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# STOCHASTIC FILTERING THEORY: A DISCUSSION OF CONCEPTS, METHODS, AND RESULTS

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Stochastic Filtering Theory: A Discussion of Concepts, Methods, and Results*)

by

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ABSTRACT

The purpose of the paper is to give an exposition of the stochastic filtering problem. A definition is proposed for a stochastic dynamical system, in terms of the conditional independence relation. The stochastic filtering problem is then defined as the determination of the conditional distribution of the state given past observations. Two methods to solve this problem are sketched. A list with stochastic dynamical systems for which the stochastic filtering problem has been resolved is given, and some examples are presented.

KEY WORDS & PHRASES: *Conditional Independence, Stochastic Dynamical Systems, Stochastic Filtering Problems.*

*) This report will be submitted for publication elsewhere.

## 1. INTRODUCTION

The purpose of this paper is to give an exposition of the problem, methods and results of stochastic filtering theory. The novelty of this paper is in the definition and application ot the concept of a stochastic dynamical system, and in the formulation that includes both sample continuous and jump process observations.

In this paper we restrict attention to observed processes on $\Omega \times T \Rightarrow R^k$. Due to space limitation we will not discuss stochastic filtering problems for infinite dimensional stochastic systems, for random fields, and quantum mechanical systems. Neither will we discuss the important practical issues of asymptotic analysis of filtering algorithms, filtering techniques, estimation bounds, and adaptive filtering. The reader is referred to the literature on these topics.

We briefly summarize the historical development of the stochastic filtering problem. Suppose given a stationary second order process specified by its mean and covariance function, that is considered to be observed. The linear observation prediction problem is to find a linear operation on the observations that yields a least squares estimate of the future observations. It has been the contribution of Wiener [44] and Kolmogorov to have reduced this problem to the problem of solving the Wiener-Hopf equation. The difficulty with this equation is that it seems impossible to solve it in its full generality.

It is the contribution of Kalman, and of Bucy, to have singled out a class of observed processes for which the linear observation prediction problem can be solved. The idea underlying their approach is the concept of a state and of a linear dynamical system, as developed by Kalman [17]. The model taken is a Gauss-Markov model, which class allows consideration of non-stationary processes. The linear stochastic filtering problem is then defined as the linear estimation of the state of this system given past observation. The resulting algorithms, known as the Kalman and the Kalman-Bucy filter for respectively discrete and continuous time processes, have found wide spread application [14, 15].

At about 1960 a generalization of the linear stochastic filtering problem has been formulated, in which the linear dynamical system is replaced by a nonlinear dynamic model driven by disturbances having Gaussian distribution. A precise definition of a stochastic dynamical system is not given. The filtering problem is then defined to be the estimation of the "state" of this model given past observations. For this problem a representation for the estimate has been derived known as the Kushner-Stratonovich formula [19, 39]. The filtering problem has only been resolved for two models.

Since about 1970 the filtering problem for counting and jump processes has received attention. A model similar to that in the proceeding paragraph has been adopted. A representation for the estimate of the "state" given past observations has been derived under various sets of assumptions [1, 2, 3, 4, 5, 32, 33, 37, 38, 42, 43, 47].

In this paper we propose a general framework for the stochastic filtering problem, based on the following principles. The objects we deal with are stochastic processes defined on a totally ordered parameter set, and, of course, specified by their distributions. At any time one has a past history that is assumed known with certainty, and an uncertain future about which one can only speak in terms of conditional distributions. Then we define a stochastic dynamical system in which the state transition function and the read-out function map into the distribution of the state and the observation respectively. The stochastic filtering problem is then defined to be the determination of the conditional distribution of the state given past observations.

The emphasis in this paper is on conceptual ideas. Therefore no proofs will be given. In section two we define the concept of a stochastic dynamical system and the filtering problem . In section three we present two methods to analyze filtering problems. Some examples are presented in section four. We close with some miscellaneous comments in section five. For a comprehensive survey of the literature up to 1974 on filtering theory the reader is referred to [12].

We assume that the reader is familiar with the concepts and results of the modern theory of stochastic processes, in particular on $\sigma$-algebra families, martingales, stochastic integrals and stochastic differential equations. We refer the reader to the references [6, 7, 10, 23, 27, 28, 50] for further details.

## 2. THE PROBLEM FORMULATION

2.1. The set-up. The objects that we will deal with are stochastic processes defined on some probability space and a totally ordered parameter set. We take as specification of these processes their distribution. Estimation will be understood to mean the determination of the conditional distribution given information.

DEFINITION 2.1 An *observed process* will be a collection

$$\{\{\Omega, F, P\}, \{T, B_T\}, \{R^k, B_k\}, \{F_t, t \in T\}, y\}$$

where $\{\Omega, F, P\}$ is a complete probability space, $T \subset R$ is an interval, $B_T$ the Borel $\sigma$-algebra of subsets of T, $\{R^k, B_k\}$ the k-dimensional Euclidean space with its Borel $\sigma$-algebra, $\{F_t, t \in T\}$ an increasing and complete family of $\sigma$-algebras, $y: \Omega \times T \to R^k$ a separable and measurable stochastic process such that $\{y_t, F_t, t \in T\}$ is adapted. Usually the distribution of y is specified by $E[\exp(iv'y_t)|F_s]$ for all s, $t \in T$, $s < t$, $v \in R^k$. For short we call $\{y_t, F_t, t \in T\}$ an observed process.

