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H.C. TIJMS, M.H. VAN HOORN & A. FEDERGRUEN

APPROXIMATIONS FOR THE STEADY-STATE PROBABILITIES IN THE MULTI-SERVER M/G/c QUEUE

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Approximations for the steady-state probabilities in the multi-server M/G/c queue *)

by

H.C. Tijms **), M.H. van Hoorn **) & A. Federgruen

ABSTRACT

For the multi-server queue with Poisson arrivals and general service times we present various approximations for the steady-state probabilities of the queue size. These approximations are computed from numerically stable recursion schemes which can be easily applied in practice. Numerical experience reveals that the approximations are very accurate with errors typically below 5%. For the delay probability the various approximations result either into the widely used Erlang delay probability or into a new approximation which improves in many cases the Erlang delay probability approximation.

KEY WORDS & PHRASES: *Multi-server* M/G/c *queue, steady-state probabilities, approximations, stable recursion schemes.*

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**) Vrije Universiteit, Amsterdam

1. *Introduction.*

Consider the M/G/c queue with $c \geq 2$ servers where customers arrive in accordance with a Poisson process with rate λ and the service times of the customers are independent and identically distributed. Denote by S the service time of a customer and let F be the probability distribution of the service time. It is assumed that $F(0) = 0$, $ES^2 < \infty$ and $\rho < 1$ where the traffic intensity ρ is defined by

$$\rho = \lambda ES/c.$$

An infinite capacity queueing system is considered. Hence an arriving customer joins the queue if he finds all c servers occupied or else he is served immediately by one of the free servers. A server will never remain idle if customers are waiting in the queue. We note that the analysis to be given carries over to the finite capacity case.

The purpose of this paper is to derive various approximations for the steady-state probabilities of the queue size. Although several good approximation formulae for the mean queue size have been obtained (e.g. Boxma, Cohen and Huffels (1979), Cosmetatos (1976) and Nozaki and Ross (1978)), the paper by Hokstad (1978) seems to be the only one so far in which approximations for the steady-state probabilities in the M/G/c queue with general service times have been obtained. These approximations derived by using the supplementary variable technique involve the Laplace transform of the service time distribution. An exact method for the steady-state probabilities in the M/D/c queue with deterministic times has been given by Crommelin (1932). This method involves the solution of an infinite system of linear equations and gives numerical difficulties when c is large and the traffic intensity is close to 1, cf. also Kühn (1976). An exact analysis for the steady-state probabilities in the $M/E_k/c$ queue with Erlangian service times was given by Heffer (1969) and Mayhugh and McGormick (1968) by using the phase method, cf. also Yu (1977). In view of the fact that the computational work required by this analysis is too sophisticated and extensive that it can be routinely done by practitioners, Hillier and Lo (1977) obtained and tabulated computational results for a number of cases of the $M/E_k/c$ queue.

In this paper we shall present various approximations for the steady-state probabilities in the M/G/c queue with general service times. These approximations are computed by numerically stable recursive schemes which can be easily applied in practice. Our numerical experience reveals that the approximations for the cumulative steady-state probabilities are very accurate with errors typically below 5% and in many cases within 0-2%. The resulting approximations for the delay probability either give the Erlang delay probability or improve in many cases this widely used approximation for the delay probability for general service times. Further, the resulting approximations for the mean queue size either coincide with or differ only by a multiplicative factor tending to 1 as $\rho \to 1$ from the accurate approximation for the mean queue size given in Nozaki and Ross(1978).

In section 2 we shall present the main lines of our approach which uses simple arguments from the theory of regenerative processes. This regenerative approach was introduced in Hordijk and Tijms (1976) for the M/G/1 queue and further studied in Federgruen and Tijms (1978) for the M/G/1 queue with variable service rate. For clarity of presentation we first discuss the M/D/c queue for which the analysis is facilitated by the fact that for the case of deterministic service times a new service cannot be completed earlier than services already in progress. Next in section 4 we treat the M/G/c queue with general service times. Finally, in section 5 we discuss some numerical results.

2. *The regenerative approach.*

We first introduce some notation. Denote the steady-state probabilities by

$$p_i = \lim_{t\to\infty} \Pr\{\text{at time } t \text{ there are } i \text{ customers in the system}\},\ i \geq 0.$$

Since $\rho < 1$, these limits exist and $\Sigma_{i=0}^{\infty} p_i=1$. Define the delay probability P_d and the mean queue size L_q by

$$P_d = 1 - \sum_{n=0}^{c-1} p_n \text{ and } L_q = \sum_{n=c}^{\infty} (n-c)p_n.$$

