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A NON-LINEAR VECTOR FINITE DIFFERENCE SCHEME

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A Non-linear vector finite difference scheme *)

by

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ABSTRACT

A non-linear vector iterative scheme of linear fractional form is investigated. Explicit expressions are given for each vector iterate and for the limiting vector. These equations arise in the bottleneck analysis of closed networks of queues.

KEY WORDS & PHRASES: *vector difference equations, queueing networks.*

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*) This report will be submitted for publication elsewhere.

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1. INTRODUCTION

The following finite difference scheme arose in a closed network of queues (see Appendix):

(1) $$y_i(k+1) = a_i[\frac{1}{k} + y_i(k)] / \sum_{j=1}^{m} a_j[\frac{1}{k} + y_j(k)] \quad 1 \le i \le m; \quad k = 1,2,3,\ldots$$

(2) $$y_i(1) = a_i / \sum_{j=1}^{m} a_j \quad 1 \le i \le m;$$

where $m \ge 2$ and every $a_i > 0$. For each k, the y's are strictly positive and sum to unity. Show that $\lim_{k\to\infty} y_i(k)$ exists for each i and find its value. Give an expression for $y_i(k)$ if $\{a_i\}_{i=1}^{m}$ are distinct.

2. SOLUTION

With the notation

$$b_k = \sum_{i=1}^{m} (a_i)^k$$

the first few iterates of (1) yield, after clearing fractions, the expressions

$$y_i(1) = a_i/b_1 \quad 1 \le i \le m$$

$$y_i(2) = [a_i b_1 + (a_i)^2]/[(b_1)^2 + b_2] \quad 1 \le i \le m$$

$$y_i(3) = \frac{\frac{(b_1)^2+b_2}{2} a_i + b_1(a_i)^2 + (a_i)^3}{\frac{(b_1)^2+b_2}{2} b_1 + b_1 b_2 + b_3} \quad 1 \le i \le m.$$

This suggest the general form

(3) $$y_i(k) = \sum_{j=1}^{k} c(k)_j (a_i)^j / \sum_{j=1}^{k} c(k)_j b_j$$

where the c's are independent of i. Insertion of (3) into (1) yields

(4) $$c(k+1)_1 = \frac{1}{k} \sum_{j=1}^{k} c(k)_j b_j \quad k \ge 1$$

$$c(k+1)_j = c(k)_{j-1} \qquad 2 \le j \le k+1, \quad k \ge 1$$

which implies

(5) $$c(k)_j = d_{k-j} \qquad 1 \le j \le k < \infty$$

and the recursion formula

(6) $$d_k = \frac{1}{k} \sum_{j=1}^{k} d_{k-j} b_j \qquad k = 1,2,3,\ldots$$

with initial condition

(7) $$d_0 = c(1)_1 = 1.$$

We may solve for $\{d_k\}_{k=0}^{\infty}$ via generating functions as follows. Let

(8) $$B^*(z) = \sum_{k=1}^{\infty} z^k b_k = \sum_{i=1}^{m} (a_i z)/(1-a_i z)$$

$$D^*(z) = \sum_{k=0}^{\infty} z^k d_k$$

with $D^*(0) = d_0 = 1$.

From (6) one obtains

$$z\frac{dD^*(z)}{dz} = \sum_{k=1}^{\infty} kz^k d_k = \sum_{k=1}^{\infty} z^k \sum_{j=1}^{k} d_{k-j} b_j$$

$$= \sum_{j=1}^{\infty} b_j z^j \left(\sum_{k=j}^{\infty} d_{k-j} z^{k-j} \right) = B^*(z) D^*(z),$$

$$\frac{d}{dz} \ln D^*(z) = \frac{B^*(z)}{z} = \sum_{i=1}^{m} a_i/(1-a_i z).$$

Integrate upwards from $z = 0$ to obtain

(9) $$D^*(z) = 1/\prod_{i=1}^{m} (1-a_i z).$$

In a similar fashion, we write (3) as

(10) $$y_i(k) = F_k(a_i)/G_k$$

and may solve for F_k and G_k via generating functions as follows. Let

$$F_k(a) = \sum_{j=1}^{k} c(k)_j (a)^j = \sum_{j=1}^{k} d_{k-j} (a)^j$$

(11) $$F^*(z,a) = \sum_{k=1}^{\infty} z^k F_k(a) = \sum_{k=1}^{\infty} \sum_{j=1}^{k} z^k d_{k-j} (a)^j$$

$$= \sum_{j=1}^{\infty} (az)^j \sum_{k=j}^{\infty} d_{k-j} z^{k-j} = \frac{azD^*(z)}{1-az}$$

$$G_k = \sum_{j=1}^{k} c(k)_j b_j = \sum_{j=1}^{k} d_{k-j} b_j$$

(12) $$G^*(z) = \sum_{k=1}^{\infty} z^k G_k = B^*(z)D^*(z)$$

$$= \left[\sum_{i=1}^{m} \frac{a_i z}{1-a_i z}\right] \frac{1}{\prod_{j=1}^{m} (1-a_j z)} .$$

