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O.J. VRIEZE

DUOPOLY MODELS, STOCHASTIC GAMES AND BIMATRIX GAMES

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Duopoly models, stochastic games and bimatrix games*)

by

O.J. Vrieze

ABSTRACT

This paper deals with a multi-period duopoly model under uncertainty. When one (or both) of the duopolists has taken an action, there is some kind of cooling-period of length T, i.e. a period in which neither of the duopolists may take any action. Next there is specified a sequence of equidistant time-points at which the duopolists may or may not take an action. As soon as one (or both) of the duopolists takes an action, there again will be a cooling period of length T, etc.
An action for a duopolist is a combination of setting a price for his products and choosing a production rate.

The state of the system is characterized by the following quantities of both duopolists: price, production rate, stock and selling rate.

An action of one (or both) of the duopolists in a state causes a random jump to a new state, where the randomness concerns the selling rate of the duopolists.

In the model production costs, stock costs and switching costs are included. The infinite horizon problem will be considered and future incomes will be discounted.

It will be shown that this duopoly model can be reformulated as a stochastic two-person game model with non-zero sum payoffs and a discount factor depending on the state and the actions of the players. It follows from the literature on stochastic games that this game possesses an equilibrium point.

Furthermore the one-period model is studied. Under suitable conditions (which seem to be quite natural) on the payoff matrices it will be shown that the resulting bimatrix game possesses a unique equilibrium point, which consists of nearly pure strategies. Extensions of this result to the multi-period case will be discussed.

Key Words & Phrases: duopoly model, two person nonzero sum stochastic game, equilibrium, bimatrix game, uniqueness of equilibrium points.

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*) This report will be submitted for publication elsewhere.

1. INTRODUCTION AND MODEL FORMULATION

Quite a lot of papers are published concerning duopoly models. The historical viewed first ones studied the non-cooperative static situation in which both duopolists once can take an action. The problem was to find an equilibrium situation. See for instance COURNOT [1], MAYBERRY [9], HENDERSON [6], SHUBIK [19], SCHNEIDER [16] and RUFFIN [13].

Later on within this field of non-cooperative attack dynamic was introduced into two directions. The first direction studied the following question: How can an equilibrium situation of a static model be reached? i.e. starting in an arbitrary situation is there an appealing path to the equilibrium situation, where this path consists of a sequence of points at which the duopolists simultaneously take an action. This question is closely related to stability questions. Literature on this subject can be found in THEOCHARIS [22] and [23], SATO-NAGATANI [14], CYERT & DE GROOT [2], VAN DER WHEEL [24].

The second direction concerns multi-period models both deterministic and stochastic. Their problems are the existence of equilibrium points (usually in the sense of NASH [10]) for e.g. the discounted model or the average payoff model and finding algorithms to obtain the equilibrium points. Some results on these subjects can be found in FRIEDMAN [4] and [5], KIRMAN & SOBEL [7] and SOBEL [20]. This last direction is for a great deal inspired by the results of the theory on stochastic games (for a survey of this field, see PARTHASARATHY [11]).

So far we have only mentioned non-cooperative entries. In the development of the duopoly theory there also is a stream of papers which attack the problem from a cooperative point of view. Contributors to this direction are STACKELBERG [21], VON NEUMAN-MORGENSTERN [25], SELTEN [17], SCARF [15] and MARSCHAK & SELTEN [8].

This paper deals with a multi-period duopoly model under uncertainty within the non-cooperative setting. In all the above mentioned papers the duopolists are assumed to take their actions simultaneously. An essential feature of our paper is that this need not be the case.

We are now going to formulate our model.
Two duopolists each have the possibility of producing (or buying) and selling

a commodity. For convenience we confine ourselves to the case with one commodity. The reader can easily verify that the multi-commodity case can be attacked along the same lines.

The state of our system is specified by the following eight quantities: $(p_1,r_1,s_1,v_1,p_2,r_2,s_2,v_2)$, where p_i is player i's price, r_i is player i's production rate, s_i is player i's stocks and v_i is player i's selling rate, $i = 1,2,$.
An action for a duopolist is a combination of fixing a price at which he will offer his commodity and choosing a production rate.

The moments upon which the duopolists may take their actions are specified as follows: When one (or both) of the duopolists has taken an action (i.e. changing his price or production rate) there is a kind of cooling-period of length T, i.e. a period in which neither of the duopolists may take any action. Next there is specified a sequence of equidistant time-points, at which the duopolists may or may not take an action. However as soon as one (or both) of the duopolists takes an action there again will be a cooling-period of length T, etc. As unit of time we take the distance between two consecutive time points, at which the duopolists may take an action. As an example we may think of a situation where both duopolists may or may not once a day take an action; in that case T = 1.

As long as neither of the duopolists takes an action the selling rates v_1 and v_2 are assumed to remain constant.
An action of one (or both) of the duopolists in a state $x = (p_1,r_1,s_1,v_1,p_2,r_2,s_2,v_2)$ causes a random jump to a new state. This randomness concerns the selling rates and may also depend on the present state x.

Let V be the set of pairs (v_1,v_2) which can occur. In our model we assume V to be finite.