In general no explicit expressions for the steady-state probabilities can be given except for the M/M/c queue. We write $p_i = p_i(\exp)$ and $P_d = P_d(\exp)$ when F is an exponential distribution function with mean ES. Denote by

$$(2.1)\qquad \Omega = \left\{ \sum_{k=0}^{c-1} \frac{(\lambda ES)^k}{k!} + \frac{(\lambda ES)^c}{c!(1-\rho)} \right\}^{-1},$$

then

$$(2.2)\qquad p_i(\exp) = \frac{(\lambda ES)^i}{i!}\,\Omega \text{ for } 0 \le i < c,\quad p_i(\exp) = \frac{(\lambda ES)^i}{c!c^{i-c}}\,\Omega \text{ for } i \ge c$$

$$(2.3)\qquad P_d(\exp) = \frac{(\lambda ES)^c}{c!(1-\rho)}\,\Omega$$

The right hand side of (2.3) is called the Erlang delay probability and this probability is widely used as an approximation for P_d when the service time has a general distribution. Numerical experience shows that the Erlang delay probability is a good approximation, cf. Palm (1957) and Krampe, Kubat and Runge (1973). A theoretical support for this empirical result may be found in the generally valid formula (e.g. cf. Nozaki and Ross (1978))

$$(2.4)\qquad \lambda ES = \sum_{n=0}^{c-1} np_n + c\{1 - \sum_{n=0}^{c-1} p_n\},$$

which relation can be directly verified from Little's formulae $L = \lambda W$ and $L_q = \lambda W_q$.

We shall now discuss the regenerative approach for obtaining a recursive scheme by which approximations for the steady-state probabilities can be computed. We need the following notation. Given that at epoch 0 a customer arrives who finds no other customers in the system, define the following random variables.

T = the next epoch at which a customer arrives who finds no other customers in the system.

T_i = amount of time during which i customers are in the system in the busy cycle $(0,T)$, $i=0,1,\ldots$

N = number of customers served in the busy cycle $(0,T)$.

N_i = number of service completion epochs at which i customers are left behind in the system in the busy cycle $(0,T)$, $i=0,1,\ldots$

By the theory of regenerative processes (cf. Ross (1970) and Stidham (1972)), we have

$$(2.5)\qquad p_i = \frac{ET_i}{ET} \text{ for } i \geq 0.$$

Moreover, since Poisson arrivals see time averages, we have that p_i is equal to the long-run expected fraction of customers who find upon arrival i other customers in the system (cf. Theorem 3 in Stidham (1972)). Further, the long-run expected fraction of customers who find upon arrival i other customers in the system is equal to the long-run expected fraction of customers who leave upon service completion i other customers behind in the system. By the theory of regenerative processes, this latter fraction is given by EN_i/EN and so

$$(2.6)\qquad p_i = \frac{EN_i}{EN} \text{ for } i \geq 0.$$

Since EN/ET equals the long-run expected average number of customers served per unit time, we have $EN/ET = \lambda$ and consequently

$$(2.7)\qquad EN_i = \lambda ET p_i \text{ for } i \geq 0.$$

Throughout the analysis to follow we make an approximation assumption. In this assumption probability distribution functions F_j^*, $j \geq 1$ appear and the various approximations to be discussed in the next sections depend on the specification of these probability distribution functions.

APPROXIMATION ASSUMPTION. *For any service completion epoch at which* j *customers are left behind in the system, the smallest of the remaining service times of the* min(j,c-1) *services already in progress has probability distribution function* F_j^* *independently of what occurred at previous service completion epochs.*

Define now the following quantities. For any $n \geq 1$, let

A_n = the expected amount of time during which n customers are in the system until the next service completion epoch given that at epoch 0 a customer arrives who finds no other customers in the system.

For any $j \geq 1$ and $n \geq j$, let

$A_{n,j}$ = the expected amount of time during which n customers are in the system until the next service completion epoch given that at epoch 0 a service completion occurs and j customers are left behind in the system where the smallest of the remaining service times of the services already in progress at epoch 0 has probability distribution function F_j^*.

We are now in a position to state our basic recursion scheme. Using the approximation assumption and Wald's equation, we have approximately

$$(2.8)\qquad ET_n \simeq A_n + \sum_{j=1}^{n} EN_j A_{n,j} \qquad \text{for } n = 1,2,\ldots,$$

and so, by (2.5.) and (2.7.)

$$(2.9)\qquad p_n ET \simeq A_n + \sum_{j=1}^{n} \lambda p_j ETA_{n,j} \qquad \text{for } n = 1,2,\ldots$$

This approximative relation suggests the following recursion scheme.

$$(2.10)\qquad q_n = A_n + \sum_{j=1}^{n} \lambda q_j A_{n,j} \qquad \text{for } n = 1,2,\ldots$$

We can recursively compute the quantities $q_1, q_2, \ldots$ from this relation. Define q_0 by

$$(2.11)\qquad q_0 = 1/\lambda$$

and note that, by $ET_0 = 1/\lambda$, we have $p_0 ET = q_0$. We can now approximate the steady-state probabilities p_i, $i \geq 0$ by

$$(2.12)\qquad p_i(\text{appr}) = q_i / \sum_{n=0}^{\infty} q_n \text{ for } i \geq 0.$$

Clearly, the approximations are determined by the quantities A_n and $A_{n,j}$ which in turn depend on the specification of the probability distribution functions F_j^*. In the next sections we shall give various approximations where we first discuss in section 3 the case of deterministic service times.

REMARK. The above approach carries over to the finite capacity case in which the queueing system has only place for $M < \infty$ customers. The relation (2.8) again applies for $n = 1,\ldots,M$ where however, the expressions for A_M and $A_{M,j}$

need some obvious modifications. The relation (2.6) is not longer valid. It now follows that the long-run expected fraction of entering customers who find i other customers in the system equals EN_i/EN for $0 \le i < M$. Hence $p_i/(1-p_M) = EN_i/EN$ for $0 \le i < M$. However, by $EN/ET = \lambda(1-p_M)$, we have that also in the finite capacity case the relation (2.7.) applies for $0 \le i < M$.

