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STATIONARY DISTRIBUTIONS FOR CONTROL POLICIES IN AN M/G/1 QUEUE WITH REMOVABLE SERVER

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Stationary distributions for control policies in an M/G/1 queue with removable server*

by

H.C. Tijms

ABSTRACT. This paper considers an M/G/1 queue where the idle fraction of the server's time is controlled by turning off the server when the system becomes empty and turning on the server when some critical congestion level is reached. For both the N-policy which turns the server on when the queue size becomes N and the D-policy which turns the server on when the workload exceeds D, we shall derive the stationary distributions of the workload and the waiting time of a customer.
Also, we extend some of the results to the case where between arrivals the workload decreases at a general rate when the server is busy.

KEY WORDS & PHRASES: *M/G/1 queue, removable server, N-policy D-policy, workload, queueing time, stationary distributions.*

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* This report will be submitted for publication elsewhere

1. Introduction

This paper deals with an M/G/1 queueing system in which the idle fraction of the server's time can be controlled by turning the server off when the system becomes empty and turning the server on as soon as the system reaches some critical congestion level. In our model customers arrive at a single-server station according to a Poisson process with rate λ. Each customer involves an amount of work which is distributed as the random variable S having probability distribution $F(x) = \Pr\{S \le x\}$ with $F(0) = 0$ and finite first moment μ. It is assumed that $\rho=\lambda\mu<1$. To utilize the idle fraction of the server's time during which he may engaged to other duties, we consider the so-called N-policy and D-policy which both turn the server off only when the system becomes empty. The N-policy is a control policy based on the queue size and turns the server on as soon as the number of customers in the system reaches the level N where N is a positive integer. The D-policy is a control policy based on the workload (= total amount of work in the system) and turns the server on as soon as the workload exceeds the level D where D is a non-negative number.

The N-policy was studied amongst others in [4], [8], [11] and [14], whereas the D-policy was considered in [2], [3], [5] and [12]. In [14] an expression for the average queue size for the N-policy was obtained (cf. also [11] for a generalization to a non-preemptive priority model) and in [4] and [8] the optimality of this policy among other control policies was discussed. For the D-policy a formula for the average workload was obtained in [3] and [12], whereas in [5] and [12] the optimality of this policy was treated. Further, we note that the model of the D-policy is closely related to a general control model in [13].

In this paper we shall give for both the N-policy and D-policy an extremely simple derivation of the stationary distribution of the workload from which distribution the stationary distribution of the waiting time of a customer will be obtained. This derivation will be based on simple renewal-theoretic arguments and will give analytical results which are tractable for numerical computations. The stationary distributions will be derived in section 3 after we have given some preliminaries in section 2. In these sections it is assumed that between arrival epochs the workload decreases linearly at unit rate when the server is on. However, in the final section 4 we shall obtain some results for a general decrease rate for the workload.

2. Preliminaries

For $n = 1,2,\ldots$, let τ_n be the arrival epoch of the nth customer and let X_n be the amount of work involved by the nth customer. Define $E^{(n)}(t)=\Pr\{\tau_n \le t\}$, then for all $n \ge 1$,

$$E^{(n)}(t) = 1 - \sum_{k=0}^{n-1} e^{-\lambda t} \frac{(\lambda t)^k}{k} \quad \text{for } t \ge 0. \tag{2.1}$$

Let $F^{(0)}(x) = 1$ for $x \ge 0$ and $F^{(0)}(x) = 0$ for $x < 0$, and denote by $F^{(n)}$ the n-fold convolution of F with itself for $n = 1,2,\ldots$. Define the renewal function $M_F(x)$ by

$$M_F(x) = \sum_{n=1}^{\infty} F^n(x) \tag{2.2}$$

Next, for any fixed $u \ge 0$, define the random variables

$$K(u) = \inf\{n \ge 1 \mid \sum_{j=1}^{n} X_j > u\}, \; T(u) = \tau_{K(u)} \text{ and } S(u) = \sum_{j=1}^{K(u)} X_j, \tag{2.3}$$

i.e. K(u) is the total number of customers whose cumulative amount of work firstly exceeds u, T(u) is the arrival epoch at which this occurs and S(u) is the total amount of work involved by these customers. Then, using well-known results from renewal theory and Wald's equation (e.g. [6] and [10]), we have for any $u \ge 0$ that