Historically the filtering problem has been motivated by the stochastic observation prediction problem, which is to determine $E[\exp(iv'y_t)|F_s^y]$ for all s, $t \in T$, $s < t$, $v \in R^k$. This problem can be embedded in the stochastic filtering problem. To define the stochastic filtering problem we need the definition of a stochastic dynamical system.

2.2. Conditional Independence. In this subsection we define a relation for a triple of $\sigma$-algebras, that will be used in the sequel.

DEFINITION 2.2. The $\sigma$-algebras $F_1$, $F_2$ are said to be *conditionally independent* given the $\sigma$-algebra G iff

$$E[x_1\, x_2|G] = E[x_1|G]\, E\, [x_2|G]$$

for any $x_1 \in L_{1b}(F_1)$, $x_2 \in L_{1b}(F_2)$. Notation $\{F_1, F_2, G\} \in CI$.

PROPOSITION 2.3. The following are equivalent:
a. $\{F_1, F_2, G\} \in CI$;
b. $E[x_1|F_2 \vee G] = E[x_1|G]$ for all $x_1 \in L_1(F_1)$;
c. $\{F_2, F_1, G\} \in CI$;
d. $\{F_1 \vee G, F_2 \vee G, G\} \in CI$.

PROOF. Omitted.

The concept of conditional independence is known in the literature [27], and is used in the study of Markov processes. The equivalent property 2.3.b. expresses that conditioning $F_1$ on $F_2 \vee G$, it is sufficient to know G only. Thus conditional independence is seen to be equivalent to a sufficiency property for $\sigma$-algebras. Sufficient $\sigma$-algebras in the Bayesian formulation of statistics have been considered in [36]. The concept of a splitting $\sigma$-algebra, as introduced by McKean [26], is also seen to be the same concept of conditional independence. The equivalence between these concepts seems to us to be particularly important for a stochastic system theory.

A publication on certain problems related to the conditional independence relation is in preparation.

2.3. Stochastic Dynamical Systems. In this subsection we propose a definition for a stochastic dynamical system. So as not to overburden the paper we consider here only systems without input.

Briefly, a dynamical system, that we will here call a deterministic dynamical system, without input is a collection

$$\Sigma = \{T, Y, \underline{Y}; X, f, g\}$$

where the state transition function $f: T \times T \times X \to X$, $x(t) = f(t, s, x(s))$, and the read-read-out map $g: T \times X \to Y$, $y(t) = g(t, x(t))$ satisfy certain conditions [17, p. 5].

With this definition in mind one way define a stochastic dynamical system as a collection

$$S\Sigma = \{\Omega, F, P, T, Y, \underline{Y}, X, f, g\}$$

such that the maps $f(t, s, x(s)) \longmapsto$ distribution of $x(t)$, and $g(t, x(t)) \longmapsto$ distribution of $y(t)$ satisfy certain conditions. This definition has been suggested by Kalman [17, p. 5]. However this definition presupposes a "finite dimensional" state space. Below we present a definition of a stochastic dynamical system that incorporates this idea.

Because we want to work with stochastic integrals and stochastic differential equations it is necessary to consider the increments of the observation process as the output of the stochastic dynamical system. The alternative is to work with the observation as the output but then one must use white noise processes in the representations. For discrete time processes this issue does not arise.

We introduce a somewhat different viewpoint on stochastic dynamical systems. Let $\{y_t, F_t, t \in T\}$ be an observed process, where $F_t$ represents past information at time $t \in T$. With the above intuitive definition of a stochastic dynamical system in mind a state process $\{x_t, t \in T\}$ based on the past $\{F_t, t \in T\}$ should be adapted $\{x_t, F_t, t \in T\}$ and such that

$$E[\exp(iu'x_\tau + iv'(y_t - y_s)) | F_s] = E[\exp(iu'x_\tau + iv'(y_t - y_s)) | F^{x_s}]$$

for all $s, t \in T$, $s < t$, $u \in R^n$, $v \in R^k$. This statement is equivalent to $\{{}_tF^{\Delta y} \vee {}_tF^x, F_t, F^{x_t}\} \in CI$ and $F^{x_t} \subset F_t$ for all $t \in T$, where ${}_tF^{\Delta y} = \sigma(\{y_s - y_t, \forall s > t\})$, ${}_tF^x = \sigma(\{x_s, \forall s > t\})$, $F^{x_t} = \sigma(\{x_t\})$.

We will take this last statement as our definition of a stochastic dynamical system.

To obtain a general formulation for stochastic dynamical systems we will work with σ-algebra families rather then with stochastic processes. Thus let $\{F_t, t \in T\}$ and $\{G_t, t \in T\}$ be σ-algebra families, $F_t$ representing past information and $G_t$ representing future information at time $t \in T$.