3. *The M/G/c queue with deterministic service times.*

We first define the well-known equilibrium distribution of F by

$$(3.1) \qquad F_e(t) = \frac{1}{ES} \int_0^t (1-F(x))dx, \quad t \ge 0.$$

In this section we now consider the case where

$$F(t) = 0 \text{ for } t < D \text{ and } F(t) = 1 \text{ for } t \ge D$$

with $D = ES$. In this case F_e is the uniform distribution function on $(0,D)$.

Since for deterministic service times a new service cannot be completed earlier than sercices already in progress, we have

$$(3.2) \qquad A_n = \int_0^\infty (1-F(t))e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} dt = \int_0^D e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} dt, \quad n \ge 1.$$

To explain this relation, note that $(1-F(t))e^{-\lambda t}(\lambda t)^{n-1}/(n-1)!$ is the probability that at epoch t the first service is still in progress and n customers are in the system given that at epoch 0 a customer arrives who finds no other customers in the system. Assuming that $F_j^*(D) = 1$ for all j, we find in the same way that

$$(3.3) \qquad A_{n,j} = \int_0^D (1-F_j^*(t))e^{-\lambda t} \frac{(\lambda t)^{n-j}}{(n-j)!} dt \text{ for } j \ge 1 \text{ and } n \ge j.$$

We first consider the following seemingly reasonable choice for F_j^*.

Case A. *For all* $j \ge 1$, *let*

$$1-F_j^*(t) = \{1-F_e(t)\}^{\min(j,c-1)} = \begin{cases} (1-t/D)^{\min(j,c-1)}, & 0 \le t < D \\ 0, & t \ge D. \end{cases}$$

That is, we assume that at any service completion epoch the remaining service times of the services still in progress are independent random variables with F_e as common probability distribution. A very similar assumption was made in Molina (1927) (cf. also Syski (1960)) and in Nozaki and Ross (1978). Note however, that in these references the assumption about remaining service times was made for services still in progress at an *arbitrary* epoch whereas we make an assumption for services still in progress at a service completion epoch. It turned out that the approximations resulting from Case A were rather unsatisfactory, in particular when the traffic intensity ρ is close to 1. For that reason we considered also the following two cases B and C.

Case B. *For* $1 \le j \le c-1$, *let*

$$1-F_j^*(t) = \begin{cases} (1-t/D)^j, & 0 \le t < D \\ 0, & t > D, \end{cases}$$

and for $j \ge c$, *let*

$$F_j^*(t) = \begin{cases} 0, & 0 \le t < D/c \\ 1, & t \ge D/c. \end{cases}$$

Case B can be motivated by replacing the M/G/c queue with service time S by an M/G/1 queue with service time S/c when all servers are occupied, cf. also Hokstad (1977) and Stoyan (1977). Case C only differs from case B by taking $F_{c-1}^*(t) = F(ct)$ with the same mean D/c as $F_{c-1}^*(t)$ in Case B.

Case C. *For* $1 \le j \le c-2$, *let*

$$1-F_j^*(t) = \begin{cases} (1-t/D)^j, & 0 \le t < D \\ 0, & t > D, \end{cases}$$

and for $j \ge c-1$, *let*

$$F_j^*(t) = \begin{cases} 0, & 0 \le t < D/c \\ 1, & t \ge D/c. \end{cases}$$

In the sequel we write $q_n = q_n^A$, $L_q(\text{appr}) = L_q^A(\text{appr})$, $p_n(\text{appr}) = p_n^A(\text{appr})$, $P_d(\text{appr}) = P_d^A(\text{appr})$ for case A, etc. However, we will suppress the dependence of the quantities $A_{n,j}$ on the case considered. The next two theorems specify the approximations resulting from the various cases.

THEOREM 3.1. *Let* Ω *be defined by* (2.1). *Then*

$$q_n^A = q_n^B = \frac{1}{\lambda}\frac{(\lambda D)^n}{n!}, \quad 0 \le n \le c-1, \tag{3.4}$$

$$q_n^C = \begin{cases} q_n^B, & 0 \le n \le c-2 \\ q_{c-1}^B\{1-\lambda \int_0^D (1-t/D)^{c-1}e^{-\lambda t}dt\}e^{\rho}, & n = c-1, \end{cases} \tag{3.5}$$

$$\sum_{n=0}^{\infty} q_n^A = \sum_{n=0}^{\infty} q_n^B = \sum_{n=0}^{\infty} q_n^C = \frac{1}{\lambda\Omega}, \tag{3.6}$$

$$L_q^A(\text{appr}) = \frac{(\lambda D)^{c+1}\Omega}{(c+1)!(1-\rho)^2},\quad L_q^B(\text{appr}) = \frac{(\lambda D)^{c+1}\Omega}{2c.c!(1-\rho)^2}\{1+(1-\rho)\,\frac{(c-1)}{c+1}\}. \tag{3.7}$$

$$L_q^C(\text{appr}) = \frac{(\lambda D)^{c+1}\Omega}{2c.c!(1-\rho)^2}. \tag{3.8}$$

