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ON STOCHASTIC DYNAMICAL SYSTEMS

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On stochastic dynamical systems *)

by

C. van Putten & J.H. van Schuppen

ABSTRACT

The conditional independence relation for a triple of σ-algebras is investigated, specifically the question of the characterization and the construction of minimal σ-algebras that make two given σ-algebras conditionally independent. A definition of a σ-algebraic stochastic dynamical system is proposed for σ-algebra families in terms of the conditional independence relation. For this σ-algebraic stochastic dynamical system the stochastic realization problem is posed. From this general formulation the corresponding concepts for stochastic processes may be deduced.

KEY WORDS & PHRASES: *conditional independence, stochastic dynamical system, stochastic realization problem.*

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1. INTRODUCTION

The purpose of this paper is to report on our research work on stochastic dynamical systems, specifically on the definition of this concept and the associated stochastic realization problem. The novelty of this paper is in the application of the concept of conditional independence, for a triple of σ-algebras. The motivation of our research work is in the problem of finding system models for arbitrary stochastic processes, and in the stochastic filtering and the stochastic control problem.

What is the stochastic realization problem and what results have been obtained so far? The stochastic realization problem has been defined as to find a representation for a Gaussian process, or equivalently a second order process, as the output of a linear dynamical system driven by Gaussian independent increment processes. This problem has been posed by Kalman (5), motivated by the formulation of a stochastic system theory and the linear stochastic filtering problem. Doob (3) in 1944 has initiated research on Gaussian processes. Faurre (4) has shown existence of realizations using the connection with linear deterministic dynamical systems, specifically the spectral factorization and the positive real lemma. The first probabilistic approach has been given by Akaike (1) using the concept of canonical variables. Picci (12,13) has extended this approach to continuous time Gaussian processes and indicated the relation with sufficient statistics and splitting σ-algebras. Lindquist and Picci (7,8,9) have resolved the strong stochastic realization problem, and clarified the problem of finding all minimal output based realizations. Ruckebusch (15,16,17) has developed a Hilbert space formulation which also applies to the infinite dimensional case.

Basic to our approach is the conditional independence relation for a triple of σ-algebras. A major problem for this relation is to characterize and to construct all minimal σ-algebras that make two given σ-algebras conditionally independent. We will present some partial results on this question. The formulation and the proofs given are definitely different from the case of Gaussian random variables or equivalently conditional independence in Hilbert space.

Our aim is to treat stochastic realization problems for arbitrary stochastic processes. Rather than working with stochastic processes we will work with the spaces they generate, namely σ-algebra families. Thus we define a σ-algebraic stochastic dynamical system in terms of the conditional independence relation. The characterizing property is the condition that future outputs and future states and past outputs and past states are conditionally independent given the current state. A definition of stochastic observability and stochastic reconstructability will also be given. For this concept we then pose the stochastic realization problem. From this formulation for σ-algebra families we may then deduce the corresponding definitions and results for stochastic processes. The approach given is in many respects similar to that presented by Ruckebusch (17) in terms of Hilbert spaces.

Since this is only a brief paper no proofs will be given. A publication on the material of this paper is in preparation. Througout this paper $\{\Omega,F,P\}$ will be a complete probability space. All sub-σ-algebras of F will be assumed to be complete. Without mentioning otherwise any random variable and stochastic process is real valued. All stochastic processes will be assumed to have separable and measurable modifications with sample paths that are right continuous with left hand limits.

A brief outline of the paper follows. In section two we present results for the conditional independence relation. The definition of a stochastic dynamical system and an example is given in section three. The stochastic realization problem is posed and commented upon in section four.

2. CONDITIONAL INDEPENDENCE

An important tool in our definition of a stochastic dynamical system is the concept of conditional independence, a concept known in the literature. For the conditional independence relation we present some results on invariance of this relation under certain operations. Furthermore we pose the problem of characterizing all σ-algebras that make two given σ-algebras conditional independence and that are minimal with respect to set inclusion. We shall conclude with an example in which all those minimal σ-algebras can be exhibited.

Below $L_b^1(F_i)$ is the collection of all bounded F_i-measurable real valued random variables $(i=1,2)$; $L^1(F_i)$ is the collection of all F_i-measurable real valued random variables with finite expectation $(i=1,2)$.

