stichting
mathematisch
centrum

MC

AFDELING MATHEMATISCHE BESLISKUNDE (DEPARTMENT OF OPERATIONS RESEARCH) BW 103/79 APRIL

A. KOLEN

ON "A CUT APPROACH TO THE RECTILINEAR DISTANCE FACILITY LOCATION PROBLEM" BY J.-C. PICARD AND H.D. RATLIFF

Preprint

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O).

AMS(MOS) subject classification scheme (1970): 90B99, 90C05, 90C35

On "A cut approach to the rectilinear distance facility location problem" by J.-C. Picard and H.D. Ratliff*)

by

Antoon Kolen

ABSTRACT

Picard and Ratliff recently proposed at cut approach to the rectilinear distance facility location problem and claimed it is fundamentally different from the direct search approach as developed by Pritsker and Ghare, Rao, Jual and Love, and Sherali and Shetty. Our objective is to show that the approach of Picard and Ratliff is essentially a direct search approach.

KEY WORDS & PHRASES: *Multifacility Location Theory, Minimum Cut, Direct search approach, Cut approach.*

---

*) This paper will be submitted for publication elsewhere.

0. INTRODUCTION

The problem we consider is that of locating n new facilities in the plane when there are m existing facilities, located at coordinates (a_i, b_i) for $i = 1,2,\ldots,m$, and the objective is to minimize the sum of weighted rectilinear distances. Let (x_j, y_j) $(j = 1,2,\ldots,n)$ denote the coordinates of the new facilities. Then the problem can be formulated as

$$\text{(P1)} \qquad \min z(X,Y) = \sum_{j=1}^{n} \sum_{i=1}^{m} w_{ji}\{|x_j-a_i| + |y_j-b_i|\} + \sum_{j=1}^{n} \sum_{k=j+1}^{n} v_{jk}\{|x_j-x_k| + |y_j-y_k|\},$$

where $w_{ji} \geq 0$, $v_{jk} \geq 0$ for all i, j, and k (we define $v_{jk} = 0$ for all $j \geq k$). This problem can be decomposed into two independent subproblems:

$$\text{(P2)} \qquad \min F(X) = \sum_{j=1}^{n} \sum_{i=1}^{m} w_{ji}|x_j-a_i| + \sum_{j=1}^{n} \sum_{k=j+1}^{n} v_{jk}|x_j-x_k|,$$

and

$$\text{(P3)} \qquad \min G(Y) = \sum_{j=1}^{n} \sum_{i=1}^{m} w_{ji}|y_j-b_i| + \sum_{j=1}^{n} \sum_{k=j+1}^{n} v_{jk}|y_j-y_k|.$$

We shall develop an algorithm to solve (P2). Without loss of generality we may assume that $0 < a_1 < a_2 < \ldots < a_m$.

It is well known [2] that there exists an optimal solution to (P2) with $x_j \in \{a_1, a_2, \ldots, a_m\}$ for all $j = 1,2,\ldots,n$. In finding an optimal solution we shall restrict ourselves to the direct search approach (PRITSKER and GHARE [5], RAO [6], JUEL and LOVE [3], and SHERALI and SHETTY [7]) and to the cut approach (PICARD and RATLIFF [4]). Other solution methods have been proposed by CABOT et al. [1], and WESOLOWSKY and LOVE [8, 9]. In Section 1 we shall discuss the direct search approach and give an efficient algorithm to solve (P2), which just like the cut approach in [4], requires the solution of at most m-1 minimum cut problems on networks with at most n+2 vertices. In Section 2 we shall discuss the relationship between the cut approach and the direct search approach to solve (P2).

1. THE DIRECT SEARCH APPROACH

The direct search approach can be described as follows. Start with some solution X^1 with $x_j^1 = a_{k_j}$ $(1 \le j \le n)$. Consider the set T of new facilities located at existing facility coordinate a_k; assume $|T| = h$. If there is a subset $S \subseteq T$ that can be moved to an adjacent existing facility coordinate (a_{k+1} or a_{k-1}) such that the new solution X^2 satisfies $F(X^2) < F(X^1)$, move that subset to the corresponding adjacent facility coordinate. Then repeat this procedure with the solution X^2.