PROOF. Fix $1 \le n \le c-1$. By (2.10.),

$$(1-\lambda A_{n,n})q_n = A_n + \sum_{j=1}^{n-1} \lambda q_j A_{n,j}. \tag{3.9}$$

For each of the cases A-C we derive from (3.3) by partial integration with respect to the function $e^{-\lambda t}$ that for $j=1,\ldots,n-1$

$$A_{n,j} = \gamma_j(n) - \frac{j}{\lambda D}\gamma_{j-1}(n) \tag{3.10}$$

where $\gamma_j(n)$ for $j=0,\ldots,n-1$ is defined by

$$\gamma_j(n) = \int_0^D (1-t/D)^j\, e^{-\lambda t}\,\frac{(\lambda t)^{n-j-1}}{(n-j-1)!}\,dt. \tag{3.11}$$

Observe from (3.2) that

$$\gamma_0(n) = A_n. \tag{3.12}$$

Hence, by (3.9)-(3.10),

$$(3.13)\qquad (1-\lambda A_{n,n})q_n = A_n + \sum_{j=1}^{n-1} \lambda q_j\{\gamma_j(n) - \frac{j}{\lambda D}\gamma_{j-1}(n)\} \text{ for } 1 \le n \le c-1.$$

Further, from (3.3) we derive by partial integration that for the cases A-B,

$$(3.14)\qquad 1-\lambda A_{n,n} = \frac{n}{D}\int_0^D (1-t/D)^{n-1}e^{-\lambda t} = \frac{n}{D}\gamma_{n-1}(n) \text{ for } 1 \le n \le c-1$$

and for case C

$$(3.15)\qquad 1-\lambda A_{n,n} = \begin{cases} (n/D)\gamma_{n-1}(n) & \text{for } 1 \le n \le c-2 \\ e^{-\lambda D/c} & \text{for } n = c-1. \end{cases}$$

Using (3.12)-(3.15), we now get (3.4)-(3.5) by induction. Next define the generating function Q(x) by

$$Q(x) = \sum_{n=1}^{\infty} q_n x^n \text{ for } |x| \le 1.$$

Using (2.10), (3.2) and (3.3) we derive for each of the cases A-C,

$$Q(x) = x\int_0^D e^{\lambda t(x-1)}dt + \sum_{j=1}^{\infty} \lambda q_j x^j \int_0^D (1-F_j^*(t))e^{\lambda t(x-1)}dt, \; |x| \le 1$$

from which we get after some algebra for each of the cases A-C,

$$(1-\rho)Q(1) = D + \lambda D \sum_{j=1}^{c-1} q_j(\frac{1}{j+1} - \frac{1}{c})$$

and

$$(1-\rho)Q'(1) = D + \lambda\frac{D^2}{2} + \lambda\sum_{j=1}^{c-1} jq_j(\frac{D}{j+1} - \frac{D}{c}) + \lambda^2 Q(1)\int_0^D t(1-F_c^*(t))dt +$$
$$+ \lambda^2 \sum_{j=1}^{c-1} q_j \int_0^D t(F_c^*(t) - F_j^*(t))dt.$$

Using these relations, (3.4)-(3.5) and (2.11), we get after some straightforward calculations the relations (3.6)-(3.8).

By Theorem 3.1. we have the remarkable result that $p_n^A(\text{appr}) = p_n^B(\text{appr}) = p_n(\text{exp})$ for $0 \le n \le c-1$ and $p_n^C(\text{appr}) = p_n(\text{exp})$ for $0 \le n \le c-2$. Consequently both $P_d^A(\text{appr})$ and $P_d^B(\text{appr})$ are equal to $P_d(\text{exp})$. However, $P_d^C(\text{appr})$ is a new approximation for the delay probability and is equal to the Erlang delay probability minus a positive correction factor. Although in case C the approximations for the steady-state probabilities violate relation (2.4.), it turns out from numerical investigations that P_d^C (appr) improves the Erlang delay probability approximation. We further have that $L_q^A(\text{appr})$ is equal to the approximation for L_q found by Molina (1927) except a multiplicative factor of $(1-\rho^{c+1})/(1-\rho^c)$. This approximation is known to be rather poor, in particular when the traffic intensity ρ is close to 1. In view of the rather unsatisfactory results we found for case A, we shall not discuss this case further. For case C we have the remarkable result that $L_q^C(\text{appr})$ is equal to the approximation for L_q given by Nozaki and Ross (1978). This approximation for the mean queue size is quite accurate. Further, we have that $L_q^B(\text{appr})$ is equal to $L_q^C(\text{appr})$ except a simple multiplicative factor tending to 1 as $\rho \to 1$. Numerical experience with the approximations for the cases B and C will be further discussed in section 5. We mention that we also investigated the case in which $F_j^*(t) = F(\min(j+1,c)t)$ for all j. However, this case yielded unsatisfactory approximations.

We conclude this section by showing that for the cases A-C the recursion (2.10) can be further simplified where we only discuss the simplifications for cases B and C.