Let us first treat the special case where $\{a_i\}_{i=1}^{m}$ are distinct. Notice $G^*(z)$ has a double pole at each $(a_j)^{-1}$, and the following partial fraction expansion:

(13) $$G^*(z) = \sum_{i=1}^{m} \left[\frac{H_i}{(1-a_i z)^2} + \frac{Q_i}{(1-a_i z)}\right]$$

where

$$H_i = \lim_{z \to 1/a_i} (1-a_i z)^2 G^*(z) = 1/ \prod_{\substack{j=1 \\ j \neq i}}^{m} (1-a_j/a_i)$$

$$Q_i = H_i\left(-1 + \sum_{j \neq i} \frac{a_j/a_i}{1-a_j/a_i}\right) + \sum_{j \neq i} \frac{H_i}{1-a_j/a_i} .$$

This implies

(14) $$G_k = \sum_{i=1}^{m} [(k+1)H_i + Q_i](a_i)^k \qquad k = 1,2,\ldots$$

Similarly, from (11) follows

(15) $$F^*(z,a_i) = \frac{a_i z}{(1-a_i z) \prod_{j=1}^{m} (1-a_j z)} \qquad 1 \le i \le m$$

with a double pole at $z = 1/a_i$ and a simple pole at all other $1/a_j$. A partial fraction expansion yields

$$(16)\qquad F^*(z,a_i) = \frac{H_i}{(1-a_i z)^2} + \frac{R_i}{1-a_i z} + \sum_{\substack{j=1\\ j\neq i}}^{m} \frac{B_{ij}}{(1-a_j z)}$$

where

$$B_{ij} = \lim_{z\to 1/a_j} (1-a_j z)\, F^*(z,a_i) = \frac{(a_i/a_j)H_j}{1-a_i/a_j}$$

$$R_i = F^*(0,a_i)-H_i-\sum_{\substack{j=1\\ j\neq i}}^{m} B_{ij} = -\Big(H_i+\sum_{j\neq i} B_{ij}\Big).$$

This implies

$$(17)\qquad F_k(a_i) = kH_i(a_i)^k - \Big(\sum_{\substack{j=1\\ j\neq i}}^{m} B_{ij}\Big)(a_i)^k + \sum_{\substack{j=1\\ j\neq i}}^{m} B_{ij}(a_j)^k \qquad k = 1,2,3,\ldots$$

Insertion of (14) and (17) into (10) produces an explicit expression for each $y_i(k)$. For the asymptotic behavior, let

$$(18)\qquad a_{max} = \max_{1\le i\le m} a_i$$

and let i_0 be the unique index with $a_{i_0} = a_{max}$.
Put

$$F_k(a_i) \sim k\, H_{i_0}(a_{max})^k \qquad a_i = a_{max}$$

$$\sim B_{ii_0}(a_{max})^k \qquad a_i \neq a_{max}$$

$$G_k \sim k\, H_{i_0}(a_{max})^k$$

into (10) to conclude

$$(19)\qquad y_i(k) \to 1 \qquad a_i = a_{max}$$

$$\to 0 \text{ as } k^{-1} \qquad a_i \neq a_{max}.$$

Returning now to the general case, retain the definition (18) and let $p \ge 1$ denote the multiplicity of a_{max} in $\{a_i\}_{i=1}^m$. Then (12) implies $G^*(z)$ has a pole of order $p+1$ at $z = 1/a_{max}$, and

(20) $$G_k \sim R\, k^p (a_{max})^k$$

where

(21) $$R = \lim_{z \to 1/a_{max}} (1-a_{max}\, z)^{p+1} G^*(z) = p \Big/ \prod_{\substack{i=1 \\ a_i<a_{max}}}^{m} (1-a_i/a_{max}).$$