$$\Pr\{K(u) = k\} = F^{(k-1)}(u) - F^{(k)}(u), \; k = 1,2,\ldots . \tag{2.4}$$

$$EK(u) = 1+M_F(u), \; E[K(u)\{K(u)-1\}] = 2M_F(u) + 2\int_0^u M_F(u-y)\, dM_F(y). \tag{2.5}$$

$$\Pr\{T(u) \le t\} = \sum_{k=1}^{\infty} E^{(k)}(t) \Pr\{K(u) = k\}, \; t \ge 0. \tag{2.6}$$

$$\Pr\{S(u) \le x\} = F(x) - \int_0^u \{1 - F(x-y)\}\, dM_F(y), \; x > u. \tag{2.7}$$

$$ET(u) = \{1 + M_F(u)\}/\lambda, \quad ES(u) = \mu\{1 + M_F(u)\}. \tag{2.8}$$

We note that in the special case of $F(x) = 1 - e^{-\eta x}$, $x \geq 0$, we have for any $u \geq 0$,

(2.9) $$\Pr\{K(u) = k\} = e^{-\eta u} \frac{(\eta u)^{k-1}}{(k-1)!} \quad \text{for } k = 1,2,\ldots, \; M_F(u) = \eta u,$$

(2.10) $$\Pr\{S(u) \leq x\} = 1 - e^{-\eta(x-u)} \quad \text{for } x > u,$$

(2.11) $$\frac{\partial \Pr\{T(u) \leq t\}}{\partial t} = \lambda e^{-(\eta u + \lambda t)} \, I_0(2\sqrt{\lambda \eta u t}), \quad t > 0,$$

where I_0 is the modified Bessel function (e.g. [1])

$$I_0(z) = \sum_{k=0}^{\infty} \frac{1}{k!k!} \left(\frac{z}{2}\right)^{2k}.$$

We now define the (defective) distribution function H by $H(x) = 0$ for $x<0$ and

(2.12) $$H(x) = \lambda \int_0^x \{1-F(y)\}dy \quad \text{for } x \geq 0$$

and its renewal function by

$$M_H(x) = \sum_{n=1}^{\infty} H^n(x)$$

where $H^{(n)}$ denotes the n-fold convolution of H with itself. For the case of $F(x)=1-e^{-\eta x}$, $x \geq 0$, we have

(2.13) $$H(x) = \frac{\lambda}{\eta}(1-e^{-\eta x}) \text{ and } \frac{dM_H(x)}{dx} = \lambda e^{-(\eta-\lambda)x} \quad \text{for } x > 0.$$

For any $z \geq 0$, define now for the queueing system considered

(2.14) $k(x;z)$ = expected amount of time during which the workload is less than or equal to z up to the first epoch at which the system is empty given that the workload is x at epoch 0, $x \geq 0$.

Observe that, by definition, for any $z \geq 0$

(2.15) $$k(0;z) = 0 \text{ and } k(x;z) = k(z;z) \text{ for } x \geq z.$$

Further, it is immediately verified that for any z the function $k(x;z)$ is continuous in $x \geq 0$. We shall now derive a formula for $k(x;z)$ where it is assumed that between arrival epochs the workload decreases linearly at unit rate when the server is on. Fix $z \geq 0$. By using standards arguments, we have for any $0 < x < z$,

$$k(x+\Delta x;z) = 1 + \lambda\Delta x[\int_0^{z-x} k(x+y;z)\,dF(y) + k(z;z)\{1-F(z-x)\}] +$$

$$+ (1-\lambda\Delta x)\,k(x;z) + 0(\Delta x),$$

so, for $0 < x < z$,

$$\frac{\partial k(x;z)}{\partial x} = 1 + \lambda\{1-F(z-x)\}\,k(z;z) - \lambda k(x;z) + \lambda\int_0^{z-x} k(x+y;z)dF(y).$$

Using partial integration, it is readily verified that, for $0 < x < z$,

$$\frac{\partial k(x;z)}{\partial x} = 1 + \lambda\{1-F(z-x)\}\,k(z;z) + \lambda\frac{\partial}{\partial x}\int_0^{z-x} k(x+y;z)\{1-F(y)\}\,dy.$$