Let $A_i(x)$ be the set of actions available to duopolists i in state x, i = 1,2. Then for each triplet (x,a_1,a_2) with x a state, $a_1 \in A_1(x)$ and $a_2 \in A_2(x)$ there is specified a set of probabilities $\{p((v_1,v_2) \mid x,a_1,a_2) \mid (v_1,v_2) \in V\}$, such that

(1.1) $$p((v_1,v_2) \mid x,a_1,a_2) \geq 0 \quad \text{and} \quad \sum_{(v_1,v_2)\in V} p((v_1,v_2) \mid x,a_1,a_2) = 1,$$

where $p(v_1,v_2) \mid x,a_1,a_2)$ is the probability that the selling rates v_1 and v_2 will occur if in state x the duopolists take actions a_1 and a_2 respectively. Note that $p((v_1,v_2) \mid x,a_1,a_2)$ only depends on both a_1 and a_2, when the duopolists take actions at the same time.

In the following we assume that p_i and r_i, $i = 1,2$, each can only accept a finite number of different values and that T is a positive integer. Furthermore negative stocks are forbidden and duopolist i has a maximum stock capacity of s_i^m, $i = 1,2$.

From the assumptions above and the assumption that V is finite we can deduce that also the stocks can only take on a finite number of different values at those moments upon which the duopolists may take actions. This may be seen as follows.
There are only a finite number of different stock changing rates (p_i-v_i) for duopolists i, which we assume to be written in decimalform. Let g_i be the greatest common divisor of all those numbers (p_i-v_i) i.e., the greatest rational number, which by division on (p_i-v_i) has an integral outcome for all (p_i-v_i). Now it is easy to see that with a starting stock s_i^0 the stock at those moments upon which the duopolists may take an action, can be expressed as $s_i^0 + n\cdot g_i$ where n is an integer. When we take into account the bounds 0 and s_i^m it follows that only a finite number of different stocks can occur. Note that the actual upperbound on the stocks equals $s_i^0 + n^m\cdot g_i$ where n^m is such that $s_i^0 + n^m\cdot g_i \leq s_i^m < s_i^0 + (n^m+1)\cdot g_i$.

From the above now we see that the state variable $x = (p_1,r_1,s_1,v_1,p_2,r_2,s_2,v_2)$ can only take on a finite number of different values at those moments upon which the duopolists may take actions. Let N be this number and number those states $1,2,\ldots,N$.

Although the stocks change continuously during a period between two consecutive actions, so the state changes continuously, this period is characterized by the state $k \in \{1,2,\ldots,N\}$ at the beginning of such a period. We therefore in the following use the somewhat abused identification, that during such a period the state is k.

When at a certain time t_0 the state has become $k = (p_1,r_1,s_1,v_1,p_2,r_2,s_2,v_2)$, belonging to $\{1,2,\ldots,N\}$, two things are possible. First if $p_i-v_i = 0$, $i = 1,2$ the stocks remain constant, Secondly if $p_i-v_i \neq 0$, $i = 1$ and/or $i = 2$, there will be at least one stock that changes in time.

As we have assumed stock bounds 0 and s_i^m, this gives us a bound on the sequence of equidistant points, at which the duopolists may or may not take an action. Let for state k K(k) be the integral such that $t_0 + T + K(k) + 1$ would be the first time point at which at least one of the duopolists has negative stock or exceeds his maximal stock capacity. So the time points at which the duopolists can take an action in state k can be numbered as $0,1,\dots,K(k)$. For the time being we assume that $p_i - v_i \neq 0$ for every state $k \in \{1,2,\dots,N\}$. Later on we shall make clear that this is not an essential assumption.

We are now going to specify the cost aspects of our model. Let $c_i(r_i)$ be the production costs per unit of time, when the production rate is r_i, $i = 1,2$. Let $b_i(s_i)$ be the stock costs per unit of time, when the stock level is s_i, $i = 1,2$. We assume that $b_i(s_i)$ is a continuous function in s_i.

Then, when at time t_0 the state has become $k = (p_1,r_1,s_1,v_1,p_2,r_2,s_2,v_2) \in \{1,2,\dots,N\}$ and if at $t > t_0$ none of the duopolists has taken an action yet, then the profit rate for player i at that time is:

$$(1.2) \qquad w_i(k,t) = p_i v_i - c_i(r_i) - b_i(s_i + (t-t_0)(p_i-v_i)).$$

Furthermore there are switching costs. If in state k player i takes action $a_i \in A_i(k)$, then he has an immediate cost $h_i(k,a_i)$. We are going to consider the discounted model over an infinite time horizon. An income at time t will be discounted by a factor $e^{-\rho t}$, with $\rho > 0$.

2. REFORMULATION OF THE MODEL AS A TWO-PERSON NONZERO-SUM STOCHASTIC GAME

What actually happens in the above duopoly model is that in a state $k \in \{1,2,\dots,N\}$ duopolist i chooses a time-point $t_i \in \{0,1,\dots,K(k)\}$ and an action $a_i \in A_i(k)$ which he wants to carry out at t_i.

If $t_1 < t_2$, then only action $a_1(k)$ is executed.

If $t_1 > t_2$, then only action $a_2(k)$ is executed.

If $t_1 = t_2$, then both $a_1(k)$ and $a_2(k)$ are executed.

In the game model the two duopolists are of course the two players.

The set of states is $S = \{1,2,\dots,N\}$, where $k \in S$ should be associated with the state k of the duopoly model.

The action space in state k for player i is $\{0,1,\dots,K(k)\} \times A_i(h)$, $i = 1,2$. An element of this action space will be denoted by $(t_i(k),a_i(k))$.