DEFINITION 2.4. A *stochastic dynamical system* is a collection

$$\{\{\Omega, F, P\}, T, \{G_t, t \in T\}, \{F_t, t \in T\}, \{H_t, t \in T\}\}$$

where $\{\Omega, F, P\}$ is a complete probability space, T a totally ordered set, $\{G_t, t \in T\}$, $\{F_t, t \in T\}$, $\{H_t, t \in T\}$ are complete sub-σ-algebra families of F, such that for all $t \in T$

$$\{G_t \vee (\vee_{s\geq t} H_s),\ F_t \vee (\vee_{\tau\leq t} H_\tau),\ H_t\} \in CI.$$

Then we call $\{H_t,\ t \in T\}$ the *state σ-algebra* at $t \in T$.
Notation $\{G_t,\ F_t,\ H_t,\ t\in T\} \in \Sigma S$.

b. If in addition there exists a stochastic process $x : \Omega \times T \to R^n$ such that $H_t = F^{x_t}$ for all $t \in T$, then we call $\{G_t,\ F_t,\ F^{x_t},\ t \in T\}$ a *finite dimensional stochastic dynamical system* and x the *state process*. Notation $\{G_t,\ F_t,\ F^{x_t}\} \in \Sigma SF$.

DEFINITION 2.5. Given the observed process $\{y_t,\ F_t,\ t \in T\}$.

a. If there exists a complete σ-algebra family $\{H_t,\ t \in T\}$ with $H_t \subset F_t$ for all $t \in T$ and $\{{}_tF^{\Delta y},\ F_t,\ H_t,\ t \in T\} \in \Sigma S$, then we call this collection a *forward stochastic dynamical system for y*. Here ${}_tF^{\Delta V} = \sigma(\{y_s-y_t,\ \forall s>t\})$.

b. If in addition there exists a process $x : \Omega \times T \to R^n$ with $\{x_t,\ F_t,\ t \in T\}$ adapted such that $\{{}_tF^{\Delta y},\ F_t,\ F^{x_t}, t \in T\} \in \Sigma SF$ then we call this collection a *finite dimensional forward stochastic dynamical system for y*. In this case we call

$$E[\exp(iv'(y_t-y_s))|F^{x_s}]:\Omega\times T\times T\times R^n \to \underline{C}_k,\quad E[\exp(iu'x_t)|F^{x_s}]:\Omega\times T\times T\times R^n \to \underline{C}_n$$

respectively the *stochastic read-out function* and the *stochastic state-transition function* of this stochastic dynamical system, where $\underline{C}_k$ is the set of characteristic functions $\underline{C}_k$: $R^k \to C$.

c. A *stochastic dynamical system representation* is a specification of the stochastic state transition function and the stochastic read out function of a stochastic dynamical system.

d. A *stochastic differential stochastic dynamical system representation* is a stochastic dynamical system representation in the form of a pair of stochastic differential equations driven by independent increment processes for the state process and the observed process.

The justification for calling the collection $\{G_t, F_t, H_t, t \in T\}$ a stochastic dynamical system is in the interpretation of the defining property, namely that $\{G_t \vee(\vee_{s\geq t}H_s),\ F_t \vee(\vee_{\tau\leq t}H_\tau),\ H_t\} \in CI$ for all $t \in T$, or, equivalently, that for all $t \in T$, $A \in G_t \vee(\vee_{s\geq t} H_s)$ we have

$$E[I_A \mid F_t \vee(\vee_{\tau\leq t}H_\tau) \vee H_t] = E[I_A \mid H_t].$$

In words this says that any event in the future information or the future states conditioned on past information and past states, depends only on the current state. Thus the two properties of a dynamical system, namely sufficiency of the state for the output and recursiveness of the state, are captured by the above definition.

The definition of a stochastic dynamical system also implies that $\{\vee_{s\geq t}H_s,\ \vee_{\tau\leq t}H_\tau,\ H_t\} \in CI$ for all $t \in T$, hence $\{H_t,\ t \in T\}$ may be called a Markovian σ-algebra family. If in addition there exists a process $x : \Omega \times T \to R^n$ such that $H_t = F^{x_t}$ for all $t \in T$, then we can conclude that $\{x_t,\ F_t,\ t \in T\}$ is a Markov process.

Note that in definition 2.4 no restriction is given on the σ-algebra family $\{H_t, t \in T\}$. The term forward in definition 2.5 is now to be understood in connection with the condition $F^{x_t} \subset F_t$; thus the state is constructed on the basis of past information. A corresponding definition can be given for a backward stochastic dynamical system, reminiscent of backward Markov models. This topic will not be elaborated here.

Having given a definition of a stochastic dynamical system the following problems arise, the stochastic realization problem, the definition of stochastic observability, and related issues. We will leave these problems to future publications, except for stating the following problem.

DEFINITION 2.6. The *stochastic realization problem.* Given an observed process $\{y_t, F_t, t \in T\}$.

a. Find, if possible, a σ-algebra family $\{H_t, t \in T\}$ with $H_t \subset F_t$ for all $t \in T$, such that $\{_tF^{\Delta y}, F_t, H_t, t \in T\} \in \Sigma S$.

b. Find, if possible, a stochastic process $x : \Omega \times T \to R^n$ with $\{x_t, F_t, t \in T\}$ adapted, such that $\{_tF^{\Delta y}, F_t, F^{x_t}, t \in T\} \in \Sigma SF$.

c. Given $\{_tF^{\Delta y}, F_t, F^{x_t}, t \in T\} \in \Sigma SF$. Find, if possible, a stochastic differential stochastic dynamical system representation for x, y.

Some examples of stochastic dynamical systems are given in section four.