If no such subset exists at any location we have found an optimal solution. A justification of the algorithm can be found in the linear programming formulation of (P2). RAO [6] has proved through the negation of various alternatives that a single non-degenerate simplex pivot can only result in the movement of a subset of new facilities at a given location to an adjacent location also coincident with some existing facility coordinate. SHERALI and SHETTY [7] showed that moving S to a_{k+1} reduces $F(X^1)$ by $r_S(X^1)(a_{k+1}-a_k)$, moving S to a_{k-1} reduces $F(X^1)$ by $\ell_S(X^1)(a_k-a_{k-1})$, where

$$r_S(X^1) = \sum_{j\in S} r_j(X^1) + \sum_{j\in S}\sum_{s\in S} (v_{js}+v_{sj}),$$

$$r_j(X^1) = \sum_{\substack{i \\ i>k_j}} w_{ji} + \sum_{\substack{s \\ k_s>k_j}} (v_{js}+v_{sj}) - \sum_{\substack{i \\ i\le k_j}} w_{ji} - \sum_{\substack{s \\ k_s\le k_j}} (v_{js}+v_{sj}),$$

$$\ell_S(X^1) = \sum_{j\in S} \ell_j(X^1) + \sum_{j\in S}\sum_{j\in S} (v_{js}+v_{sj}),$$

$$\ell_j(X^1) = \sum_{\substack{i \\ i<k_j}} w_{ji} + \sum_{\substack{s \\ k_s<k_j}} (v_{js}+v_{sj}) - \sum_{\substack{i \\ i\ge k_j}} w_{ji} - \sum_{\substack{s \\ k_s\ge k_j}} (v_{js}+v_{sj}).$$

The difficulty lies in establishing whether there is a subset that can be moved to an adjacent location such that the value of the objective function reduces. JUEL and LOVE [3] solve this problem by checking each subset explicitly. SHERALI and SHETTY [7] solve this problem by maximizing

$r_S(X^1)(\ell_S(X^1))$ over all subsets $S \subseteq T$ using the following quadratic zero-one formulation:

$$\text{maximize} \sum_{i=1}^{h} t_i r_i(X^1)(\text{or } \ell_i(X^1)) + \sum_{i=1}^{h}\sum_{j=1}^{h} t_i t_j (v_{ij}+v_{ji})$$

$$\text{s.t.} \quad t_i \in \{0,1\},\ i = 1,2,\ldots,h.$$

This problem can be solved by adapting an algorithm of CABOT and FRANCIS ([7]). However, Sherali and Shetty fail to observe that maximizing $r_S(X^1)$ over all subsets $S \subseteq T$ is actually a minimum cut problem on a network with h+2 vertices.

This can be seen as follows. We have

$$r_S(X^1) = \sum_{j\in S}\Big[\sum_{\substack{i\\ i>k_j}} w_{ji} + \sum_{\substack{s\\ k_s>k_j}} (v_{js}+v_{sj})\Big] -$$

$$- \sum_{j\in S}\Big[\sum_{\substack{i\\ i\le k_j}} w_{ji} + \sum_{\substack{s\\ k_s<k_j}} (v_{js}+v_{sj})\Big] -$$

$$- \sum_{j\in S}\sum_{\substack{s\\ k_s=k_j}} (v_{js}+v_{sj}) + \sum_{j\in S}\sum_{s\in S} (v_{js}+v_{sj}).$$

Defining $\bar{S} = T\backslash S$ and

$$C = \sum_{j\in T}\Big[\sum_{\substack{i\\ i>k_j}} w_{ji} + \sum_{\substack{s\\ k_s>k_j}} (v_{js}+v_{sj})\Big]$$

(note that C is a constant which does not depend on the subsets S of T), we can rewrite $r_S(X^1)$ as

$$r_S(X^1) = C - \sum_{j\in\bar{S}}\Big[\sum_{\substack{i\\ i>k_j}} w_{ji} + \sum_{\substack{s\\ k_s>k_j}} (v_{js}+v_{sj})\Big] -$$

$$- \sum_{j \in S} \left[\sum_{\substack{i \\ i \le k_j}} w_{ji} + \sum_{\substack{s \\ k_s < k_j}} (v_{js}+v_{sj}) \right] - \sum_{j \in S} \sum_{s \in \bar{S}} (v_{js}+v_{sj}).$$