Similarly, (15) implies $F^*(z,a_i)$ has a pole at $z = 1/a_{max}$, of order p+1 if $a_i = a_{max}$ and of order p if $a_i < a_{max}$. Consequently

(22) $$F_k(a_{max}) \sim S\, k^p (a_{max})^k$$

where

(23) $$S = \lim_{z \to 1/a_{max}} (1-a_{max}\, z)^{p+1} F^*(z,a_{max}) = R/p.$$

(24) $$F_k(a_i) \sim T_i k^{p-1} (a_{max})^k \qquad \text{if } a_i < a_{max}$$

where T_i is independent of k. Insertion of (20-24) into (10) now leads to

(25) $$y_i(k) \to \frac{1}{p} \qquad \text{if } a_i = a_{max} \quad \text{(there are p such i's)}$$

$$\to 0 \quad \text{as } k^{-1} \quad \text{if } a_i < a_{max}.$$

This completes the proof that $\lim_{k\to\infty} y_i(k)$ exists.

3. APPENDIX: DERIVATION OF THE FINITE-DIFFERENCE EQUATION [1,2,3,]

Consider a closed network of queues with m servers and $k \geq 1$ customers. Each customer has a mean service time S_i at server i, and when this service is completed he has probability p_{ij} of going next to server j. Here $p_{ij} > 0$, $\sum_{j=1}^{m} p_{ij} = 1$ and $(p_{i,j})$ is assumed to be an irreducible Markov chain with a unique equilibrium distibution $[\pi_i]$ satisfying $\pi P = \pi$, $\Sigma_{i=1}^{m}\, \pi_i = 1$ and $\pi_i > 0$ for all $i = 1,\ldots,m$.

The steady-state quantities of interest are $\lambda_i(k)$ = throughout at server i (customers served per unit time), $N_i(k)$ = mean number of customers in service or on queue at server i, and $W_i(k)$ = mean waiting time (queueing plus service) experienced by customers at server i. These satisfy the three sets of equations

(A-1) $N_i(k) = \lambda_i(k)W_i(k)$ $1 \le i \le m$

(A-2) $\lambda_i(k) = k\pi_i / \left[\sum_{j=1}^{m} \pi_j W_j(k)\right]$ $1 \le i \le m$

(A-3) $W_i(k) = S_i + S_i N_i(k-1)$ $1 \le i \le m; \quad k \ge 1$

with the understanding

(A-4) $N_i(0) = 0$ $1 \le i \le m.$

Equation (A-1) is Little's formula applied to server i. Equation (A-2) says that each of the k customers has a mean time between visits to server i of $(\pi, W(k))/\pi_i$. Together these two equations show

(A-5) $\sum_{i=1}^{m} N_i(k) = k.$

Equation (A-3) expresses the waiting time W_i as the sum of the service time S_i and queueing time $\bar{N}_i S_i$ where $\bar{N}_i$ is the mean number of customers on queue when a customer arrives. First-come first-served queue discipline and exponentially-distributed service times (with mean S_i) are assumed; with the approximation $\bar{N}_i = N_i(k-1)$, equation (A-3) is obtained.[3]

If λ_i is eliminated via (A-2) and then W_i is eliminated via (A-3), the following equations for N_i are obtained:

(A-6) $$N_i(k) = \frac{k\pi_i S_i[1+N_i(k-1)]}{\sum_{j=1}^{m} \pi_j S_j[1+N_j(k-1)]} \qquad 1 \le i \le m.$$

Letting $a_i = \pi_i S_i > 0$ and $y_i(k) = N_i(k)/k$, (A-6) reduces to (1) with boundary conditions (2).

The variable $y_i(k)$ is the *fraction* of the K customers who are located at server i. These satisfy $\sum_{i=1}^{m} y_i(k) = 1$ due to (A-5). The asymptotic result (25) says that as the number of customers becomes very large, almost all of them queue up at the set of most-heavily congested servers, who "pace" the rest of the system.

REFERENCES

[1] BUZEN, J.P., *Computational Algorithms for Closed Queueing Networks with Exponential Servers*, CACM 16, Sept. 1973, pp. 521-531.

[2] GORDON, W.T., G.F. NEWELL, *Closed Queueing Systems with Exponential Servers*, Operations Res. 15 (1967) pp. 252-265.

[3] REISER, M., S.S. LAVENBERG, *Mean Value Analysis of Closed Multichain Queueing Networks*, IBM Research Report RC 7023 March 1978, Yorktown Heights, New York.