We point out that the above approach to stochastic cynamical systems differs essentially from what should be called linear stochastic dynamical systems. There the objects are second order stochastic processes, specified by their first and second moment; the spaces are the Hilbert spaces generated by linear operations on these processes; and the conditioning operation is the Hilbert space projection operation. This formulation is more or less implicit in Kalman's work [16], and has been formalized in the work by Faurre, Akaike, Picci, Lindquist and Ruckebush. For references see [20, 21, 22, 29]. The definitions given here have been inspired by these publications, in particular by the work by Picci [29].

2.4. The Filtering Problem. With the concept of a stochastic dynamical system defined, we can now present the definition of the stochastic filtering problem.

DEFINITION 2.7. Given the observed process $\{y_t, F_t, t \in T\}$ and suppose that $\{_tF^{\Delta y}, F_t, F^{x_t}, t \in T\} \in \Sigma SF$.

a. The *stochastic filtering problem* is to determine the conditional characteristic function

$$E[\exp(iu'x\ ) \mid F^y_t]$$

for all $t \in T$, $u \in R^n$.

b. If there exists a process $z : \Omega \times T \to R^m$ with $\{z_t, F^y_t, t \in T\}$ adapted, such that

$\{F^{x_t}, F^y_t, F^{z_t}, t \in T\} \in \Sigma SF$, then we call this collection a *finite dimensional stochastic dynamical filter system* for the above defined stochastic filtering problem. For short, we call this collection a *filter system*, and z the *filter state*.

To determine the conditional characteristic function in 2.7.a. will be understood as to exhibit the function from the past of the observations to the characteristic function. We will use the term stochastic filtering problem rather than the term stochastic reconstruction problem, which term is suggested by the analogy with deterministic dynamical system theory [17].

A filter system has the two properties

$$E[\exp(iu'x_t)|F^y_t] = E[\exp(iu'x_t)|F^{z_t}],$$

$$E[\exp(iw'z_t)|F^y_s] = E[\exp(iw'z_t)|F^{z_s}],$$

or $z_t$ is a sufficient variable in estimating $x_t$ given $F^y_t$, and $\{z_t, F^y_t, t \in T\}$ is a Markov process. The last statement implies intuitively that z can be computed recursively, but this aspect we have been unable to clarify yet. Clearly the existence of a finite dimensional filter system is important for the practical application of this theory. It is not at all clear that the filter state will be $E[x_t|F^y_t]$.

It can be shown that the stochastic observation prediction problem can be embedded in the stochastic filtering problem. Here we will not consider the stochastic prediction and the stochastic smoothing problem, which are to determine

$$E[\exp(iu'x_t)|F^y_s]$$

for $t > s$ and $t < s$ respectively.

A method to solve the stochastic filtering problem is to reduce it to the problem of solving an equation for the conditional characteristic function.

## 3. METHODS

In this section we present two methods for the stochastic filtering problem, both of which yield equations for the conditional characteristic function.

3.1. *The Semi-Martingale Representation Method.* We start by defining two concepts from the theory of stochastic process.

DEFINITION 3.1. The process $\{x_t, F_t, t \in T\}$ is called an *uniformly integrable semi-martingale* iff x has a decomposition as $x = x_0+a+m$ where $x_0 \in L_1(F_0)$, $\{a_t, F_t, t \in T\} \in V_1$ is of integrable variation, $a_0 = 0$, $\{m_t, F_t, t \in T\} \in M_{1u}$ is an uniformly integrable martingale, $m_0 = 0$. Notation $\{x_t, F_t, t \in T\} \in SM_{1u}$.

The above class of semi-martingales is a sub-class of those introduced in [28], to which the reader is referred for further details. The class of semi-martingales has proven to be an extremely general class of processes, that is closed under a large number of operations.

DEFINITION 3.2. Let $\{y_t, F_t, t \in T\}$ be an observed process. We say that the *martingale representation condition* holds for the class $M_{1uloc}\{F^y_t, t \in T\}$ if there exists a sample continuous local martingale $m^c$ and a positive integer valued random measure p such that if $m \in M_{1uloc}\{F^y_t, t \in T\}$ then m has a representation as

$$m = m_0+(h.m^c)+(\phi.(p-\bar{p}))+(\psi.p),$$

for certain predictable processes h, $\phi$, $\psi$.

Here the expressions on the right hand side are stochastic integrals, we refer to [10, 28] for details. It is a rather deep and important result in stochastic integration theory that the martingale representation condition is satisfied for a large number of observed processes.

We formulate a sub-problem of the stochastic filtering problem.

DEFINITION 3.3. Given the observed process $\{y_t, F_t, t \in T\}$ and assume that the martingale representation condition is satisfied for $M_{1uloc}\{F^y_t, t \in T\}$. Let $\{x_t, F_t, t \in T\} \in SM_{1u}$. The *semi-martingale representation problem* is to give a decomposition for the projection of x on $\{F^y_t, t \in T\}$.

The solution to this problem is provided by the following ideas. The projection of x on $\{F^y_t, t \in T\}$ is defined to be $\hat{x} = \{E[x_t|F^y_t], F^y_t, t \in T\}$ which is again a semi-martingale, say with decomposition $\hat{x} = \hat{x}_0+\bar{a}+\bar{m}$. A relation between a and $\bar{a}$ can be given. Then the martingale representation condition is invoked to obtain a representation for $\bar{m}$. Finally the processes in this martingale representation can be determined. The above method has been proposed in [9]. Note the analogy with linear stochastic filtering theory.