Hence, for some constant b_z,

(2.16) $$k(x;z) = d(x;z) + b_z + \int_0^{z-x} k(x+y;z)\,dH(y) \quad \text{for } 0 \leq x \leq z,$$

where $H(y)$ is defined by (2.12) and

$$d(x;z) = x + \lambda k(z;z)\int_0^{x}\{1-F(z-y)\}dy = x + k(z;z)\{H(z)-H(z-x)\},\ 0 \leq x \leq z.$$

The equation (2.16) is a (defective) renewal equation whose unique solution is given by (e.g. [6] and [9]),

(2.17) $$k(x;z) = d(x;z) + b_z + \int_0^{z-x}\{d(x+y;z)+b_z\}\,dM_H(y),\ 0 \leq x \leq z.$$

The constant b_z and the value $k(z;z)$ follow by putting $x=z$ in (2.17) and using the boundary condition $k(0;z) = 0$.

For the special case where $F(x) = 1-e^{-\eta x}$, $x \geq 0$, we find by using (2.13) for any $z \geq 0$ that

(2.18) $$k(x;z) = \frac{x\eta}{\eta-\lambda} + \frac{\lambda}{(\eta-\lambda)^2}\{e^{-(\eta-\lambda)z} - e^{-(\eta-\lambda)(z-x)}\},\ 0 \leq x \leq z.$$

3. The stationary distributions

In this section we shall derive the stationary distributions of the workload and the waiting time of a customer, where it is assumed that between arrival epochs the workload decreases linearly at unit rate when the server is on. Further, we suppose that the customers are served in order of arrival. Observe that the distribution of the workload is independent of the latter assumption.

From now on we assume for ease that the system becomes empty at epoch 0. We say that the system is in *phase* 1(0) when the server is on (off). We now define the following random variables.

(3.1) $V(t)$ = workload at time t, $t \geq 0$,

$A(t)$ = i when the system is in phase i at time t, $t \geq 0$,

$W(t)$ = amount of time a customer would have to wait until his service starts if he arrived at time t, $t \geq 0$,

D_n = amount of time the nth customer has to wait until his service starts, $n \geq 1$.

Observe that each of the processes $\{(V(t), A(t)), t \geq 0\}$, $\{W(t), t \geq 0\}$ and $\{W_n, n \geq 1\}$ is regnerative where the epochs at which the system becomes empty are regeneration epochs. Define now a cycle as the time interval between two successive epochs at which the system becomes empty, and let

(3.2) β = expected length of one cycle,

$\gamma_i(z)$ = expected amount of time that during one cycle the workload is less than or equal to z and the system is in phase i, $z \geq 0$ and $i = 0,1$.

Then, by a standard result in the theory of regenerative processes (e.g. Proposition 5.9 in [9] and Theorem 1 in [10]), we have both for the N-policy and D-policy that for any $z \geq 0$ and $i = 0,1$,

long-run expected fraction of time during which the workload is less than or equal to z and the system is in phase i $= \dfrac{\gamma_i(z)}{\beta}$.

Since the length of one cycle has an absolutely continuous distribution, it follows from Theorem 1 in [10] that for both the N-policy and the D-policy

(3.3) $$\lim_{t\to\infty} \Pr\{V(t) \le z,\ A(t) = i\} = \frac{\gamma_i(z)}{\beta} \quad \text{for all } z \ge 0 \text{ and } i = 0,1,$$

which gives the jointly stationary distribution of the workload and the phase of the system.

From this result we can get the stationary distribution of the waiting time of a customer as follows. Using Theorem 3 in [10] which roughly states that Poisson arrivals see the system in the same way as a random observer, we have for both control policies

(3.4) $$\lim_{n\to\infty} \Pr\{D_n \le y\} = \lim_{t\to\infty} \Pr\{W(t) \le y\} \quad \text{for } y \ge 0.$$

Using the fact that for the N-policy we have for $j = 0,\dots,N-1$ that $1/\lambda\beta$ gives the expected fraction of time during which j customers are present and the system is in phase 0, we find for the N-policy

(3.5) $$\lim_{t\to\infty} \Pr\{W(t) \le y\} = \frac{1}{\lambda\beta} \sum_{j=0}^{N-1} (F^{(j-1)} * E^{(N-j)})(y) + \frac{\gamma_1(y)}{\beta} \quad \text{for } y \ge 0$$

where $F^{(j-1)} * E^{(N-j)}$ denotes the convolution of the distribution functions $F^{(j-1)}$ and $E^{(N-j)}$. Recalling definition (2.3) of T(u), we similarly find for the D-policy

(3.6) $$\lim_{t\to\infty} \Pr\{W(t) \le y\} = \int_0^{\min[y,D]} \frac{d\gamma_0(z)}{\beta} [1-F(D-z) + \int_0^{D-z} \Pr\{T(D-z-x) \le y-z\}\, dF(x)] + \frac{\gamma_1(y)}{\beta}, \quad y \ge 0.$$

Hence it remains to determine the quantities β and $\gamma_i(z)$ for both control policies.