If in the duopoly model at time t_0 the state has become k and if the duopolists choose action time points $t_1(k)$ and $t_2(k)$ and actions $a_1(k)$ and $a_2(k)$ then the discounted profit for duopolist i from t_0 until the next action time point equals:

(2.1)
$$V_i(k,t_0,t_1(k),t_2(k),a_1(k),a_2(k)) =$$

$$= \int_{t_0}^{t_0+T+\min(t_1(k),t_2(k))} e^{-\rho t}\, w_i(k,t)dt - \bar{h}_i(k,a_i(k))\cdot e^{-\rho(t_0+T+\min(t_1(k),t_2(k)))},$$

where

$$\bar{h}_i(k,a_i(k)) = 0 \quad \text{if} \quad t_i(k) > \min(t_1(k),t_2(k))$$

and

$$\bar{h}_i(k,a_i(k)) = h_i(k,a_i(k)) \quad \text{if} \quad t_i(k) = \min(t_1(k),t_2(k)).$$

If we substitute (1.2) into (2.1) and set $\tau = T+\min(t_1(k),t_2(k))$ this yields:

(2.2)
$$V_i(k,t_0,t_1(k),t_2(k),a_1(k),a_2(k)) =$$

$$= e^{-\rho t_0}\left[\frac{1-e^{-\rho\tau}}{\rho}\right](p_i v_i - c_i(r_i)) - e^{-\rho t_0}\left[\int_0^{\tau} e^{-\rho t'}\cdot b_i(s_i+t'(p_i-v_i))dt'\right]$$
$$- e^{-\rho t_0}\left[\bar{h}_i(k,a_i(k))\, e^{-\rho\tau}\right].$$

Now if in the game model in state k the players choose actions $(t_1(k),a_1(k))$ and $(t_2(k),a_2(k))$ then from (2.2) it may be easy understandable that as an immediate payoff to player i we define

$$g_i(k,(t_1(k),a_1(k)),(t_2(k),a_2(k)) = \left[\frac{1-e^{-\rho\tau}}{\rho}\right](p_i v_i - c_i(r_i))$$

$$- \int_0^{\tau} e^{-\rho t'} \cdot b_i(s_i + t'(p_i - v_i))dt' - \bar{h}_i(k,a_i(k)) \cdot e^{-\rho\tau},$$

where

$$\tau = T + \min(t_1(k),t_2(k)).$$

The discount factor that belongs to the actions $(t_1(k),a_1(k))$ and $(t_2(k),a_2(k))$ in state k equals $\beta(k,(t_1(k),a_1(k)),(t_2(k),a_2(k))) = e^{-\rho(T+\min(t_1(k),t_2(k))}$, i.e. an immediate payoff in the next state must be multiplied by $\beta(k,(t_1(k),a_1(k)),(t_2(k),a_2(k)))$ for calculating its value at the time point at which the system has entered the present state.

The transition probabilities belonging to the actions $(t_1(k),a_1(k))$ and $(t_2(k),a_2(k))$ in state k for the game model, equal the transition probabilities as specified under (1.1) in the duopoly model.

With this the game model is specified. The equivalence with the duopoly model is easily verified.

A stationary strategy π for player 1 in the game model is a vector $(\pi_1,\pi_2,\ldots,\pi_N)$, where π_k is a probability distribution on the set of pure actions for player 1 in state k. For a strategy π for player 1 we denote the probability with which he chooses pure action $(t_1(k),a_1(k))$ in state k by $\pi(t_1(k),a_1(k))$.

Analogue notations hold for a stationary strategy ρ for player 2.

If the players play the stationary strategies π and ρ respectively, then the total expected discounted payoff to player i can be found as the unique solution of the following set of equations:

(2.3) $$V_i(k,\pi,\rho) =$$

$$\sum_{(t_1(k),a_1(k))} \sum_{(t_2(k),a_2(k))} \pi_k(t_1(k),a_1(k)) \cdot$$

$$\cdot \rho_k(t_2(k),a_2(k))\Big\{g_i(k,(t_1(k),a_1(k)),(t_2(k),a_2(k))) +$$

$$+ \beta(k,(t_1(k),a_1(k)),(t_2(k),a_2(k))) \cdot \sum_{\ell=1}^{N} p(\ell|k,a_1(k),a_2(k)) \cdot$$

$$\cdot V_i(\ell,\pi,\rho)\Big\}, \qquad k = 1,2,\dots,N.$$

Here $V_i(k,\pi,\rho)$ denotes the total expected discounted payoff to player i, when the game starts in state k, the players use stationary strategies π and ρ respectively and the game moves on over an infinite horizon, $k = 1,2,\dots,N$.

DEFINITION 2.1. A pair of strategies (π^*,ρ^*) is called an *equilibrium point* if and only if

$$V_2(k,\pi^*,\rho^*) \geq V_2(k,\pi^*,\rho), \qquad k = 1,2,\dots,N, \ \forall\rho,$$

and

$$V_1(k,\pi^*,\rho^*) \geq V_1(k,\pi,\rho^*), \qquad k = 1,2,\dots,N, \ \forall\pi.$$

Now we can state our main theorem.