We will not attempt to solve the above problem here. Below we present two canonical cases. Special cases and generalizations may be found in [1, 3, 4, 9, 23, 32, 33, 40, 41, 42, 43, 48].

THEOREM 3.4. Let the observed process $\{y_t, F_t\ t \in T\}$ and $x : \Omega \times T \to R^n$ satisfy

$$\begin{aligned} dx_t &= f_t dt+dm_t,\ x_0, \\ dy_t &= h_t dt+dw_t,\ y_0, \end{aligned}$$

where $w : \Omega \times T \to R^k$, $\{w_t, F_t, t \in T\}$ is a standard Brownian motion process, $h : \Omega \times T \to R^k$, $\{h_t, F_t, t \in T\} \in SM_{1u}$ with $E[\int_T \|h_s\|^2 ds] < \infty$, $m : \Omega \times T \to R^n$, $\{m_t, F_t, t \in T\} \in M_2$, $f : \Omega \times T \to R^n$, $\{f_t, F_t, t \in T\} \in SM_{1u}$ with $E[\int_T \|f_s\|^2 ds] < \infty$, and that $\|x_t(\omega)\| \leq 1$ for all $(\omega, t) \in \Omega \times T$.

a. Then the martingale representation condition holds for the class $M_{1uloc}\{F^y_t, t \in T\}$.

b. There exists a process $\phi : \Omega \times T \to R^{n+k}$, $\{\phi_t, F_t, t \in T\} \in L_1(t) \cap SM_{1u}$ such that $<m, w>_t = \int_0^t \phi_s ds$.

c. The solution to the semi-martingale representation problem is given by

$$d\hat{x}_t = \hat{f}_t dt + [\hat{\Sigma}_t^{xh} + \hat{\phi}_t]\,(dy_t - \hat{h}_t dt),\ \hat{x}_0 = E[x_0 | F_0^y],$$

$$\hat{\Sigma}_t^{xh} = E[(x_t - \hat{x}_t)(h_t - \hat{h}_t)' | F_t^y],$$

where the hat symbol denotes the projection of a semi-martingale on the σ-algebra $\{F_t^y,\ t \in T\}$.

PROOF [9, 23]

The formula of 3.4.c. is known in the literature as the Kushner-Stratonovich formula.

THEOREM 3.5. Let the observed process $\{y_t, F_t, t \in T\}$ and $x : \Omega \times T \to R^n$ satisfy

$$x_t = x_0 + a_t + m_t,$$
$$p(w,\ dt \times dv) = h(t,\ v)\ \mu(dt,\ dv) + q(w,\ dt \times dv)$$

where y is a pure jump process, p its associated jump measure, $\{h(t, v), F_t, t \in T, v \in R^k\}$ predictable, $\{\mu((0, t]xA), F_t^y, t \in T, A \in B_k\}$ predictable, $\{q(w, (0, t]xA), F_t, t \in T, A \in B_k\} \in M_{1uloc}$, $x \in SM_{1u}$ with $m \in M_2$.

a. Then the martingale representation condition holds for $M_{1uloc}\{F_t^y, t \in T\}$.
b. There exists a predictable process $\{\psi(t,v), F_t, t \in T, v \in R^k\}$ such that

$$< m,\ q(w,\ (0,\ t]xA) >_t = \int_0^t \int_A \psi(s,\ v)\ h(s,\ v)\ \mu(ds,\ dv).$$

c. The solution to the semi-martingale representation problem is given by

$$\hat{x}_t = \hat{x}_0 + \bar{a}_t + \int_0^t \int_{R^k} k(s,\ v)\ \bar{q}(w,\ ds \times dv),$$

$$\bar{q}(w,\ dt \times dv) = (p(w,\ dt \times dv) - \hat{h}(t,\ v)\ \mu(dt,\ dv),$$

$$k(t,v) = \left(E[(x_t - \hat{x}_t)(h(t,\ v) - \hat{h}(t,\ v))' | F_t^y] + E[\psi(t,\ v) h(t,\ v) | F_t^y]\right) \hat{h}(t,\ v)^{-1},$$

of which a predictable version is taken.

PROOF [1, section 5].

We return to the stochastic filtering problem. Let $\{{}_tF^{\Delta y}, F_t, F^{xt}, t \in T\} \in \Sigma SF$, and suppose that the state process x is a semi-martingale. Then it can be shown that for all $u \in R^n$ the process $\{\exp(iu'xt), F_t, t \in T\} \in SM_{1u}$ is a semi-martingale. Depending on the availability of the solution to the semi-martingale representation problem for the stochastic system under consideration, one obtains the semi-martingale decomposition for the process $\hat{c}(u) = \{E[\exp(iu'x_t) | F_t^y], F_t^y, t \in T\}$. In general one can express the processes in the decomposition as operations on $\hat{c}(u)$, so that this representation becomes a genuine equation for the conditional characteristic function. One is then faced with the question how to obtain a solution for this equation.