(i) *The N-policy*. Since the expected length of one busy period in the standard M/G/1 queue without removable server is equal to $\mu/(1-\rho)$, we have the well-known result

(3.7) $$\beta = N/\lambda + N\mu/(1-\rho) = N/\lambda(1-\rho).$$

Clearly, using (2.3), (2.4), (2.14) and (2.15), we have for any $z \ge 0$

$$(3.8)\qquad \gamma_0(z) = E[\min \tau_{K(z)},\tau_N] = \sum_{k=1}^{N-1} \frac{k}{\lambda} \Pr\{K(u) = k\} + \frac{N}{\lambda} \Pr\{K(u) \geq N\} = \frac{1}{\lambda} \sum_{k=0}^{N-1} F^{(k)}(z)$$

and

$$(3.9)\qquad \gamma_1(z) = \int_0^\infty k(x;z)\, dF^{(N)}(x) = \int_0^z k(x;z)\, dF^{(N)}(x) + k(z;z)\{1-F^{(N)}(z)\}.$$

For the special case of $F(x) = 1-e^{-\eta x}$, $x \geq 0$ these formulae become

$$\gamma_0(z) = \frac{N}{\lambda} - \sum_{j=0}^{N-1} \left(\frac{N-1-j}{\lambda}\right) e^{-\eta z} \frac{(\eta z)^j}{j!},\quad z \geq 0$$

and

$$\gamma_1(z) = \frac{N}{\eta-\lambda} [1 - \sum_{j=0}^{N} e^{-\eta z} \frac{(\eta z)^j}{j!}] + \{\frac{\eta z}{\eta-\lambda} - \frac{\lambda}{(\eta-\lambda)^2}\} \sum_{j=0}^{N-1} e^{-\eta z} \frac{(\eta z)^j}{j!} + \frac{\lambda}{(\eta-\lambda)^2} e^{-(\eta-\lambda)z} [1-(\frac{\eta}{\lambda})^N \{1-\sum_{j=0}^{N-1} e^{-\lambda z} \frac{(\lambda z)^j}{j!}\}],\quad z \geq 0.$$

(ii) *The D-policy*. It is well-known for the M/G/1 queue that $x/(1-\rho)$ gives the expected time until the system becomes empty when the workload is equal to x at epoch 0. Hence, using (2.3) and (2.8), we get the well-known result

$$(3.10)\qquad \beta = \frac{1+M_F(D)}{\lambda} + \frac{\mu\{1+M_F(D)\}}{1-\rho} = \frac{1+M_F(D)}{\lambda(1-\rho)}.$$

Clearly, using (2.3), (2.7), (2.8), (2.14) and (2.15), we have

$$(3.11)\qquad \gamma_0(z) = ET(z) = \{1 + M_F(z)\}/\lambda \quad \text{for } z \geq 0,$$

and

$$(3.12)\qquad \gamma_1(z) = \int_D^\infty k(x;z)\, d\Pr\{S(D) \leq x\} =$$

$$= \begin{cases} k(z;z) & \text{for } 0 \leq z \leq D \\ \int_D^z k(x;z)\, d\Pr\{S(D) \leq x\} + k(z;z) \Pr\{S(D) > z\} & \text{for } z \leq D. \end{cases}$$

For the special case of $F(x) = 1-e^{-\eta x}$, $x \geq 0$ we find by using (2.10) and (2.18),

$\beta = (1+\eta D)/\lambda(1-\rho)$, $\gamma_0(z) = (1+\eta z)/\lambda$ for $z \geq 0$

$$\gamma_1(z) = \begin{cases} \frac{\eta z}{\eta-\lambda} + \frac{\lambda}{(\eta-\lambda)^2}\{e^{-(\eta-\lambda)z} - 1\} & \text{for } 0 \leq z \leq D, \\ \frac{1+\eta D}{\eta-\lambda} + \frac{\lambda}{(\eta-\lambda)^2} e^{-(\eta-\lambda)z} - \frac{\eta}{(\eta-\lambda)^2} e^{-(\eta-\lambda)(z-D)} & \text{for } z > D. \end{cases}$$