Definition 2.1. Let $\{\Omega,F,P\}$ be a probability space and F_1, F_2, G sub-σ-algebras of F. F_1 and F_2 are said to be *conditionally independent* given G iff

$E[x_1x_2 \mid G] = E[x_1 \mid G]E[x_2 \mid G]$

for all $x_1 \in L_b^1(F_1)$, $x_2 \in L_b^1(F_2)$.

Notation $(F_1,F_2,G) \in CI(P)$. G is said to be *splitting* w.r.t. F_1 and F_2. □

Example 2.2. F, F_1 and $F_1 \vee F_2$ (the smallest σ-algebra containing both F_1 and F_2) are splitting σ-algebras w.r.t. F_1 and F_2.

Proposition 2.3. The following statements are equivalent.

(a) $(F_1,F_2,G) \in CI(P)$;

(b) $(F_2,F_1,G) \in CI(P)$;

(c) $E[x_1 \mid F_2 \vee G] = E[x_1 \mid G]$ for all $x_1 \in L^1(F_1)$. □

If G is the σ-algebra containing all sets N of F with $P(N) = 0$ or $P(N) = 1$ then $E[x \mid G] = Ex$ for $x \in L^1(F)$ and consequently $(F_1,F_2,G) \in CI(P)$ is equivalent to independence of F_1 and F_2.

The concept of conditional independence is used in the study of Markov processes. The equivalent property 2.3(c) expresses that conditioning F_1 on $F_2 \vee G$, it is sufficient to know G only. Thus conditional independence is seen to be equivalent to a sufficiency property for σ-algebras. Sufficient σ-algebras in the Bayesian formulation of statistics have been considered in (18).

Example 2.4. If H_1, H_2, H_3 are independent σ-algebras, then H_3 is a splitting σ-algebra w.r.t. $H_1 \vee H_3$ and $H_2 \vee H_3$.

Conditional independence is preserved under certain changes in F_1, F_2, G and P as the following propositions show. First we introduce the operations of projection of σ-algebras.

Definition 2.5. Let F_1, F_2 be sub-σ-algebras of F, then $\sigma(F_1 \mid F_2)$ is defined to be the smallest σ-algebra with respect to which all conditional expectations $E[x_1 \mid F_2]$, $x_1 \in L^1(F_1)$ are measurable

$\sigma(F_1 \mid F_2) = \sigma(\{E[x_1 \mid F_2] \mid x_1 \in L^1(F_1)\})$. □

Proposition 2.6. Let F_1, F_2, G, H be sub-σ-algebras of F.

(a) If $(F_1,F_2,G) \in CI(P)$ and $G \subset H \subset G \vee F_1$, then $(F_1,F_2,H) \in CI(P)$;

(b) If $(F_1,F_2,G) \in CI(P)$, then $(F_1,F_2,\sigma(F_1 \mid G)) \in CI(P)$. □

In proving propositions like the one above the following proposition may be useful.

Proposition 2.7. Let F_1, F_2, G, H be sub-σ-algebras of F.

(a) $(F_1,F_2,G) \in CI(P)$ iff $\sigma(F_1 \mid F_2 \vee G) \subset G$;

(b) $(F_1,F_2,G) \in CI(P)$ iff $(F_1',F_2',G) \in CI(P)$ for all σ-algebras $F_1' \subset F_1 \vee G$ and $F_2' \subset F_2 \vee G$.

(c) $(F_1,F_2 \vee H,G) \in CI(P)$ iff $(F_1,F_2,G) \in CI(P)$ and $(F_1,H,F_2 \vee G) \in CI(P)$. □

Necessary and sufficient conditions for the preservation of conditional independence under measure transformation are given by

Proposition 2.8. Given the sub-σ-algebras F_1, F_2, G and the probability measures P_1, P_0 on (Ω,F), assume that:

(a) $P_1 << P_0$ on $F_1 \vee F_2 \vee G$, with $\rho = dP_1/dP_0$,

(b) $(F_1,F_2,G) \in CI(P_0)$.

Then $(F_1,F_2,G) \in CI(P_1)$ iff $\rho = \rho_1\rho_2$ a.s. P_0, where $\rho_1 \in L^+(F_1 \vee G,P_0)$, $\rho_2 \in L^+(F_2 \vee G,P_0)$. □

We now define the property of minimality of splitting σ-algebras, which is particularly important for the stochastic ralization problem.

