y_{i-1})\right) \leq \varepsilon \quad \text{a.s.}$$

Since this holds for all $\varepsilon > 0$, the result follows. □

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