THEOREM 2.2. *The duopoly model as specified in section* 1 *possesses an equilibrium point of stationary strategies.*

PROOF. From the above shown similarity between the duopoly model and the game model we see, that it suffices if we proof the theorem for the game model. But that is a well-known result in stochastic game theory. ROGERS [12] was the first who showed this. He restricts himself to the class of stationary strategies. That in these models an equilibrium point within the class of stationary strategies is also equilibrium point within the class of behaviour strategies can be found in VRIEZE [26]. □

If we allow for a certain state k the possibility $p_i - v_i = 0$, $i = 1,2$, so the sequence of points at which the players may take an action is not bounded, then in the game model player i has in state k a countable number of pure actions. Also in that case the game model and so the duopoly model possesses an equilibrium point of stationary strategies, but now we must base ourself on a heavier theorem as e.g. can be found in VRIEZE [26] or FEDERGRÜN [3].

It is known that in general this game model possesses more than one equilibrium point, so a natural question thrusts upon itself, namely, are there conditions under which the above model has a unique equilibrium point and furthermore how close stand these conditions to reality. The following section let in some light on these questions.

3. UNIQUE EQUILIBRIUM POINTS

We first consider the one-period case, then the problem reduces to a bimatrix game.

Let player 1 have m pure actions and player 2 n.

Let A be the m×n-matrix of payoffs for player 1 and B be the m×n-matrix of payoffs for player 2.

We now state some properties for payoff matrices, which will be very useful in formulating conditions under which a bimatrix game has a unique equilibrium point.

P_1: a payoff matrix for player 1 (player 2) is said to be one-peaked in the columns (rows) if for each column j (row i) the following holds:
there exist a unique i_0 (j_0), such that $a_{i_0 j} > a_{ij}$, $\forall i \neq i_0$ $(b_{ij_0} > b_{ij}, \forall j \neq j_0)$.

In the following for a one-peaked payoff matrix for player 1 (player 2) we denote with i_j (j_i) the row (column) such that $a_{i_j j} > a_{ij}$, $\forall i \neq i_j$, $j = 1,\dots,n$ $(b_{ij_i} > b_{ij}, \forall j \neq j_i, i = 1,\dots,m)$

P_2: a one-peaked payoff matrix for player 1 (player 2) is said to be slow peak decreasing (slow peak increasing) if the following holds: or $i_j = i_{j-1}$ or $i_j = i_{j-1}-1$, $j = 2,\dots,n$ (or $j_i = j_{i-1}$ or $j_i = j_{i-1}+1$, $i = 2,\dots,m$).

P_3: a payoff matrix for player 1 (player 2) is said to be concave in the columns iff:
$a_{ij}-a_{i-1j} > a_{i+1j}-a_{ij}$, $i = 2,\dots,m-1$; $j = 1,\dots,n$
$(b_{ij}-b_{ij-1} > b_{ij+1}-b_{ij}$, $j = 2,\dots,n-1$; $i = 1,\dots,m)$.

THEOREM 3.1. *A bimatrix game such that the payoff matrices* A *and* B *obey the properties* P_1, P_2 *and* P_3 *possesses a unique equilibrium point* (x^*,y^*), *where both* x^* *and* y^* *are concentrated at at most two pure actions.*

PROOF. As A and B obey the properties P_1 and P_2 exactly one of the following two possibilities must occur.

a) There exist a row i_0 and a column j_0 such that

$$a_{i_0j_0} > a_{ij_0}, \ \forall i \neq i_0 \quad \text{and} \quad b_{i_0j_0} > b_{i_0j}, \ \forall j \neq j_0.$$

b) There exist a row i_0 and a column j_0, such that

$$b_{i_0j_0} > b_{i_0j}, \ \forall j \neq j_0; \qquad b_{i_0+1\ j_0+1} > b_{i_0+1j}, \ \forall j \neq j_0+1.$$

$$a_{i_0j_{0+1}} > a_{ij_0+1}, \ \forall i \neq i_0; \qquad a_{i_0+1\ j_0} > a_{ij_0}, \ \forall i \neq i_0+1.$$

In case a) we clearly have an equilibrium point in pure strategies.
If b) holds, consider the following 2×2-bimatrix game (A^o,B^o):

$$A^o = \begin{pmatrix} a_{i_0j_0} & a_{i_0j_0+1} \\ a_{i_0+1\ j_0} & a_{i_0+1\ j_0+1} \end{pmatrix} \qquad B^o = \begin{pmatrix} b_{i_0j_0} & b_{i_0j_0+1} \\ b_{i_0+1\ j_0} & b_{i_0+1\ j_0+1} \end{pmatrix}.$$

It is easy to check that this bimatrix game does not possess an equilibrium point in pure strategies. Furthermore, as this game is non-degenerate there is a unique completely mixed equilibrium point. Let this equilibrium point be (x_0^*,y_0^*), with $x_0^* = (x_{i_0}^*,x_{i_0+1}^*)$ and $y_0^* = (y_{j_0}^*,y_{j_0+1}^*)$. Note that x_0^* is the unique optimal strategie for player 1 in the matrix game B^o, where player 1 is a minimizing player and that y_0^* is the unique optimal strategie for player 2 in the matrix game A^o, where player 2 is a minimizing player.

It can now be verified (property P_3) that the strategies

$$x^* = (0,\ldots,0,x_{i_0}^*,x_{i_0+1}^*,0,\ldots,0)$$

and

$$y^* = (0,\ldots,0,y_{j_0}^*,y_{j_0+1}^*,0,\ldots,0)$$

form an equilibrium point for the game (A,B).
So both to case a) and to case b) there corresponds an equilibrium point.