Definition 2.9. A σ-algebra G is called a *minimal splitting σ-algebra* w.r.t. F_1 and F_2 iff

(a) $(F_1,F_2,H) \in CI(P)$ and $H \subset G$ imply $H = G$. □

The main problem here is to characterize all minimal σ-algebras that make two given σ-algebras conditionally independent, and to devise a procedure to construct such minimal σ-algebras. We have not yet succeeded in resolving this problem. Below we state some preliminary results.

The next proposition already has been stated in (18).

Proposition 2.10. $\sigma(F_1 \mid F_2)$ and $\sigma(F_2 \mid F_1)$ are minimal splitting σ-algebras w.r.t. F_1 and F_2. □

It is not true that every minimal splitting σ-algebra is contained in $F_1 \vee F_2$ as the following example shows.

Example 2.11. Let x_1, x_2, x_3 be independent nontrivial random variables and $F_1 = \sigma\{x_1+x_2\}$, $F_2 = \sigma\{x_2+x_3\}$, $G = \sigma\{x_2\}$. Then $(F_1,F_2,G) \in CI(P)$, $G \not\subset F_1 \vee F_2$. G is minimal if $x_2 = 1_A$ and $0 < P(A) < 1$.

Proposition 2.12. A minimal splitting σ-algebra G w.r.t. F_1 and F_2 has the properties $\sigma(F_1 \mid G) = G$ and $\sigma(F_2 \mid G) = G$. □

Example 2.13. Consider the probability space $(\mathbb{N}, 2^{\mathbb{N}},P)$ with $\mathbb{N} = \{1,2,\dots\}$, $2^{\mathbb{N}}$ the σ-algebra of all subsets of $\mathbb{N}$ and P a probability measure on $2^{\mathbb{N}}$ satisfying $P(\{n\}) > 0$ for all $n \in \mathbb{N}$. Every σ-algebra F in $2^{\mathbb{N}}$ may be characterized by a partition of $\mathbb{N}$, π_F, giving the atoms of F. Let us define σ-algebras F_1 and F_2 by their partitions.

$$\pi_{F_1} = \{1\},\{2,3,4\},\{5\},\{6,7,8\},\dots ,$$

$$\pi_{F_2} = \{1,2\},\{3\},\{4,5,6\},\{7\},\{8,9,10\},\dots .$$

Let $\langle n\rangle_F$ be the atom of n in F, then $E[1_{\langle n\rangle_F} \mid G](m) = P(\langle n\rangle_F \cap \langle m\rangle_G)/P(\langle m\rangle_G)$.

We then have the following propositions about splitting and minimal splitting σ-algebras w.r.t. F_1 and F_2.

Proposition 2.13.1. A σ-algebra G is splitting w.r.t. F_1 and F_2 iff

(a) $\langle 2n-1\rangle_G \cap \langle 2m-1\rangle_G = \emptyset$ for all $n \neq m$, $n,m \in \mathbb{N}$ and

(b) $\langle 2n\rangle_G \subset \{2n-2,2n-1,2n,2n+1,2n+2\} \cap \mathbb{N}$ for all $n \in \mathbb{N}$.

Proposition 2.13.2. A σ-algebra G is minimal splitting w.r.t. F_1 and F_2 iff

(a) $\langle 2n-1\rangle_G \cap \langle 2m-1\rangle_G = \emptyset$ for all $n \neq m$, $n,m \in \mathbb{N}$ and

(b) a sequence $\{k_n\}_{n=1}^{\infty}$ exists, $k_n \in \{-1,1\}$ for $n \in \mathbb{N}$ such that $\{2n\} \subset \langle 2n+k_n\rangle_G$ for all $n \in \mathbb{N}$.

We remark that the concept of conditional independence for a triple of σ-algebras is different from the concept of conditional independence for Hilbert spaces, as used in (10,17). The extension of the proofs from Hilbert space formulation to the σ-algebra formulation is nontrivial mainly because one cannot take an orthogonal complement with respect to a σ-algebra as one can with respect to a subspace in a Hilbert space.

3. STOCHASTIC DYNAMICAL SYSTEMS

In this section we define stochastic dynamical systems. We motivate our definition with the following well-known model.