Hence finding the subset S of T which maximizes $r_S(X^1)$ over all subsets S of T is equivalent to finding the subset S of T which minimizes.

$$\sum_{j \in \bar{S}} e_j + \sum_{j \in S} f_j + \sum_{j \in S} \sum_{s \in \bar{S}} c_{js},$$

where

$$e_j = \sum_{\substack{i \\ i > k_j}} w_{ji} + \sum_{\substack{s \\ k_s > k_j}} (v_{js}+v_{sj}),$$

$$f_j = \sum_{\substack{i \\ i \le k_j}} w_{ji} + \sum_{\substack{s \\ k_s < k_j}} (v_{js}+v_{sj}),$$

and

$$c_{js} = v_{js} + v_{sj}.$$

This is equivalent to finding a minimum (s,t) cut in the following network. (fig. 1). We have vertices s, t and 1,2,...,h, and arcs (s,j) with capacity e_j (j = 1,2,...,h), arcs (j,t) with capacity f_j (j = 1,2,...,h) and arcs (j,k) of capacity c_{jk} $(1 \le j < k \le n)$.

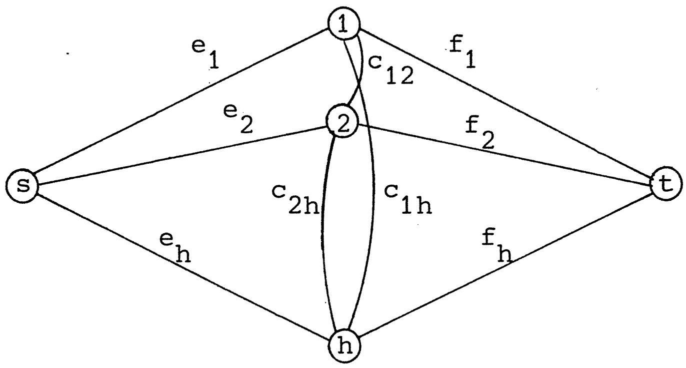

Fig. 1: Network corresponding to the problem of maximizing $r_S(X^1)$ over all subsets S of T.

As initial solution for the direct search approach both JUEL and LOVE [3], and SHERALI and SHETTY [7] take the solution one gets by locating each new facility with respect to existing facilities using the algorithm for the single facility rectilinear location problem (see [2]).

We will give an algorithm that starts with all new facilities located at the same existing facility coordinate a_q $(1 \le q \le m)$.

<u>Direct Search Algorithm</u>

STEP 1: Start with all new facilities located at a_q
Set $i = q$, $j = q$.

STEP 2: Let X^i be the current solution.
If $i = m$, then go to Step 3.
If $i < m$, then determine the subset S of the set T_i of new facilities located at a_i such that $r_S(X^i) = \max_{V \le T_i} r_V(X^i)$.
If $r_S(X^i) \le 0$, then go to Step 3. Otherwise move S to a_{i+1}.
Set $i = i+1$ and go to Step 2.

STEP 3: Let X^j be the current solution.
If $j = 1$, then stop; we have an optimal solution.
If $j > 1$, then determine the subset S of the set T_j of new facilities located at a_j such that $\ell_S(X^j) = \max_{V \le T_j} \ell_V(X^j)$.
If $\ell_S(X^j) \le 0$, then stop; we have an optimal solution.
Otherwise move S to a_{j-1}. Set $j = j-1$ and go to Step 3.

Before proving that this algorithm produces an optimal solution, we make the following observations

<u>OBSERVATION 1</u>: Let X^1 be a given solution, and let T be the set of new facilities located at a_k. Let X^2 be the solution resulting from X^1 by moving the subset $S_1 \subseteq T$ from a_k to a_{k+1}. Then the following equalities hold

(a) $r_{S_1 \cup S_2}(X^1) = r_{S_1}(X^1) + r_{S_2}(X^2)$, where S_1 and S_2 are disjoint subsets of T.