We now claim that in both cases the corresponding equilibrium point is the unique equilibrium point for the game (A,B).
In the following we concentrate upon case b) for in case a) the same reasoning can be held with i_0 substituted for i_0+1 and j_0 substituted for j_0+1.

Assume that $(\bar{x},\bar{y})$ is an equilibrium point for the game (A,B).
Let

$$x_1 = \sum_{i=1}^{i_0-1} \bar{x}_i, \qquad x_2 = \sum_{i=i_0+2}^{m} \bar{x}_i,$$

$$y_1 = \sum_{j=1}^{j_0-1} \bar{y}_j \quad \text{and} \quad y_2 = \sum_{j=j_0+2}^{n} \bar{y}_j.$$

We need four steps.

Step 1. Let $\bar{x} = x^*$. From property P_3 it can be seen that y^* is strictly better for player 2 against $\bar{x}$ than every other strategy, so $\bar{y} = y^*$. In the same way: if $\bar{y} = y^*$, then $\bar{x} = x^*$.

Step 2. Let $\bar{x}$ be such that $x_1 = 0$ and $x_2 \neq 0$. From property P_3 it follows that $\bar{y}$ must be such that $y_1 = 0$, for if $y_1 \neq 0$ then transferring this weight to column j_0 would be strictly better for player 2. But now it follows that player 1 yields more if he transfer the weight x_2 to the row i_0+1, so there exists no $\bar{y}$ such that for this $\bar{x}$ $(\bar{x},\bar{y})$ is an equilibrium point.

Step 3. Let $\bar{x}$ be such that $x_2 = 0$ and $x_1 \neq 0$. Then $\bar{y}$ must be such that $y_2 = 0$ (else transferring to j_0+1), from which we see that player 1 gets more if he transfers the weight x_1 to the row i_0, so also this $(\bar{x},\bar{y})$ cannot be an equilibrium point.
An analogue reasoning as in the steps 2 and 3 shows that $\bar{y}$ can be neither such that $y_1 = 0$, $y_2 \neq 0$, nor such that $y_1 \neq 0$, $y_2 = 0$.

Step 4. Let $\bar{x}$ be such that $x_1 \neq 0$ and $x_2 \neq 0$.
Let i_1 be a row with $i_1 < i_0$ and $\bar{x}_{i_1} \neq 0$ and let i_2 be a row with $i_2 > i_0+1$ and $x_{i_2} \neq 0$. In order $(\bar{x},\bar{y})$ to be an equilibrium point, we must have:

(3.1) $\quad A_{i_1.}\cdot\bar{y} = A_{i_2.}\cdot\bar{y} = \bar{x}^T A\bar{y}$ and $A_{i.}\cdot\bar{y} \le \bar{x}^T A\bar{y}$,

$i \ne i_1$ and $i \ne i_2$.

Here $A_{i.}$ denotes the i-th row of A.
Let j_1 be such that $i_{j_1} = i_1$ and $i_{j_1-1} = i_1+1$. Such a j_1 exists for else row i_1+1 would be strictly better for player 1 than row i_1.
Now from (3.1) we derive

(3.2) $$\sum_{j=j_1}^{n} (a_{i_1 j}-a_{i_1+1 j})\bar{y}_j \ge \sum_{j=1}^{j_1-1} (a_{i_1+1 j}-a_{i_1 j})\bar{y}_j.$$

Note that each term in the summations of (3.2) is positive.
Now using several times property P_3 we get:

$$\sum_{j=j_1}^{n} (a_{i_1 j}-a_{i_1+2 j})\bar{y}_j > 2\sum_{j=j_1}^{n} (a_{i_1 j}-a_{i_1+1 j})\bar{y}_j \ge$$

$$\ge 2\sum_{j=1}^{j_1-1} (a_{i_1+1 j}-a_{i_1 j})\bar{y}_j > \sum_{j=1}^{j_1-1} (a_{i_1+2 j}-a_{i_1 j})\bar{y}_j.$$

So

$$A_{i_1.}\cdot\bar{y} > A_{i_1+2.}\,\bar{y}.$$

In the same way we can show

(3.3) $$A_{i_1.}\cdot\bar{y} > A_{i_1+k.}\cdot\bar{y}, \qquad k = 2,3,\ldots,m-i_1.$$

As $i_2 \in \{i_1+k \mid k=2,3,\ldots,m-i\}$ we see that (3.3) contradicts (3.1), so again $(\bar{x},\bar{y})$ cannot be an equilibrium point.
Combining these four steps yields the conclusion that (x^*,y^*) is the only equilibrium point for the game (A,B). □

In regarding the properties P_1, P_2 and P_3 the properties P_1 and P_3 seem to be quite natural for a duopoly situation. Property P_2 is the strongest one. A reasoning which makes this property somewhat acceptable is the following: the set of pure actions for a player must be seen as the lattice points of the original action space, which is assumed to be an interval. As it is reasonable to assume the payoff functions to be continuous on these original action spaces it is imaginable that the lattice may be chosen in such a wat that two consecutive points for a player give only small deviations in the payoff values for the other player so that his best replies against these two points respectively must be close together.

We now go back to the stochastic game. We are going to state two theorems concerning uniqueness of equilibrium points in two-person non-zero stochastic games.