REMARK. The averages $V = \lim_{t\to\infty} t^{-1}E[\int_0^t V(s)ds]$ and $W_q = \lim_{n\to\infty} n^{-1}E\{\sum_{k=1}^n D_n\}$ may be determined from the obtained stationary distributions. However, these averages can be much more easily determined in a direct way. Therefore denote by L the average expected number of customers in the system. For the N-policy a formula for L is easily obtained and given by (cf. [11] and [14]),

$$L = \rho + \frac{\lambda^2\mu^{(2)}}{2(1-\rho)} + \frac{N-1}{2}, \tag{3.13}$$

where it is assumed that the second moment $\mu^{(2)}$ of F is finite. Next we obtain for the N-policy the averages W_q and V from

$$L = \lambda[W_q + \mu] \text{ and } V = \lambda\{\mu W_q + \tfrac{1}{2}\mu^{(2)}\}, \tag{3.14}$$

where the second relation in (3.14) follows by using the same simple arguments as on pp. 556-558 in [10]. For the D-policy, we have (see [3] and [12]),

$$V = \frac{\lambda\mu^{(2)}}{2(1-\rho)} + D - \frac{\{D+\int_0^D M_F(y)dy\}}{1+M_F(D)} . \tag{3.15}$$

However, for the D-policy the second relation in (3.14) does not hold since under this policy the waiting time of a customer may depend on his service time. The fundamental relation $L = \lambda[W_q+\mu]$ does apply for the D-policy and a formula for L is easily obtained as follows. Using the well-known results that in the standard M/G/1 queue the expected length of one busy period is $\mu(1-\rho)$ and the total expected amount of time spent by customers in the system during one busy period is $\mu/(1-\rho) + \lambda\mu^{(2)}/2(1-\rho)^2$, we easily find that under the D-policy the total expected amount of time spent by customers in the system during one cycle is given by

$$\sum_{k=1}^{\infty} \tfrac{1}{2}[\frac{k(k-1)}{\lambda} + k\{\frac{\mu}{1-\rho} + \frac{\lambda\mu^{(2)}}{2(1-\rho)^2}\} + \tfrac{1}{2}k(k-1)\frac{\mu}{1-\rho}] \Pr\{K(D) = k\},$$

so, since L is the ratio of this quantity and the expected length β of one cycle and using (2.5) and (3.10), we get for the D-policy

$$(3.16) \qquad L = \rho + \frac{\lambda^2\mu^{(2)}}{2(1-\rho)} + \frac{M_F(D) + \int_0^D M_F(D-y)\,dM_F(y)}{1+M_F(D)} .$$

4. A general decrease rate for the workload.

In this section we consider the case in which between arrival epochs the workload decreases at rate $r(x)$ when the workload equals x and the server is on, where $r(x)$ is a continuous function for $x > 0$ so that

$$r(0) = 0, \ \inf_{x>0} r(x) > 0 \text{ and } \lim_{x\to\infty} r(x) > \rho .$$

To derive the stationary distribution of the workload, we first note that, by using the same quantities as defined in the previous sections, the relations (3.3), (3.8), (3.9), (3.11) and (3.12) also apply to the present case. Hence it remains to determine β and the function $k(x;z)$. Clearly, letting $t(x) = \lim_{z\to\infty} k(x;z)$ for $x \geq 0$, we have

$$(4.1) \qquad \beta = \begin{cases} N/\lambda + \int_0^\infty t(x)\, dF^{(N)}(x) & \text{for the N-policy} \\ [1+M_F(D)]/\lambda + \int_D^\infty t(x)\, d\Pr\{S(D) \leq x\} & \text{for the D-policy.} \end{cases}$$

Consider now the determination of $k(x;z)$. Fix $z \geq 0$. We again have that $k(x;z)$ is continuous in $x \geq 0$ with $k(0;z) = 0$ and $k(x;z) = k(z;z)$ for $x \geq z$. Similarly as in section 2, we find for $0 < x < z$

$$(4.2) \qquad r(x)\frac{\partial k(x;z)}{\partial x} = 1 + \lambda\{1-F(z-x)\} - \lambda k(x;z) + \lambda \int_0^{z-x} k(x+y;z)\,dF(y).$$