THEOREM 3.2. *For all* $n \ge c$,

$$q_n^B = \lambda q_{c-1}^B \int_0^D (1-t/D)^{c-1} e^{-\lambda t} \frac{(\lambda t)^{n-c}}{(n-c)!}\, dt + \sum_{j=c}^{n} q_j^B \Big(1 - \sum_{k=0}^{n-j} e^{-\rho} \frac{\rho^k}{k!}\Big) \tag{3.16}$$

and

$$q_n^C = \lambda q_{c-2}^C \int_0^D (1-t/D)^{c-2} e^{-\lambda t} \frac{(\lambda t)^{n-c+1}}{(n-c+1)!}\, dt + \sum_{j=c-1}^{n} q_j^C \Big(1 - \sum_{k=0}^{n-j} e^{-\rho} \frac{\rho^k}{k!}\Big). \tag{3.17}$$

PROOF. Consider case B. We first note that for $j \ge c$ and $n \ge j$,

$$A_{n,j} = \int_0^{D/C} e^{-\lambda t} \frac{(\lambda t)^{n-j}}{(n-j)!}\, dt = (1/\lambda)\Big(1 - \sum_{k=0}^{n-j} e^{-\rho} \frac{\rho^k}{k!}\Big). \tag{3.18}$$

In the same way as (3.10), we derive from (3.3) that for any $n \geq c$,

$$A_{n,j} = \gamma_j(n) - \frac{j}{\lambda D}\gamma_{j-1}(n) \text{ for } j=1,\dots,c-1$$

where $\gamma_j(n)$ is again defined by (3.11.). Together this relation and (3.4.) imply

$$(3.19) \qquad A_n + \sum_{j=1}^{c-1} \lambda q_j^B A_{n,j} = \lambda q_{c-1}^B \gamma_{c-1}(n) \text{ for } n \geq c.$$

By (2.10), (3.18)-(3.19) we get (3.16). In the same way we derive (3.17).

4. *The M/G/c queue with general service times.*

For the case of general service times a new service may be completed earlier than services already in progress at the beginning of this new service. This phenomenon has no effect on the determination of the quantities $A_{n,j}$ for $j \geq c$ but complicates the determination of the quantities A_n and $A_{n,j}$ for $j < c$.

Let F_e be defined by (3.1). The approximations resulting from the extension of Case A in section 3 to general service times appeared to be rather unsatisfactory and will not be further discussed. To give the generalisation of case B in section 3, we first make the following observation. If at epoch 0 a new service is started when $j \geq c$ customers are present and the smallest of the remaining service times of the $c-1$ services already in progress at epoch 0 has probability distribution function $F_j^*(t)$, then the probability that at epoch t both these remaining $c-1$ services and this new service will be still in progress is given by $(1-F_j^*(t))(1-F(t))$. To generalize case B in section 2, choose now the probability distribution functions F_j^* such that

$$(4.1) \qquad (1-F_j^*(t))(1-F(t)) = 1-F(ct) \text{ for } j \geq c.$$

That is, for any $j \geq c$, $(1-F_j^*(t))(1-F(t)) = \Pr\{S/c>t\}$ for all t where S is the service time in the M/G/c queue. Therefore we consider as generalization of case B in section 3 the following case.

Case B. *For* $1 \le j \le c-1$, *let*

$$1-F_j^*(t) = (1-F_e(t))^j \text{ for all } t$$

and for $j \ge c$, *let*

$$(1-F_j^*(t))(1-F(t)) = 1-F(ct) \text{ for all } t.$$

Note that for case B the approximation assumption exactly holds for the M/M/c queue. We have the following results.

THEOREM 4.1. *For case* B,

$$q_n^B = \frac{1}{\lambda}\frac{(\lambda ES)^n}{n!}, \quad 0 \le n \le c-1 \tag{4.2}$$

$$q_n^B = \lambda q_{c-1}^B \int_0^\infty (1-F_e(t))^{c-1}(1-F(t))e^{-\lambda t}\frac{(\lambda t)^{n-c}}{(n-c)!}\,dt + \tag{4.3}$$

$$+ \sum_{j=c}^{n} \lambda q_j^B \int_0^\infty (1-F(ct))e^{-\lambda t}\frac{(\lambda t)^{n-j}}{(n-j)!}\,dt, \quad n \ge c,$$

$$\sum_{n=0}^{\infty} q_n^B = \frac{1}{\lambda\Omega}, \tag{4.4}$$

$$L_q^B(\text{appr}) = \frac{\lambda^2(\lambda ES)^{c-1}ES^2\Omega}{2c.c!(1-\rho)^2}\{1+(1-\rho)(\frac{2cES}{ES^2}\int_0^\infty (1-F_e(t))^c dt - 1)\}. \tag{4.5}$$

PROOF. We can easily give expressions for the quantities A_1, $A_{n,n}$ and $A_{n,j}$ for $j \ge c$. By the same argument as given below (3.2), we find

$$A_1 = \int_0^\infty (1-F(t)e^{-\lambda t}dt, \tag{4.6}$$

$$A_{n,n} = \int_0^\infty (1-F_e(t))^n e^{-\lambda t}dt = \tag{4.7}$$

$$= \frac{1}{\lambda} - \frac{n}{\lambda ES}\int_0^\infty (1-F_e(t))^{n-1}(1-F(t))e^{-\lambda t}dt, \quad 1 \le n \le c-1,$$

$$A_{n,j} = \int_0^\infty (1-F(ct))e^{-\lambda t}\frac{(\lambda t)^{n-j}}{(n-j)!}, \quad n \ge j \ge c, \tag{4.8}$$

where the second equality in (4.7) is obtained by partial integration and using (3.1). By (2.10), (4.6) and (4.7) with n=1 we have