A two-person non-zero sum stochastic game will be denoted by a fixed-tuple (S,A,B,P,β) where:

$S = \{1,2,\ldots,N\}$ is the set of states.

$A = (A_1,\ldots,A_N)$ with A_k a $(m_k \times n_k)$-matrix, such that $a_{ij}(k)$ is the payoff to player 1 in state k if player 1 chooses row i and player 2 chooses column j, $i = 1,2,\ldots,m_k$; $j = 1,2,\ldots,n_k$.

$B = (B_1,\ldots,B_n)$, the same as A but for player 2.

$P = \{p(\ell|k,i,j) \mid i = 1,\ldots,m_k;\ j = 1,\ldots,n_k;\ \ell = 1,\ldots,N;\ k = 1,\ldots,N\}$ is the set of transition probabilities, i.e. if in state k the joint players action is (i,j) then the probability that the system moves to state ℓ is $p(\ell|k,i,j)$.

$\beta = \{\beta_{i,j}(k) \mid i = i,\ldots,m_k,\ j = 1,\ldots,n_k,\ k = 1,\ldots,N\}$ is the set of discount factors, where $\beta_{ij}(k)$ belongs to state k and actions (i,j).

DEFINITION 3.2. For a two-person non-zero sum stochastic game (S,A,B,P,β) and for each $\bar{v}_1 \in \mathbb{R}^n$ and $\bar{v}_2 \in \mathbb{R}^n$ the elements of the following set of N bimatrix games are called the *dummy bimatrix games* with $\bar{v}_1$ and $\bar{v}_2$:

$$G_k(\bar{v}_1,\bar{v}_2) = (A_k(\bar{v}_1),\Gamma_k(\bar{v}_2)),$$

where

$$\alpha_{ij}(k) = a_{ij}(k) + \beta_{ij}(k) \sum_{\ell=1}^{N} p(\ell|k,i,j)v_1(\ell),$$

and

$$\gamma_{ij}(k) = b_{ij}(k) + \beta_{ij}(k) \sum_{\ell=1}^{N} p(\ell|k,i,j)v_2(\ell),$$

$$i = 1,\ldots,m_k;\ j = 1,2,\ldots,n_k;\ k = 1,\ldots,N.$$

We need the following lemma.

LEMMA 3.3. *A pair of stationary strategies* (x^*,y^*) *with* $x^* = (\bar{x}_1^*,\ldots,\bar{x}_N^*)$ *and* $y^* = (\bar{y}_1^*,\ldots,\bar{y}_N^*)$ *is an equilibrium point for a game* (S,A,B,P,β) *if and only if* $(\bar{x}_k^*,\bar{y}_k^*)$ *is an equilibrium point in the associated dummy bimatrix game* $G_k(\bar{V}_1^*,\bar{V}_2^*)$, $k = 1,\ldots,N$, *where* $\bar{V}_i^*$ *is the total expected discounted payoff to player* i, i = 1,2, *under the strategies* (x^*,y^*) *in the stochastic game.*

PROOF. The proof of this lemma can be found in FEDERGRÜN [3] (lemma 2.3) and also in VRIEZE [26] (the proof of theorem 2.1), although they both have $\beta_{ij}(k) = \beta$, $\forall i$, $\forall j$, $\forall k$. For the proof this is not an essential assumption. □

Let for a stochastic game (S,A,B,P,β) and for a vector $(\bar{v}_1,\bar{v}_2) \in \mathbb{R}^{2N}$ $T_k(\bar{v}_1,\bar{v}_2)$ denote the set of equilibrium points for dummy bimatrix game $G_k(\bar{v}_1,\bar{v}_2)$ and let $U_k(\bar{v}_1,\bar{v}_2)$ denote the set of payoff pairs associated with these equilibrium points.

DEFINITION 3.4. A pair of vectors $(\bar{v}_1,\bar{v}_2) \in \mathbb{R}^{2N}$ and $(\bar{w}_1,\bar{w}_2) \in \mathbb{R}^{2N}$ is called *contracting* with respect to a game (S,A,B,P,β) if and only if for each $(\bar{v}_1^o,\bar{v}_2^o) \in \times_{k=1}^N U_k(\bar{v}_1,\bar{v}_2)$ and each $(\bar{w}_1^o,\bar{w}_2^o) \in \times_{k=1}^N U_k(\bar{w}_1,\bar{w}_2)$ the following holds:

$$\|(\bar{v}_1^o,\bar{v}_2^o)-(\bar{w}_1^o,\bar{w}_2^o)\| \le \alpha\cdot\|(\bar{v}_1,\bar{v}_2)-(\bar{w}_1,\bar{w}_2)\|,$$

with $0 \le \alpha < 1$ and $\|\cdot\|$ denoting the sup norm in $\mathbb{R}^{2N}$.

THEOREM 3.5. *If for a stochastic game* (S,A,B,P,β) *each pair of vectors* $(\bar{v}_1,\bar{v}_2) \in \mathbb{R}^{2N}$ *is contracting, then for this stochastic game every equilibrium point yields the same payoffs for both players.*

PROOF. Let (x^*,y^*) and $(\tilde{x},\tilde{y})$ be two equilibrium points for the stochastic game and let the associated payoff pairs be $(\bar{v}_1^*,\bar{v}_2^*)$, respectively $(\tilde{\bar{w}}_1,\tilde{\bar{w}}_2)$. From lemma 3.3 it follows that $(\bar{x}_k^*,\bar{y}_k^*)$ is equilibrium point in the dummy bimatrix game $G_k(\bar{v}_1^*,\bar{v}_2^*)$ and that $(\tilde{\bar{x}}_k,\tilde{\bar{y}}_k)$ is equilibrium point in the dummy bimatrix game $G_k(\tilde{\bar{w}}_1,\tilde{\bar{w}}_2)$.