(b) $r_{S_1 \setminus S_3}(X^1) = r_{S_1}(X^1) + \ell_{S_3}(X^2)$, where S_3 is a subset of S_1.

<u>OBSERVATION 2</u>: With X^1, X^2, S_1, S_2, S_3 analogously to Observation 1 we get

(a) $\ell_{S_1 \cup S_2}(X^1) = \ell_{S_1}(X^1) + \ell_{S_2}(X^2)$, and

(b) $\ell_{S_1 \setminus S_3}(X^1) = \ell_{S_1}(X^1) + r_{S_3}(X^2)$.

OBSERVATION 3: Let X^1 be a given solution, and let T be the set of new facilities located at a_k. The calculation of $r_S(X^1)$ does not depend on the actual location of the new facilities, but depends only on which of these facilities have a coordinate less than or equal to a_k and which have a coordinate greater than a_k. The calculation of $\ell_S(X^1)$ does not depend on the actual location of the new facilities but depends only on which of these facilities have a coordinate greater than or equal to a_k and which have a coordinate less than a_k.

We are now ready to prove the following theorem.

THEOREM 1. *The direct search algorithm produces an optimal solution.*

PROOF. Suppose that in Step 2 of the algorithm a subset S of T_i has been moved from a_i to a_{i+1}. We claim that $r_V(X^{i+1}) \leq 0$ for all $V \subseteq T_i \setminus S$ and $\ell_W(X^{i+1}) \leq 0$ for all $W \subseteq S$. This follows from Observation 1(a) and 1(b), using $r_S(X^i) = \max_{V \subseteq T_i} r_V(X^i)$. Since both the set of new facilities which have a coordinate greater than a_i and the set of new facilities which have a coordinate less than or equal to a_i are invariant during the algorithm we find using Observation 3 that $r_V(X) = r_V(X^{i+1})$ for all solutions X succeding X^{i+1} in the algorithm. Similarly $\ell_W(X) = \ell_W(X^{i+1})$ for all solutions X succeeding X^{i+1} in the algorithm. A similar argument holds for Step 3 of the algorithm. Therefore at the end of the algorithm no subset can be moved to an adjacent location such that the value of the objective function is reduced, and we have an optimal solution. □

2 THE CUT APPROACH

We next discuss the cut approach of PICARD and RATLIFF [4] in terms of the direct search approach.

DEFINITION. Let X^k be the solution defined by $x_j^k = a_k$ for all j. Then S is called a *maximal subset* with respect to a_k if $r_S(X^k) = \max_{V \subseteq \{1,2,\ldots,n\}} r_V(X^k)$.

LEMMA 1. *Let* X *be the solution defined by* $x_j = a_k$ *for all* $j \in S_1$ *and* $x_j = a_{k+1}$ *for all* $j \in S_2 = \{1,2,\ldots,n\} \setminus S_1$. *If* $r_V(X) \leq 0$ *for all* $V \subseteq S_1$ *and* $\ell_W(X) \leq 0$ *for all* $W \subseteq S_2$, *then* S_2 *is a maximal subset with respect to* a_k.

PROOF. Let H be a subset of $\{1,2,\ldots,n\}$ and let X^* be the solution defined by $x_j^* = a_{k+1}$ for all $j \in H$ and $x_j^* = a_k$ otherwise. We shall show that $F(X) \leq F(X^*)$. There are subsets $W \subseteq S_2$ and $V \subseteq S_1$ such that $H = (S_2 \setminus W) \cup V$. We can get X^* from X by first moving W to a_k and then moving V to a_{k+1}, or vice versa. These two movements reduce $F(X)$ by

$$[r_V(X) + \ell_W(X) - 2 \sum_{j \in W} \sum_{s \in V} (v_{js}+v_{sj})](a_{k+1}-a_k).$$

Since

$$r_V(X) + \ell_W(X) - 2 \sum_{j \in W} \sum_{s \in V} (v_{js}+v_{sj}) \leq 0$$

it follows that $F(X) \leq F(X^*)$. Since

$$F(X) = F(X^k) - r_{S_2}(X^k)(a_{k+1}-a_k)$$

and

$$F(X^*) = F(X^k) - r_H(X^k)(a_{k+1}-a_k)$$

it follows that

$$r_{S_2}(X^k) \geq r_H(X^k).$$

This is true for all subsets H. We have proved that S_2 is a maximal subset with respect to a_k. □

We now give an alternative proof of Theorem 1 from [4].