(4.9) $\quad q_1^B = A_1/(1-\lambda A_{1,1}) = ES,$

which verifies (4.2) for n=1. The determination of A_n for $n \geq 1$ and $A_{n,j}$ for $j < c$ is more complicated by the fact that a new service started during the execution of other services may be completed earlier than these services. Put for abbreviation for any $m = 0,1,\ldots,$ $k=1,2,\ldots$ and $t > 0$

$$a_{m,k}(t) = \int_0^t \int_0^{y_1} \ldots \int_0^{y_{k-1}} dy_1 \ldots dy_k (1-F(y_1))\ldots(1-F(y_k)) \frac{(\lambda y_k)^m}{m!}.$$

Observe that for any $m \geq 0$

$$\frac{da_{m,k}(t)}{dt} = (1-F(t))a_{m,k-1}(t) \quad \text{for } k \geq 1,\ t > 0, \tag{4.10}$$

where we define

$$a_{m,0}(t) = \frac{(\lambda t)^m}{m!} \text{ for } m \geq 0,\ t > 0. \tag{4.11}$$

Now, fix j and n with $1 \leq j < n \leq c-1$. Suppose that at epoch 0 there are j customers in the system and j services in progress. Under the condition that the smallest of the remaining service timese of these j services is equal to t and that in (0,t) there are started n-j new services at times $0 < t_1 < \ldots < t_{n-j} < t$, the expected amount of time that n customers are in the system during (0,t] until the first service completion, is given by

$$p_x(t) = \int_0^{t-t_{n-j}} (1-F(y))(1-F(y+x_{n-j}))\ldots(1-F(y+x_{n-j}+\ldots+x_2))e^{-\lambda y}dy,$$

where $x_i = t_i - t_{i-1}$ for $1 \leq i \leq n-j$ with $t_0=0$. Using this observation, it is now easily seen that for $1 \leq j < n \leq c-1$,

$$A_{n,j} = \int_0^\infty dF_j^*(t) \int_0^t \int_0^{t-x_1} \ldots \int_0^{t-(x_1+\ldots+x_{n-j-1})} dx_1 \ldots dx_{n-j} \times$$
$$\times \{\lambda^{n-j} e^{-\lambda(x_1+\ldots+x_{n-j})} p_x(t)\}.$$

Now, let $\alpha(t)$ be an increasing continuous function with $\alpha(0) = 0$ and $\alpha(\infty) < \infty$, then for any probability distribution function G concentrated on $(0,\infty)$, we have by partial integration

$$\int_0^\infty \alpha(t)dG(t) = \int_0^\infty (1-G(t))\, d\alpha(t).$$

Using this relation, we find after some algebra that

$$(4.12) \qquad A_{n,j} = \int_0^\infty (1-F_e(t))^j e^{-\lambda t}\lambda^{n-j} a_{0,n-j}(t)dt, \quad 1 \le j < n \le c-1.$$

By taking j=1 and replacing $F_1^*(t) = F_e(t)$ by $F(t)$ in this relation, we find

$$(4.13) \qquad A_n = \int_0^\infty (1-F(t))e^{-\lambda t}\lambda^{n-1} a_{0,n-1}(t)\, dt, \quad 1 < n \le c-1.$$

In the same way as (4.12) -(4.13) , we derive

$$(4.14) \qquad A_{n+c,j} = \int_0^\infty (1-F_e(t))^j e^{-\lambda t}\lambda^{c-j} a_{n,c-j}(t)\, dt, \quad 1 \le j \le c-1,\ n \ge 0,$$

$$(4.15) \qquad A_{n+c} = \int_0^\infty (1-F(t))e^{-\lambda t}\lambda^{c-1} a_{n,c-1}(t)\, dt, \quad n \ge 0.$$

We shall now verify (4.2) -(4.3) . Put for abbreviation, for $1 < n \le c-1$ and $0 \le k \le n-1$,

$$(4.16) \qquad \gamma_k(n) = \int_0^\infty (1-F_e(t))^k (1-F(t))\, e^{-\lambda t}\lambda^{n-k-1} a_{0,n-k-1}(t)\, dt.$$

Using (4.10) , we easily derive from (4.12) by partial integration with respect to the function $e^{-\lambda t}$ that

$$(4.17) \qquad A_{n,j} = \gamma_j(n) - \frac{j}{\lambda ES}\gamma_{j-1}(n), \quad 1 \le j < n \le c-1.$$

Further, observe that, by (4.13) and (4.7) ,

$$(4.18) \qquad \gamma_0(n) = A_n \text{ and } 1-\lambda A_{n,n} = \frac{n}{ES}\gamma_{n-1}(n), \quad 1 < n \le c-1.$$