There are few results on this equation for the conditional characteristic function. To be specific, one would want conditions for the existence and uniqueness of the solution, and methods to determine the solution. We mention a few cases in which the equation can be resolved. The first case is where the state process is a finite state Markov process. The second case is where the state process is a discrete state Markov process, see [31]. The third case is for the linear Gaussian, Gauss-Markov model that underlies the Kalman-Bucy filter. The method consists of converting the equation to an equation for the conditional moments, which may then be solved by using properties of the Gaussian distribution; see [13] for details. Yet another method is to extent the results for the discrete time case by a limiting argument, but one would hope for a more direct approach.

3.2. *The Measure Transformation Method.* A second method to obtain an equation for the conditional characteristic function is the measure transformation method introduced by Zakai [49].

The idea of this approach is to perform a measure transformation, such that under the new measure the processes x and y are independent. An equation for the operator of conditioning on $F_t^y$ then readily follows, which equation has to be solved. The advantage of the method is that the independence of x and y under the new measure makes the calculations involved easier.

The only assumption necessary for the application of this method is the absolute continuity, for which conditions are available in the literature. The generator for the state process, which is a Markov process by the stochastic dynamical system assumption, is not needed.

The resulting equation obtained by this method can be converted into a semimartingale representation as obtained in section 3.1.

The application of this method to sample continuous observed processes may be found in [45, 49] of which we give a summary below. For jump processes the method can be found in [1, 2, 5, 6].

THEOREM 3.6. Given the observed process $\{y_t, F_t, t \in T\}$ and the process $x : \Omega \times T \to R^n$ satisfying

1. $E[\exp(iv'(y_t - y_s))|F_s \vee F_s^x] = \exp(iv'\int_s^t C(\tau)x_\tau d\tau - \frac{1}{2}v' I_k (t-s)v)$ where $s, t \in T$, $s < t$, $v \in R^k$, $C : T \to R^{k\times n}$;
2. $\{x_t, F_t, t \in T\}$ is a Markov process such that

$$E[\int_T \| C(\tau)x_\tau \|^2 d\tau] < \infty.$$

a. Then $\{{}_tF^{\Delta y}, F_t, F^{x_t}, t \in T\} \in \Sigma SF$.

b. There exists a probability measure $P_0 : F \to [0,1]$ such that

1. $P \ll P_0$ on $F$ with $\rho_t = E_0[dP/\ dP_0|F_t]$,

$$\rho_t = \exp(\int_0^t x_s' C'(s)\ dy_s - \tfrac{1}{2}\int_0^t x_s' C'(s)\ C(s)\ x_s ds);$$

2. under $P_0$ $\{y_t, F_t, t \in T\}$ is standard Brownian motion;
3. under $P_0$ $F_T^x$, $F_T^y$ are independent;
4. $P = P_0$ on $F_T^x$.

Then $E[\exp(iu'x_t)|F_t^y] = E_0[\exp(iu'x_t)\rho_t|F_t^y]/E_0[\rho_t|F_t^y]$ a.s.

c. We have the equation

$$E_0[\exp(iu'x_t)\ \rho_t|F_t^y] = E[\exp(iu'x_t)]$$

$$+\int_0^t E_0[\rho_s\ E_0[\exp(iu'x_t)|F^{x_s}]x_s'|F_s^y]C'(s)\ dy_s.$$

PROOF [45, 6.5].

4. EXAMPLES

In this section we indicate some stochastic dynamical systems for which the stochastic filtering problem has been resolved. In the list below we summarize the stochastic dynamical system by the conditional distribution for the observed process and the character of the state process.

The stochastic filtering problem has been resolved for the following stochastic dynamical systems.

1. The Gaussian, Gauss-Markov system, yielding the Kalman-Bucy filter [15, 23].
2. The Gaussian, Finite State Markov process system, Wonham [46].
3. The Poisson, Finite State Markov process system, Segall [34], Rudemo [30].
4. The Poisson, Gamma system, Frost [51], see theorem 4.3. below.
5. The jump process with Gaussian kernel in its dual predictable measure, with Gauss-Markov state process. Reference Fishman, Snyder [8].
6. The observed process is a function of the state process, while the state process is a Markov process with a discrete state space, Rudemo [31].
7. The Gaussian, Bilinear system, as presented by Marcus, Willsky [25].

We remind the reader that we have excluded stochastic filtering problems on geometric structures, algebraic structures, and partially ordered sets. No claim is made that the above list is complete.

Below we present the solutions to the stochastic filtering problem for three stochastic dynamical systems.

THEOREM 4.1. The linear Gaussian - Gauss Markov system.
Given the observed process $\{y_t, F_t, t \in T\}$, $x : \Omega \times T \to R^n$, and assume that

1. $E[\exp(iv'(y_t - y_s))|F_s V_s F^x] = \exp(iv'\int_s^t C(\tau)x_\tau d\tau - \tfrac{1}{2}v' I_k (t-s)v)$

   for $s, t \in T$, $s < t$, $v \in R^k$, $C : T \to R^{k\times n}$, $I_n \in R^{k\times k}$ the unit matrix, $y_0 = 0$;