THEOREM 2. X^0 *is an optimal solution to* (P2) *iff* $S_k = \{j \mid x_j^0 > a_k\}$ *is a maximal subset with respect to* a_k *for all* $k = 1,2,\ldots,m-1$.

PROOF. Let S_k be a maximal subset with respct to a_k for each $k = 1,2,\ldots,m-1$. Let $X^{(k)}$ be the solution defined by $x_j^{(k)} = a_{k+1}$ for all $j \in S_k$ and $x_j^{(k)} = a_k$ otherwise. By Observation 3 we have

$$r_V(X^0) = r_V(X^{(k)}) \qquad \text{for all } V \subseteq \{1,2,\ldots,n\}\backslash S_k$$

and

$$\ell_W(X^0) = \ell_W(X^{(k)}) \qquad \text{for all } W \subseteq S_k.$$

Since S_k is a maximal subset we find using Observation 1 and 2 that $r_V(X^0) \leq 0$ for all subsets V of the set of new facilities located at a_k in X^0, and $\ell_W(X^0) \leq 0$ for all subsets W of the set of new facilities located at a_{k+1} in X^0. By repeating this argument for all $k = 1,2,\ldots,m-1$ we have proved that no subset of new facilities located at the same location can be moved to an adjacent location such that the objective function is reduced. Hence X^0 is an optimal solution. Let X^0 be an optimal solution and let $k_0 \in \{1,2,\ldots,m-1\}$ be fixed. Assume X^0 is given by $x_j^0 = a_{k_v}$ for all $j \in T_v$ $(-q \leq v \leq -1,\ 1 \leq v \leq p)$ where $k_{-q} < \ldots < k_{-1} \leq k_0 < k_1 < \ldots < k_p$ and $T_{-q},\ldots,T_{-1},T_1,\ldots,T_p$ form a partition of $\{1,2,\ldots,n\}$.

Let $\hat{T}$ be a subset of S_{k_0}. Then $\hat{T} = \cup_{v=1}^{p} \hat{T}_v$ with $\hat{T}_v \subseteq T_v$. Let $\hat{W}$ be a subset of $\{1,2,\ldots,n\}\backslash S_{k_0}$. Then $\hat{W} = \cup_{v=1}^{q} \hat{T}_{-v}$ with $\hat{T}_{-v} \subseteq T_{-v}$. Let $X^{(k_0)}$ be the solution defined by

$$x_j^{(k_0)} = a_{k_0+1} \qquad \text{for all } j \in S_{k_0} \text{ and } x_j^{(k_0)} = a_{k_0} \text{ otherwise.}$$

We shall prove below that

$$\ell_{\hat{T}}(X^{(k_0)}) \leq \sum_{v=1}^{p} \ell_{\hat{T}_v}(X^0)$$

and

$$r_W(X^{(k_0)}) \leq \sum_{v=1}^{q} r_{\hat{T}_{-v}}(X^0).$$

Since X^0 is an optimal solution we have $\ell_{\hat{T}_v}(X^0) \le 0$ for all $v = 1,2,\dots,p$ and $r_{\hat{T}_{-v}}(X^0) \le 0$ for all $v = 1,2,\dots,q$, and therefore $\ell_{\hat{T}}(X^{(k_0)}) \le 0$ and $r_{\hat{W}}(X^{(k_0)}) \le 0$. It follows from Lemma 1 that S_{k_0} is a maximal subset with respect to a_{k_0}.