Definition 3.1. Given $T = R_+$, random variables x_0, y_0, Brownian motion processes v, w, with F^{x_0}, F^{y_0}, F^v_∞, F^w_∞ independent, $\alpha, \gamma \in R$, and processes x, y defined by

$dx_t = \alpha x_t dt + dv_t, x_0,$

$dy_t = \gamma x_t dt + dw_t, y_0.$

Let $F_t = \sigma(\{x_0, y_0, v_s, w_s, \forall s \le t\})$, $F^x_t = \sigma(\{x_s, \forall s \le t\})$, ${}_tF^x = \sigma(\{x_s, \forall s \ge t\})$, $F^{\Delta y}_t = \sigma(\{y_s - y_t, \forall s < t\})$, ${}_tF^{\Delta y} = \sigma(\{y_s - y_t, \forall s > t\})$. □

Proposition 3.2. For the model of definition 3.1 we have $({}_tF^{\Delta y} \vee {}_tF^x, F_t, F^{x_t}) \in CI$ for all $t \in T$, which in turn implies that $({}_tF^{\Delta y} \vee {}_tF^x, F^{\Delta y}_t \vee F^x_t, F^{x_t}) \in CI$ for all $t \in T$.

Proof. A calculation using the conditional characteristic function. □

The second result says that any event in the future observation increments and future states conditioned on past observation increments and past states depends only on the current state. This property is the intuitive notion of a dynamical system, hence we will generalize this formulation. For a continuous time stochastic differential model it seems necessary to work with the increments of the observed process in the above proposition.

To formulate a rather general definition of a stochastic dynamical system we will not work with stochastic processes but with the spaces they generate, that is, with families of a σ-algebras.

Definition 3.3. (a) A *σ-algebraic stochastic dynamical system* is a collection

$\{\Omega, F, T, \Gamma, \{P_\gamma, \gamma \in \Gamma\}, \{G_t, H_t, t \in T\}\}$

such that for all $\gamma \in \Gamma$, $t \in T$

$(G^+_t \vee H^+_t, G^-_t \vee H^-_t, H_t) \in CI(P_\gamma).$

Here $\{\Omega, F\}$ is a measurable space, T a totally ordered index set, Γ a control index set such that for all $\gamma \in \Gamma$ there exists a probability measure P_γ: $F \to [0,1]$ with $\{\Omega, F, P_\gamma\}$ complete, $\{G_t, H_t, t \in T\}$ are two σ-algebra families complete with respect to P_γ for all $\gamma \in \Gamma$, and $G^+_t = \vee_{s>t} G_s$, $G^-_t = \vee_{s<t} G_s$, $H^+_t = \vee_{s>t} H_s$, $H^-_t = \vee_{s<t} H_s$. We call $\{G_t, t \in T\}$ the *output σ-algebra family*, and $\{H_t, t \in T\}$ the *state σ-algebra family*. A σ-algebraic stochastic dynamical system is denoted by $\{G^+_t, G^-_t, H_t, t \in T\} \in \Sigma S\{P_\gamma, \gamma \in \Gamma\}$.

(b) A σ-algebraic stochastic dynamical system $\{G^+_t, G^-_t, H_t, t \in T\} \in \Sigma S\{P_\gamma, \gamma \in \Gamma\}$ is called:

output based: iff $H_t \subset \vee_{s \in T}(G^+_s \vee G^-_s)$ for all $t \in T$;

past output based: iff $H_t \subset G^-_t$ for all $t \in T$;

future output based: iff $H_t \subset G^+_t$ for all $t \in T$;

external based: iff $H_t \not\subset \vee_{s \in T}(G^+_s \vee G^-_s)$ for at least one $t \in T$;

stochastic observable: iff $\sigma(G^+_t \mid H_t) = H_t$ for all $t \in T$;

stochastic reconstructable: iff $\sigma(G^-_t \mid H_t) = H_t$ for all $t \in T$;

(c) For $\{G^+_t, G^-_t, H_t, t \in T\} \in \Sigma S\{P_\gamma, \gamma \in \Gamma\}$ we define the *stochastic state transition function* as the map