2. $\{x_t, F_t, t \in T\}$ a Gauss-Markov process such that $E(x_t) = 0$, $Q(t, s) = E[x_t x_s'] > 0$ for all $s, t \in T$, $x$ is almost surely sample continuous, $Q : T\times T \to R^{n\times n}$ is differentiable and

$$dQ(t, s)/dt = A(t)Q(t, s),$$

$$A(t)Q(t, t)+Q(t, t)A'(t) \le -dQ(t, t)/dt.$$

   a. Then $\{{}_tF^{\Delta y}, F_t, F^{x_t}, t \in T\} \in \Sigma SF$.

   b. There exists $m \in Z_+$ and independent standard Brownian motion processes $v : \Omega \times T \to R^m$, $w : \Omega \times T \to R^k$ such that we have the representation

$$dx_t = A(t)\, x_t dt+Q(t, 0)B(t)dv_t,\ x_0,$$

$$dy_t = C(t)\, x_t dt+dw_t,\ y_0 = 0,$$

   where $B : T \to R^{n\times m}$ is a full rank solution to

$$B(t)B'(t) = Q(t,0)^{-1}[dQ(t, t)/dt-A(t)Q(t, t)-Q(t, t)A'(t)]Q^{-1}(t, 0).$$

   c. The solution to the stochastic filtering problem for the stochastic system of a. is given by

$$E[\exp(iu'x_t)|F_t^y] = \exp(iu'\hat{x}_t-\tfrac{1}{2}u'\Sigma(t)u),$$

$$d\hat{x}_t = A(t)\hat{x}_t dt+\Sigma(t)C'(t)(dy_t-C(t)\hat{x}_t dt),\ \hat{x}_0 = E(x_0),$$

$$d\Sigma(t)/dt = A(t)\Sigma(t)+\Sigma(t)A'(t)+Q(t,0)B(t)B'(t)Q'(t,0)-\Sigma(t)C(t)C'(t)\Sigma(t),$$

$$\Sigma(0) = E[(x_0-E(x_0))(x_0-E(x_0))'].$$

   d. $\{F^{x_t}, F_t^y, F^{\hat{x}_t}, t \in T\}$ is a finite dimensional filter system, known as the Kalman-Bucy filter system.

PROOF. The results of a. and b. are easily established. For c. see [23]. Then d. follows.

THEOREM 4.2. The *Poisson-FSMP system*. Given the observed process $\{y_t, F_t, t \in T\}$, $y : \Omega \times T \to R$ and $x : \Omega \times T \to R^n$ and assume that

1. $E[\exp(iv'(y_t-y_s))|F_s \vee {}_sF^x] = \exp(\int_s^t C(\tau)x_\tau dt(e^{iv}-1))$ for $s, t \in T$, $s < t$, $v \in R$, $y_0 = 0$, $C : T \to R^{1\times n}$;

2. $\{x_t, F_t, t \in T\}$ is a finite state Markov process, say with state space $X = \{x_1, x_2,\ldots x_m\}\subset(0,\infty)^n$; let $z : \Omega \times T \to R^m$, $z_t^i = I_{\{x_t=x_i\}}$, $\phi : T \times T \to R^{m\times m}$

$$\phi^{ij}(t, s) = E[z_t^i z_s^j]/E[z_s^j] \quad \text{if} \quad E[z_s^j] > 0,\ s \le t,$$
$$= 0 \qquad \text{otherwise};$$

   assume that $\phi(t, s) > 0$ for all $s, t \in T$, and that $\phi(\cdot,0) : T \to R^{m\times m}$ is differentiable, say with

$$d\phi(t, 0)/dt = A(t)\phi(t,0)$$

   for $A : T \to R^{m\times m}$; let $D = (x_1, x_2,\ldots x_m)$.

a. Then $\{{}_tF^{\Delta y}, F_t, F^{x_t}, t \in T\} \in \Sigma SF$.

b. There exist processes m, m' such that we have the representation

$$dz_t = A(t)z_t dt+\phi(t, 0)dm'_t, \; z_0,$$

$$dy_t = C(t)Dz(t)dt+dm_t, \; y_0 = 0$$

where $\{m_t, F_t, t \in T\} \in M_1$, $\{m'_t, F_t, t \in T\} \in M_1$.

c. The solution to the stochastic filtering problem for the stochastic system of definition a. is

$$E[\exp(iu'x_t)|F^y_t] = \sum_{i=1}^{m} \exp(iu'x_i)\, \hat{z}^i_t,$$

$$d\hat{z}_t = A(t)\hat{z}_t dt+\hat{k}_{t-}(D\hat{z}_{t-})^{-1}(dy_t-C(t)D\hat{z}_t dt), \; \hat{z}_0 = E(z_0),$$

$$\hat{k}_t = [\text{diagonal}(\hat{z}_t)-\hat{z}_t\hat{z}'_t]D'C'(t).$$

PROOF. The results of a. and b. follow from the theory for stochastic dynamical systems. For c. see [30, 34].