In general we cannot analytically solve this equation, except for the special case of

$$F(x) = 1 - e^{-\eta x} \text{ and } r(x) = ax+b \text{ for } x > 0 \text{ where } a,b > 0.$$

To do this, we first apply the following transformation used also in [7]. Suppressing the dependency on z, define the function h(x) for $0 \le x \le z$ by

$$(4.3)\qquad k(x;z) = e^{\eta x}\frac{dh(x)}{dx}\ ,\ h(z) = 0$$

We can then write (4.2) in the equivalent form

$$(4.4)\qquad (ax+b)\frac{d^2h(x)}{dx^2} + (a\eta x+b\eta+\lambda)\frac{dh(x)}{dx} + \lambda\eta h(x) = e^{-\eta x}\{1+\lambda e^{-\eta(z-x)}k(z;z)\},$$
$$\text{for } 0 < x < z.$$

To solve this second-order linear differential equation, let

$$(4.5)\qquad g(x) = e^{-\eta x}h(x) \text{ for } x > 0.$$

Then (4.4) becomes

$$(4.6)\qquad (ax+b)\{\frac{dg^2(x)}{dx^2} - \eta\frac{dg(x)}{dx}\} + \lambda\frac{dg(x)}{dx} = 1+\lambda e^{-\eta(z-x)}k(z;z),\ 0 < x < z.$$

We now define the new variable t, the functions f(t) and a(t), and the constant κ by

$$(4.7)\qquad t = \eta x + \frac{\eta b}{a},\ f(t) = g(\frac{t}{\eta} - \frac{b}{a}),\ a(t) = \frac{1}{\eta a}\{1+\lambda e^{-\eta(z-t/\eta+b/a)}\}k(z,z),\ \kappa = \frac{\lambda}{a}.$$

Then (4.6) is equivalent to

$$(4.8)\qquad t\frac{d^2f(t)}{dt^2} + (\kappa-t)\frac{df}{dt} = a(t),\ \frac{\eta b}{a} < t < \eta z + \frac{\eta b}{a}.$$

The homogeneous part of (4.8) is a special case of Kummer's differential equation, cf. chapter 13 in [1]. However, by the special form of (4.8), it is easy to determine directly the general solution of (4.8) for $t > 0$. Therefore, that (4.8) for $t > 0$ can be written as

$$(4.9)\qquad \frac{df(t)}{dt} = \phi(t),\ t > 0$$

$$(4.10)\qquad \frac{td\phi(t)}{dt} + (\kappa-t)\ \phi(t) = a(t),\ t > 0.$$

It is standard to verify that, for some constant c_1, the general solution of (4.10) equals

$$(4.11)\qquad \phi(t) = \{\int_0^t a(u)e^{-u}\,u^{\kappa-1}\,du + c_1\}\,e^t\,t^{-\kappa} =$$

$$= e^t\{\frac{\Gamma(\kappa)}{\mu a}\,\gamma^*(\kappa,t) + \frac{1}{\eta}\,e^{-\eta(z+b/a)}\,k(z;z) + c_1 t^{-\kappa}\},\ t \geq 0,$$

where (cf. chapter 6 in [1]),

$$\gamma^*(\kappa,t) = \frac{t^{-\kappa}}{\Gamma(\kappa)} \int_0^t e^{-u}u^{\kappa-1}du,\ t > 0.$$

Next, using (4.9), (4.7), (4.5) and (4.3), we find

$$(4.12)\qquad k(x;z) = e^{2\eta x}\,\{\eta g(x) + \frac{dg(x)}{dx}\},\ 0 \leq x \leq z,$$

where, for some constant c_2,

$$g(x) = \int_1^{\eta x+\eta b/a} \phi(y)dy + c_2 \quad \text{for } 0 \leq x \leq z.$$

The constants c_1, c_2 and the value $k(z;z)$ follow by putting x=z in (4.12) and using the boundary conditions $k(0;z) = 0$ and $g(z) = 0$.

Finally, we would remark that although relation (3.4) holds it seems a formidable task to determine the stationary distribution of the waiting time of a customer except for the case of N = 1 or D = 0 where this distribution is the same as the stationary distribution of the workload.

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