$$\ell_{\hat{T}_v}(X^0) = \sum_{j\in\hat{T}_v} [\sum_{\substack{i\\ i<k_v}} w_{ji} + \sum_{\substack{s\\ k_s<k_v}} (v_{js}+v_{sj}) - \sum_{\substack{i\\ i\ge k_v}} w_{ji} -$$

$$- \sum_{\substack{s\\ k_s\ge k_v}} (v_{js}+v_{sj}) + \sum_{j\in\hat{T}_v} \sum_{s\in\hat{T}_v} (v_{js}+v_{sj}).$$

$$\ell_{\hat{T}}(X^{(k_0)}) = \sum_{j\in\hat{T}} [\sum_{\substack{i\\ i\le k_0}} w_{ji} + \sum_{\substack{s\\ k_s\le k_0}} (v_{js}+v_{sj}) - \sum_{\substack{i\\ i>k_0}} w_{ji}$$

$$- \sum_{\substack{s\\ k_s>k_0}} (v_{js}+v_{sj})] + \sum_{j\in\hat{T}} \sum_{s\in\hat{T}} (v_{js}+v_{sj}).$$

$$\sum_{v=1}^{p} \ell_{\hat{T}_v}(X^0) - \ell_{\hat{T}}(X^{(k_0)}) = \sum_{v=1}^{p} \sum_{\substack{r=1\\ r\ne v}}^{p} \sum_{j\in\hat{T}_v} \sum_{s\in T_r\setminus\hat{T}_r} (V_{js}+v_{sj}) +$$

$$2 \sum_{v=1}^{p} \sum_{j\in\hat{T}_v} \sum_{\substack{i\\ k_0<i<k_v}} w_{ji} \ge 0$$

Similarly

$$\sum_{v=1}^{q} r_{\hat{T}_{-v}}(X^0) - r_{\hat{W}}(X^{(k_0)}) = \sum_{v=1}^{q} \sum_{\substack{r=1\\ r\ne v}}^{q} \sum_{j\in\hat{T}_{-v}} \sum_{s\in T_{-r}\setminus\hat{T}_r} (V_{js}+v_{sj}) +$$

$$2 \sum_{v=1}^{q} \sum_{j\in\hat{T}_{-v}} \sum_{k_{-v}<i\le k_0} w_{ji} \ge 0. \qquad \square$$

Our direct search algorithm constructs an optimal solution satisfying the conditions of Theorem 2 in PICARD and RATLIFF [4]. Their Theorem 3 is now easy to prove by using Observation 3. We have also proved their Theorem 4 since our direct search algorithm only depends on the quantities $r_S(X)$ and $\ell_S(X)$ which are independent of the distances between existing facilities.

REFERENCES

[1] CABOT, V., R.L. FRANCIS & M.A. STARY, *"A network flow solution of a rectilinear distance facility location problem"*, AIIE Trans. 2, 132-141 (1970).

[2] FRANCIS, R.L. & J.A. WHITE, *"Facility layout and location"*, Prentice-Hall, Inc.

[3] JUEL, H. & R.F. LOVE, *"An efficient computational procedure for solving the multi-facility rectilinear facilities location problem"*, Opl. Res. Q. 27, 697-703 (1976).

[4] PICARD, J.C. & H.D. RATLIFF, *"A cut approach to the rectilinear distance facility location problem,* Opns. Res. 26, 422-433 (1978).

[5] PRITSKER, A.A.B. & P.M. GHARE, *"Locating new facilities with respect to existing facilities"*, AIIE Trans. 2, 290-297 (1970).

[6] RAO, M.R., *"On the direct search approach to the rectilinear facilities location problem"*, AIIE Trans. 5, 256-264 (1973).

[7] SHERALI, H.D. & C.M. SHETTY, *"A primal simplex based solution procedure for the rectilinear distance multifacility location problem"*. J. Opl. Res. Soc. vol. 29, 373-381 (1978).

[8] WESOLOWSKY, G.O. & R.F. LOVE, *"The optimal location of new facilities using rectangular distances"*, Opns. Res. 19, 124-130 (1971).

[9] WESOLOWSKY, G.O. & R.F. LOVE, *"A non-linear approximation method for solving a generalized rectangular distance Weber problem"*, Management Sci. 18, 656-663 (1972).