$(s, t, H_s, \gamma) \mapsto P_\gamma \mid H_t, \quad s < t,$

and the *stochastic read-out function* as the map

$(t, H_t, \gamma) \mapsto P_\gamma \mid G_t,$

where $P_\gamma \mid H_t$ denotes the restriction of P_γ to H_t. □

The characterizing property of a σ-algebraic stochastic dynamical system implies by 2.3(c) that for any $\gamma \in \Gamma$, $t \in T$, $x \in L^1(G^+_t \vee H^+_t)$

$E_\gamma[x \mid G^-_t \vee H^-_t \vee H_t] = E_\gamma[x \mid H_t].$

In words, any event in the future output and the future states conditioned on past output and past states depends only on the current state. Note that in the above definition the roles of past and future are interchangable, due to the symmetry in the conditional independence relation. Also note that $\{G^+_t, G^-_t, H_t, t \in T\} \in \Sigma S\{P_\gamma, \gamma \in \Gamma\}$ implies that for all $t \in T$, we have that $(H^+_t, H^-_t, H_t) \in CI(P_\gamma)$, hence we will call $\{H_t, t \in T\}$ a Markovian σ-algebra family. So far we have given little attention to the stochastic control aspect in the above definition.

The following equivalent condition is sometimes useful.

<u>Proposition 3.4</u>. $\{G_t^+,G_t^-,H_t,t\in T\} \in \Sigma S\{P_\gamma,\gamma\in\Gamma\}$ iff
(1) $(H_t^+,G_t^-\vee H_t^-,H_t) \in CI(P_\gamma)$ for all $t \in T$, $\gamma \in \Gamma$;
(2) $(G_t^+,G_t^-\vee H_t^-,H_t\vee H_t^+) \in CI(P_\gamma)$ for all $t \in T$, $\gamma \in \Gamma$.

<u>Proof</u>. Apply 2.7(c). □

We specialize the above definitions to continuous time stochastic processes.

<u>Definition 3.5</u>. A *continuous time finite dimensional stochastic dynamical system* is a collection

$\{\Omega,F,T,\Gamma,\{P_\gamma,\gamma\in\Gamma\}, \{R^k,B_k\}, \{R^n,B_n\}\}$

such that for all $\gamma \in \Gamma$, $t \in T$, we have

$({}_tF^{\Delta y}\vee{}_tF^x, F_t^{\Delta y}\vee F_t^x, F^{x_t}) \in CI(P_\gamma)$

or, equivalently, that

$\{{}_tF^{\Delta y}, F_t^{\Delta y}, F^{x_t}, t\in T\} \in \Sigma S\{P_\gamma,\gamma\in\Gamma\}$.

Here $\{\Omega,F\}$ is a measurable space, $T \subset R$ is an interval with its Borel measurable subsets, Γ a control index set such that for all $\gamma \in \Gamma$ there exists a probability measure $P_\gamma\colon F \to [0,1]$, $y\colon \Omega \times T \to R^k$, $x\colon \Omega \times T \to R^n$ are stochastic processes, and $F_t^x = \sigma(\{x_s,\forall s\le t\})$, ${}_tF^x = \sigma(\{x_s,\forall s\ge t\})$, $F_t^{\Delta y} = \sigma(\{y_s-y_t,\forall s<t\})$, ${}_tF^{\Delta y} = \sigma(\{y_s-y_t,\forall s>t\})$. We call y the *output process*, and x the *state process*. A representation of this object is denoted by

$\{{}_tF^{\Delta y}, F_t^{\Delta y}, F^{x_t}, t\in T\} \in \Sigma SFC\{P_\gamma,\gamma\in\Gamma\}$. □

A ΣSFC representation is called output based, external based, stochastic observable, stochastic reconstructable iff the corresponding σ-algebaric stochastic dynamical system has these properties. The above definition, although similar to the definition of a σ-algebraic stochastic dynamical system, differs from it in several aspects. For the future and past output σ-algebra families we have taken those generated by the future and past increments of the output process, or $G_t^+ = {}_tF^{\Delta y}$, $G_t^- = F_t^{\Delta y}$. Because of this choice it is not clear how to define the output σ-algebra family $\{G_t,t\in T\}$. As remarked earlier, a stochastic differential model require us to work with the increments of the output process. Of course other conventions are possible, which will lead to different representations.
The term finite dimensional in the above definition refers mainly to the fact that the state σ-algebra family $\{H_t,t\in T\}$ has a generating process x taking values in a finite dimensional Euclidean space. If the Borel measurable function $f\colon R^n \to R^n$ is injective then it can be shown that $F^{x_t} = F^{f(x_t)}$ for all $t \in T$, hence the state process x is unique up to injective transformations. As remarked earlier a stochastic dynamical system has the property that $\{F^{x_t},t\in T\}$ is a Markovian σ-algebra, hence x is a Markov process. A discrete time finite dimensional stochastic dynamical system on $T \subset Z$ may now be defined by the condition that $\{{}_tF^y, F_t^y, F^{x_t}, t\in T\} \in \Sigma S\{P_\gamma,\gamma\in\Gamma\}$ with ${}_tF^y = \sigma(\{y_s,\forall s>t\})$, $F_t^y = \sigma(\{y_s,\forall s<t\})$. For such a stochastic dynamical system we may define the *stochastic state transition function* as