THEOREM 4.3. Let the observed process $\{y_t, F_t, t \in T\}$ with $k = 1$, and $x : \Omega \times T \to R_+$ satisfy

1. $E[\exp(iv(y_t-y_s))|F_s \vee {}_sF^x] = \exp(\int_s^t x_\tau d\tau\,(e^{iv}-1))$ for $s, t \in T$, $s < t$, $v \in R$, $y_0 = 0$;
2. $\{x_t, F_t, t \in T\}$ is a Markov process of the form $x_t = e^{\alpha t}x_0$ where $x_0 : \Omega \to R_+$ has a Gamma distribution with parameters $r, \beta \in (0, \infty)$, and $\alpha \in R_-$.

a. Then $\{{}_tF^{\Delta y}, F_t, F^{x_t}, t \in T\} \in \Sigma SF$, and we have the representation

$$dx_t = \alpha x_t dt, \quad x_0$$

$$dy_t = x_t dt+dm_t, \; y_o = 0,$$

where $\{m_t, F_t, t \in T\} \in M_1$. Also y is a counting process.

b. The solution to the stochastic filtering problem is given by

$$E[\exp(iux_t)|F^y_t] = (1-iu\beta(t))^{-(y_t+r)},$$

$$d\beta(r)/dt = \alpha\beta(t)-\beta^2(t), \; \beta(0) = \beta.$$

Then $\hat{x}_t = \beta(t)(y_t+r)$.

c. A recursive equation for $\hat{x}$ is given by

$$d\hat{x}_t = \alpha\hat{x}_t dt+\beta(t)(dy_t-\hat{x}_t dt), \; \hat{x}_0 = r\beta,$$

$$d\beta(t)/dt = \alpha\beta(t)-\beta^2(t), \; \beta(0) = \beta.$$

PROOF. The solution in b. can be found in [51] for the case $\alpha = 0$. See also [32]. Attempts to generalize the above solution to a larger class of stochastic dynamical systems have proven futile.

## 5. COMMENTS

5.1. *Research on the Stochastic Filtering Problem.* Here we give a few remarks on the stochastic filtering problem that may be relevant to future research efforts in this area.

The practical application of this theory seems to demand finite dimensional filter systems as solutions to stochastic filtering problems. It seems extremely unlikely that the solution to the stochastic filtering problem for arbitrary stochastic dynamical systems will be a finite dimensional filter system. The question should therefore be posed: find all stochastic dynamical systems that yield finite dimensional filter systems. One would hope that a resolution of this question also would yield structural information that may be used in filtering techniques.

We mention a few ideas that may be used to resolve the above question. It seems worthwile to require that the conditional distribution $E[\exp(iu'x_t)|F^y_t]$ is invariant in time, in other words is of the same type for all $t \in T$. If the underlying dynamics are linear, this probably will lead to the class of infinitely divisible distributions. One way the distribution may be made invariant is to choose a pair of conjugate distributions [52] for the stochastic dynamical system.

5.2. *Open Problems.* We mention a few issues that are relevant to the future development of a stochastic filtering theory.

1. The formulation of a general theory for stochastic dynamical systems. The issues here are general definitions, the stochastic realization problem, the formulation of the concepts of stochastic observability and stochastic controllability, etc.
2. The question of which classes of stochastic dynamical systems yield finite dimensional stochastic dynamical filter systems.
3. The investigation of equations for the conditional characteristic function. The issues here are the existence and uniqueness of solutions, and techniques to solve these equations.
4. The extension of the ideas presented in this paper to infinite dimensional stochastic systems, to systems defined on geometric structures, and stochastic systems on partially ordered sets.

5.3. *On Stochastic Filtering and Stochastic Control.* Since this paper is presented at a stochastic control oriented meeting we briefly indicate the relation between stochastic filtering and stochastic control.

Suppose given $\{\Omega, F, P\}$, $\{F_t, t \in T\}$, an observed process $\{y_t, F_t, t \in T\}$, and an input process $\{u_t, F_t, t \in T\}$ belonging to a class U of admissable input processes. If there exists a process $\{x_t, F_t, t \in T\}$ such that for all $t \in T$

$$\{{}_tF^{\Delta y} \vee {}_tF^x,\ F_t,\ F^{x_t} \vee {}_tF^u\} \in CI$$

then we call the collection $\{{}_tF^{\Delta y}, F_t, F^{x_t} \vee {}_tF^u, t \in T\}$, a stochastic dynamical system, with input.

The stochastic filtering problem in the context of control is to determine $E[\exp(iw'x_t)|F_t^y \vee F_t^u]$ for all $t \in T$, $w \in R^n$; if there exists a process $\{z_t, F_t^y \vee F_t^u, t \in T\}$ such that $\{F^{x_t}, F_t^y \vee F_t^u, F^{z_t} \vee_t F^u, t \in T\}$ is a stochastic dynamical system, then we call this collection a filter system.
The filter separation property is said to hold iff $E[\exp(iw'x_t)|F_t^y \vee F_t^u] = E[\exp(iw'x_t)|F_t^y]$ for all $t \in T$, $w \in R^n$.

Given a cost function $C : \Omega \times U \to R_+$. The stochastic control problem is to find $u^* \in U$ such that

$$E[c(u^*)|F_t^y \vee F_t^{u^*}] \leq E[c(u)|F_t^y \vee F_t^u]$$

for all $u \in U$ such that $u_s^* = u_s$ for $s \in [0, t]$, and all $t \in T$.
The control separation property is said to hold iff there is no loss in cost in restricting attention to controls adapted to the $\sigma$-algebra generated by $\{E[\exp(iw'x_t)|F_t^y \vee F_t^u], t \in T\}$. If both separation properties hold, then the control process will be a function of the filter state.

The above remark should be considered to be a first sketch of a general formulation.

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