From equation (2.3) we see that $(\bar{x}_k^*,\bar{y}_k^*)$ in $G_k(\bar{v}_1^*,\bar{v}_2^*)$ yields a payoff pair $(\bar{v}_1^*(k),\bar{v}_2^*(k))$ and analogue $(\tilde{\bar{x}}_k,\tilde{\bar{y}}_k)$ in $G_k(\tilde{\bar{w}}_1,\tilde{\bar{w}}_2)$ yields a payoff pair

$(\tilde{\bar{w}}_1(k),\tilde{\bar{w}}_2(k))$. So it is clear that $(\bar{v}_1^*,\bar{v}_2^*) \in X_{k=1}^N\ U_k(\bar{v}_1^*,\bar{v}_2^*)$ and $(\tilde{\bar{w}}_1,\tilde{\bar{w}}_2) \in X_{k=1}^N\ U_k(\tilde{\bar{w}}_1,\tilde{\bar{w}}_2)$.

Using the contracting property with $(\bar{v}_1^*,\bar{v}_2^*)$ and $(\tilde{\bar{w}}_1,\tilde{\bar{w}}_2)$ we get

$$\|(\bar{v}_1^*,\bar{v}_2^*)-(\tilde{\bar{w}}_1,\tilde{\bar{w}}_2)\| \le \alpha\|(\bar{v}_1^*,\bar{v}_2^*)-(\tilde{\bar{w}}_1,\tilde{\bar{w}}_2)\|,$$

which can only be true if $(\bar{v}_1^*,\bar{v}_2^*) \equiv (\tilde{\bar{w}}_1,\tilde{\bar{w}}_2)$. □

Note that theorem 3.5 states that in the case where each pair of vectors $(\bar{v}_1,\bar{v}_2) \in \mathbb{R}^{2N}$ and $(\bar{w}_1,\bar{w}_2) \in \mathbb{R}^{2N}$ are contracting the map $(\bar{v}_1,\bar{v}_2) \to X_{k=1}^n\ U_k(\bar{v}_1,\bar{v}_2)$ has exactly one fixed point.

Note also that it is enough if the contracting property holds for each pair of vectors $(\bar{v}_1,\bar{v}_2)$ and $(\bar{w}_1,\bar{w}_2)$, such that $\|\bar{v}_1\| \le M_1/(1-\beta)$, $\|\bar{v}_2\| \le M_2/(1-\beta)$, $\|\bar{w}_1\| \le M_1/(1-\beta)$ and $\|\bar{w}_2\| \le M_2/(1-\beta)$, where $M_1 = \max_{i,j,k} |a_{ij}(k)|$, $M_2 = \max_{i,j,k} |b_{ij}(k)|$ and $\beta = \max_{i,j,k} \beta_{ij}(k)$.

<u>THEOREM 3.6</u>. *Let for a stochastic game* (S,A,B,P,β) *the following hold:*

a) *For each* $(\bar{v}_1,\bar{v}_2) \in \mathbb{R}^{2N}$ *such that* $\|\bar{v}_1\| \le M_1/(1-\beta)$ *and* $\|\bar{v}_2\| \le M_2/(1-\beta)$ *each dummy bimatrix game* $G_k(\bar{v}_1,\bar{v}_2)$, $k = 1,\ldots,N$ *obey the properties* P_1, P_2 *and* P_3.

b) *In addition to* a)*: for each* k *the dummy bimatrix games* $G_k(\bar{v}_1,\bar{v}_2)$ *have the same structure for each* $(\bar{v}_1,\bar{v}_2)$ *such that* $\|\bar{v}_1\| \le M_1/(1-\beta)$ *and* $\|\bar{v}_2\| \le M_2/(1-\beta)$, *i.e. or* $G_k(\bar{v}_1,\bar{v}_2)$ *has a unique pure equilibrium point which is the same for each* $(\bar{v}_1,\bar{v}_2)$, *or* $G_k(\bar{v}_1,\bar{v}_2)$ *has a unique equilibrium point where player 1 uses two consecutive rows and player 2 uses two consecutive columns, such that these two rows and two columns are the same for each* $(\bar{v}_1,\bar{v}_2)$ *(the weights on them need not be the same).*

If a) *and* b) *hold, then the stochastic game has a unique equilibrium point.*

<u>PROOF</u>. We will show that each pair of vectors $(\bar{v}_1,\bar{v}_2) \in \mathbb{R}^{2N}$ and $(\bar{w}_1,\bar{w}_2) \in \mathbb{R}^{2N}$ such that $\|\bar{v}_1\|,\|\bar{w}_1\| \le M_1/(1-\beta)$ and $\|\bar{v}_2\|,\|\bar{w}_2\| \le M_2/(1-\beta)$ are contracting with contraction radius β. Then theorem 3.5 tells us that the map $(\bar{v}_1,\bar{v}_2) \to X_{k=1}^N\ U_k(\bar{v}_1,\bar{v}_2)$ ($U_k(\bar{v}_1,\bar{v}_2)$ contains but one element, $k = 1,\ldots,N$) has a unique fixed point $(\bar{v}_1^*,\bar{v}_2^*)$ and from condition a) in the theorem we see that $G_k(\bar{v}_1^*,\bar{v}_2^*)$ has a unique equilibrium point, $k = 1,\ldots,N$, which by

lemma 3.3 constitutes an equilibrium point for the stochastic game which therefore must also be unique.