$(s,t,x_s,\gamma) \to E_\gamma[\exp(iu'x_t) \mid F^{x_s}]$,

and the *stochastic read-out function* as

$(t,x_t,\gamma) \to E_\gamma[\exp(iv'y_t) \mid F^{x_t}]$.

For a finite dimensional stochastic dynamical system it seems more natural to work with the conditional characteristic function than with the conditional measure to define the stochastic state transition function as is done in 3.3. Alternatively one may define a stochastic dynamical system as a collection of spaces and maps with the condition that the stochastic state transition function f and the stochastic read-out function g are such that

$f\colon x_s \to$ distribution on x_t, $s < t$,

$g\colon x_t \to$ disrtibution on y_t.

This definition has been suggested by Kalman (6, p. 5 footnote) and is the natural extension of the definition of a deterministic dynamical system.
As an example we present the well-known model with Brownian motion noise. To allow dependence of the measure on a control index set we use the measure transformation technique introduced by Benes (2).

<u>Definition 3.6</u>. Given $\{\Omega,F,P_0\}$, $T = R_+$, two independent Brownian motion processes x, y, a control index set Γ, for each $\gamma \in \Gamma$ measurable functions $f_\gamma\colon T \times R \to R$, $h_\gamma\colon T \times R \to R$. Define for each $\gamma \in \Gamma$ the process

$\rho_t(\gamma) = 1 + \int_0^t \rho_s(\gamma)[f_\gamma(s,x_s)dx_s + h_\gamma(s,x_s)dy_s]$.

Assume Γ is such that for all $\gamma \in \Gamma$, $E_0[\rho_\infty(\gamma)] = 1$. Define for each $\gamma \in \Gamma$ the probability measure P_γ:

$F \to [0,1]$ by $dP_\gamma/dP_0 = \rho_\infty(\gamma)$. □

Proposition 3.7. For the model of definition 3.6 we have that

$\{\Omega,F,T,\Gamma,\{P_\gamma,\gamma\in\Gamma\},\{R,B\},\{R,B\}\} \in \Sigma SFC$

is a continuous time finite dimensional stochastic dynamical system. With respect to P_γ we have the representations,

$$dx_t = f_\gamma(t,x_t)dt + dv_t,$$

$$dy_t = h_\gamma(t,x_t)dt + dw_t,$$

where v, w are Brownian motion processes. □

Proof. For $t \in T$ we have

$({}_tF^{\Delta y} \vee {}_tF^{x}, F_t^{\Delta y} \vee F_t^{x}, F^{x_t}) \in CI(P_0)$.

By the expression for $\rho(\gamma)$ and 2.8, we conclude that

$({}_tF^{\Delta y} \vee {}_tF^{x}, F_t^{\Delta y} \vee F_t^{x}, F^{x_t}) \in CI(P_\gamma)$ for all $\gamma \in \Gamma$. □

We present one result on the stochastic observability of a stochastic dynamical system.