By (2.10) and (4.17) ,

$$(4.19) \qquad (1-\lambda A_{n,n})q_n = A_n + \sum_{j=1}^{n-1} \lambda q_j(\gamma_j(n) - \frac{j}{\lambda ES}\gamma_{j-1}(n)), \quad 1 < n \le c-1$$

Using (4.9) , (4.18) -(4.19) we get (4.2) by induction. In the same way as (4.17) we derive

(4.20) $$A_{n+c,j} = \delta_j(n) - \frac{j}{\lambda ES}\,\delta_{j-1}(n) \text{ for } 1 \le j \le c-1,\ n \ge 0$$

where $\delta_j(n)$ for $0 \le j \le c-1$ and $n \ge 0$ is defined by

(4.21) $$\delta_j(n) = \int_0^\infty (1-F_e(t))^j(1-F(t))e^{-\lambda t}\lambda^{c-j-1}a_{n,c-j-1}(t)dt.$$

Observe that

(4.22) $$A_{n+c} = \delta_0(n) \text{ for } n \ge 0.$$

By (4.2), (4.20) and (4.22),

$$A_{n+c} + \sum_{j=1}^{c-1} \lambda q_j^B A_{n+c,j} = \lambda q_{c-1}^B \delta_{c-1}(n) \text{ for } n \ge 0.$$

Together this relation, (2.10), (4.8) and (4.14) give (4.3). Finally, using generating functions, we derive (4.4)-(4.5) from (4.2)-(4.3).

Next we consider the generalisation of Case C in section 3.

Case C. *For* $1 \le j \le c-2$, *let*

$$1-F_j^*(t) = (1-F_e(t))^j \text{ for all } t.$$

Further, let

$$F_{c-1}^*(t) = F(\gamma^{-1}ESt) \text{ for all } t,$$

where γ *is defined by*

(4.23) $$\gamma = \int_0^\infty (1-F_e(t))^{c-1}dt,$$

and for $j \ge c$, *let*

$$(1-F_j^*(t))(1-F(t)) = 1-F(ct) \text{ for all } t.$$

Note that in both cases B and C the probability distribution function

$F^*_{c-1}(t)$ has the same mean γ where $\gamma \geq ES/c$. The cases B and C are identical when F is exponential. Put for abbreviation,

$$\eta_1 = 1 - \int_0^\infty (1-F_e(t))^{c-1}\lambda e^{-\lambda t}dt, \quad \eta_2 = 1 - \int_0^\infty (1-F(\gamma^{-1}ESt))\lambda e^{-\lambda t}dt,$$

$$\xi_1 = \int_0^\infty (1-F_e(t))^{c-1}(1-G(t))dt, \quad \xi_2 = \int_0^\infty (1-F(\gamma^{-1}ESt))(1-G(t))dt,$$

where the probability distribution function G is defined by

$$G(t) = \int_0^t F(t)\lambda e^{-\lambda(t-y)}dy, \quad t \geq 0.$$

We can give η_1, η_2, ξ_1 and ξ_2 probabilistic interpretations. The quantity ξ_1 represents the expected time until the first service completion epoch given that at epoch 0 there are c-1 customers in the system and c-1 services in progress where the smallest of the remaining service times of these services has survivor function $(1-F_e(t))^{c-1}$. A similar interpretation holds for ξ_2. In view of these interpretations we may expect that both ξ_1 and ξ_2 are approximately equal to γ in many cases, in particular when c is sufficiently large. Note that $\xi_1 = \xi_2 = \gamma = ES/c$ when the service times are deterministic.

By making slight modifications on the proof of Theorem 4.1, we find the relations below for q_n^C and next, by using generating functions, we get after some algebra the formula (4.24) below.

THEOREM 4.2. *For case* C,

$$q_n^C = \frac{1}{\lambda}\frac{(\lambda ES)^n}{n!}, \quad 0 \leq n \leq c-1,$$

$$q_{c-1}^C = \frac{1}{\lambda}\frac{(\lambda ES)^{c-1}}{(c-1)!}\,\frac{\eta_1}{\eta_2},$$

$$q_n^C = \lambda q_{c-2}^C \int_0^\infty (1-F_e(t))^{c-2}(1-F(t))e^{-\lambda t}\,\lambda \int_0^t (1-F(y))\,\frac{(\lambda y)^{n-c}}{(n-c)!}\,dy\,dt + \\ + \sum_{j=c-1}^{n} \lambda q_j^C A_{n,j}^C, \quad n \geq c,$$

where for all $n \geq c$,

$$A^C_{n,c-1} = \int_0^\infty (1-F(\gamma^{-1}ESt))e^{-\lambda t}\lambda \int_0^t (1-F(y)) \frac{(\lambda y)^{n-c}}{(n-c)!}\, dy\, dt$$

$$A^C_{n,j} = \int_0^\infty (1-F(ct))e^{-\lambda t} \frac{(\lambda t)^{n-j}}{(n-j)!}\, dt, \qquad c \le j \le n.$$

Moreover,

$$(4.24) \qquad \sum_{n=0}^\infty q_n^C = \frac{1}{\lambda\Omega} + \frac{(\lambda ES)^{c-1}}{(c-1)!\eta_2(1-\rho)} \{\eta_1(\xi_2 - \frac{ES}{c}) - \eta_2(\xi_1 - \frac{ES}{c})\}.$$