Fix $k \in \{1,\dots,N\}$. We only consider the case where player 1 must use in the k-th dummy bimatrix game two rows, say i_1 and i_1+1 and player 2 two columns, say j_1 and j_1+1. The other case can be treated quite analogue.

Let $G_k^o(\bar{v}_1,\bar{v}_2) = (A_k^o(\bar{v}_1),B_k^o(\bar{v}_2))$ denote the restriction of $G_k(\bar{v}_1,\bar{v}_2)$ to the rows i_1 and i_1+1 and the columns j_1 and j_1+1, so $(A_k^o(\bar{v}_1),B_k^o(\bar{v}_2))$ is a (2×2)-bimatrix game. From condition a) and the proof of theorem 3.1 we see that this (2×2)-bimatrix game has a unique equilibrium point $(\bar{x}_0^*(\bar{v}_1,\bar{v}_2), \bar{y}_0^*(\bar{v}_1,\bar{v}_2))$ such that $\bar{x}_0^*(\bar{v}_1,\bar{v}_2)$ is the unique completely mixed optimal strategy for player 1 in the matrix game $B_k^o(\bar{v}_2)$ (player 1 the minimizing player) and $\bar{y}_0^*(\bar{v}_1,\bar{v}_2)$ is the unique completely mixed optimal strategy for player 2 in the matrix game $A_k^o(\bar{v}_1)$ (player 2 the minimizing player). But this means that the payoffs to the players 1 and 2, which belong to the unique equilibrium point of $G_k(\bar{v}_1,\bar{v}_2)$, equal $\text{val}\{A_k^o(\bar{v}_1)\}$ and $\text{val}\{B_k^{o^T}(\bar{v}_2)\}$ respectively, where $B_k^{o^T}(\bar{v}_2)$ denotes the transpose matrix of $B_k^o(\bar{v}_2)$ and val{matrix} denotes the value of a matrix game in the usual sense.

As in the proof of theorem 3.1 the pair $(\bar{x}_0^*(\bar{v}_1,\bar{v}_2),\bar{y}_0^*(\bar{v}_1,\bar{v}_2))$ can be extended to the unique equilibrium pair of $G_k(\bar{v}_1,\bar{v}_2)$ and the payoffs for the players 1 and 2 belonging to this equilibrium pair of $G_k(\bar{v}_2,\bar{v}_2)$ are the same, so $\text{val}\{A_k^o(\bar{v}_1)\}$ and $\text{val}\{B_k^o(\bar{v}_2)\}$ respectively.
Now let $(\bar{v}_1,\bar{v}_2)$ and $(\bar{w}_1,\bar{w}_2)$ as desired and let the corresponding equilibrium points be $(\bar{x}_k(\bar{v}_1,\bar{v}_2),\bar{y}_k(\bar{v}_1,\bar{v}_2))$ and $(\bar{x}_k(\bar{w}_1,\bar{w}_2),\bar{y}_k(\bar{w}_1,\bar{w}_2))$ respectively. Then

$$(3.4) \qquad |\bar{x}_k^T(\bar{v}_1,\bar{v}_2)\cdot A_k(\bar{v}_1)\cdot\bar{y}_k(\bar{v}_1,\bar{v}_2) - \bar{x}_k^T(\bar{w}_1,\bar{w}_2)\cdot A_k(\bar{w}_1)\cdot y_k(\bar{w}_1,\bar{w}_2)| =$$

$$= |\text{val}\{A_k^o(\bar{v}_1)\} - \text{val}\{A_k^o(\bar{w}_1)\}| \le \beta\|\bar{v}_1-\bar{w}_1\|.$$

The last inequality follows from the theory of zero sum stochastic games (see e.g. SHAPLEY [18]).

Similarly

$$(3.5) \qquad \bar{x}_k^T(\bar{v}_1,\bar{v}_2)\cdot B_k(\bar{v}_2)\cdot\bar{y}_k(\bar{v}_1,\bar{v}_2) - \bar{x}_k^T(\bar{w}_1,\bar{w}_2)\cdot B_k(\bar{w})\cdot\bar{y}_k(\bar{w}_1,\bar{w}_2)| =$$

$$= |\text{val}\{B_k^{o^T}(\bar{v}_2)\} - \text{val}\{B_k^{o^T}(\bar{w}_2)\}| \le \beta\|\bar{v}_2-\bar{w}_2\|.$$

As $0 \leq \beta < 1$ the contracting property follows from (3.4) and (3.5). □

We conclude with the remark that the two conditions in theorem 3.6 are met if the matrices $A_k = \{a_{ij}(k)\}$ and $B_k = \{b_{ij}(k)\}$ obey the properties P_1, P_2 and P_3, $k = 1,\ldots,N$ and furthermore if $p(\ell|k,i,j)$ for all k and ℓ does not depend on i and j.

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