Proposition 3.8. Given

$\{{}_tF^{\Delta y}, F_t^{\Delta y}, F^{x_t}, t\in T\} \in \Sigma SFC\{P_\gamma,\gamma\in\Gamma\}$

with the representation

$$dx_t = Ax_t dt + dv_t, x_0,$$

$$dy_t = Cx_t dt + dw_t, y_0,$$

where $T = R_+$, $\Gamma = \{\gamma_0\}$ a set with one element, $x: \Omega \times T \to R^n$, $y: \Omega \times T \to R^k$, and v, w standard independent Brownian motion processes. If $Q' \triangleq (C', A'C', \ldots, (A')^{n-1}C')$, $\mathrm{rank}(Q) = n$, then this stochastic dynamical system is stochastic observable. □

Proof. For $s < t$ we have

$$E[\exp(iv'(y_t-y_s)) \mid F^{x_s}] =$$

$$= \exp(iv' \textstyle\int_s^t C\phi(t,\tau)d\tau x_s - \frac{1}{2} v'v(t-s))$$

$$E[\exp(iv' \textstyle\int_s^t C \int_s^\tau \phi(\tau,r)dv_r d\tau)]$$

Thus $\mathrm{rank}(Q) = n$ implies that

$x_s \to E[\exp(iv'(y_t-y_s)) \mid F^{x_s}]$

is bijective, hence $F^{x_s} = \sigma(\sigma(y_t-y_s) \mid F^{x_s})$ and $F^{x_s} = \sigma({}_tF^{\Delta y} \mid F^{x_s})$. □

4. THE STOCHASTIC REALIZATION PROBLEM

In this section we formulate the stochastic realization problem and present some preliminary results.

Definition 4.1. Given the collection

$\{\Omega,F,T,\Gamma,\{P_\gamma,\gamma\in\Gamma\}, \{G_t,L_t,K_t,t\in T\}\}$

where $\{G_t,L_t,K_t,t\in T\}$ are σ-algebra families with $L_t \subset K_t$ for all $t \in T$, and the other symbols are as defined in 3.3.

The *σ-algebra stochastic realization problem* for this collection is to find, if possible, a σ-algebra family $\{H_t,t\in T\}$ such that

(1) $\{G_t^+,G_t^-,H_t,t\in T\} \in \Sigma S\{P_\gamma,\gamma\in\Gamma\}$

(2) $L_t \subset H_t \subset K_t$ for all $t \in T$.

Then we call ΣS a *realization* of the above collection. We call such a realization *minimal* iff for all $t \in T$, H_t is minimal in

$(G_t^+,G_t^-,H_t) \in CI(P_\gamma)$ for all $\gamma \in \Gamma$. □

If $L_t \subset G_t^- \subset K_t$ for all $t \in T$, then it is easily established that $\{G_t^-,t\in T\}$ is a solution to the σ-algebraic stochastic realization problem. However this solution will in general not be minimal. The main problem therefore is to find minimal realizations. We note that the stochastic realization problem for the collection $\{\Omega,F,R,\{\gamma\},\{P_\gamma\},\{G_t,G_t,F,t\in T\}\}$ is the Markov extension problem as posed by Rozanov (14).

Proposition 4.2. Let

$\{G_t^+,G_t^-,H_t,t\in T\} \in \Sigma S\{P_\gamma,\gamma\in\Gamma\}$.

If $\{H_t,t\in T\}$ is minimal then ΣS is stochastic observable and stochastic reconstructable. □

Proof. Apply 2.12. □

We now specialize the stochastic realization problem to the stochastic processes case.

Definition 4.3. Given a stochastic process $y: \Omega \times T \to R^k$ and a σ-algebra family $\{F_t,t\in T\}$.

(a) The *stochastic realization problem for* y is to find, if possible, a stochastic process $x: \Omega \times T \to R^n$ such that:
   (1) $\{{}_tF^{\Delta y}, F_t^{\Delta y}, F^{x_t}, t\in T\} \in \Sigma SFC\{P\}$;
   (2) $F^{x_t} \subset F_t$ for all $t \in T$.

(b) Given a ΣSFC representation with output and state process y, x. The *strong stochastic representation problem* is to find, if possible, stochastic difference equations driven by independent random variables in the discrete time case, stochastic differential equations driven by independent increment processes in the continuous time case, both yielding processes y_1, x_1, such that $y_1 = y$, $x_1 = x$ in the sense of indistinguishable processes. The *weak stochastic representation problem* is the above stated problem where we require only that $y_1 = y$, $x_1 = x$ in the sense of probabilistic equivalence. □

Although we have made some progress with the above defined problems, they have not yet been resolved.

5. CONCLUSION AND FINAL REMARKS

In this paper a definition of a stochastic dynamical system has been given, and the stochastic realization problem has been posed. Research on the questions posed in this paper is in progress.

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