We omit the rather lengthy formula for L_q^C(appr). For both the M/D/c queue and the M/M/c queue we have that $\Sigma_{n=0}^\infty q_n^C = 1/\lambda\Omega$, but this relation is not generally valid so that in general p_n^C(appr) differs from p_n(exp) for all $0 \le n \le c-1$. Since any p_n^C(appr) for ~~$n \ge c$ involves the evaluation of~~ two double integrals, we have that Case C requires in general more computational work than Case B. The recursion schem for Case B can be rather easily used in practice. Another simple recursion scheme applies to the following case that was derived from the recursion scheme for Case B by replacing in (4.3) the survivor function $(1-F_e(t))^{c-1}(1-F(t))$ by $1-F(ct)$ in accordance with (4.1). More precisely we define

Case D. *Let the sequence* $\{q_n^D, n \ge 0\}$ *be given by*

$$(4.25) \qquad q_n^D = \frac{1}{\lambda}\frac{(\lambda ES)^n}{n!}, \qquad 0 \le n \le c-1$$

$$(4.26) \qquad q_n^D = \lambda q_{c-1}^D \int_0^\infty (1-F(ct))e^{-\lambda t} \frac{(\lambda t)^{n-c}}{(n-c)!}\, dt + \sum_{j=c}^n \lambda q_j^D \int_0^\infty (1-F(ct))e^{-\lambda t}\frac{(\lambda t)^{n-j}}{(n-j)!}\, dt, \qquad n \ge c.$$

Note that when F is exponential Case D is identical to Case B. Using generating functions, we easily derive

THEOREM 4.3. *For Case* D,

$$(4.27) \qquad \sum_{n=0}^\infty q_n^D = \frac{1}{\lambda\Omega},$$

$$(4.28) \qquad L_q^D(\text{appr}) = \frac{\lambda^2(\lambda ES)^{c-1}ES^2\Omega}{2c.c!(1-\rho)^2}.$$

We easily obtain from (4.26) that for $|x| \le 1$

$$(4.29) \quad \sum_{n=c}^{\infty} p_n^D(\text{appr})x^{n-c} = p_{c-1}^D(\text{appr})\{1-\tilde{F}(\lambda(1-x)/c)\}/\{\tilde{F}(\lambda(1-x)/c)-x\}$$

where $\tilde{F}$ denotes the Laplace-Stieltjes transform of F. Now, by (4.25), (4.27), (4.29) and the relations (17)-(19) in Hokstad (1978), we find that the approximations for Case D agree with those obtained in Hokstad (1978) by a completely different approach which requires that F has a density. Clearly the recursion relation (4.26) is much better suited for computational purposes than the representation (4.29) found in Hokstad (1978).

We further have the remarkable result that $L_q^D(\text{appr})$ is equal to the approximation for L_q given by Nozaki and Ross (1978). Note that $L_q^B(\text{appr})$ is equal to this approximation except a multiplicative factor which tends to 1 as $\rho \to 1$. We further have that $p_n^B(\text{appr}) = p_n^D(\text{appr}) = p_n(\text{exp})$ for $0 \le n \le c-1$ so that both $P_d^B(\text{appr})$ and $P_d^D(\text{appr})$ are equal to the Erlang delay probability. For case C we have as approximation for the delay probability

$$(4.30) \quad P_d^C(\text{appr}) = \beta P_d(\text{exp}) + 1 - \beta - \frac{\beta(\eta_1-\eta_2)}{\eta_2} \frac{(\lambda ES)^{c-1}}{(c-1)!} \Omega,$$

where

$$(4.31) \quad \beta = [1 + \frac{(\lambda ES)^{c-1}\lambda\Omega}{(c-1)!\eta_2(1-\rho)} \{\eta_1(\xi_2 - \frac{ES}{c}) - \eta_2(\xi_1 - \frac{ES}{c})\}]^{-1}$$

Note that for deterministic service times $\xi_1 = \xi_2 = ES/c$ and so $\beta = 1$.

5. *Numerical results.*

In this section we discuss our numerical experience with the various approximations. We consider both the M/D/c and $M/E_k/c$ queueing systems for which exact numerical results for the steady-state probabilities are available. The tables 5.1 and 5.2 concern the M/D/c queue where we have chosen the service time D=1. For the delay probability table 5.1 compares the Erlang delay probability approximation (first number in box), the approximation $P_d^C(\text{appr})$ (second number in box) and the exact value (third number in box) for a range of values for ρ and c. The exact values were taken from Kühn (1976) who computed the exact values by solving the system of linear equations given by Crommelin (1932). It turns out that P_d^C(appr) which is below P_d(exp) considerably improves the Erlang delay probability approximation P_d(exp) in all cases considered. The approximation P_d^C(appr) for the delay probability is very accurate with errors typically 0-5% for all values of ρ when $c \le 50$ and with errors as large as 7-9% for ρ close to 1 when $c \ge 100$ which latter error percentage is still within the maximum tolerance used in most practical design problems. For the cumulative steady-state probabilities $\Sigma_{i=0}^{n} p_i$ with $n \ge c-1$, table 5.2 compares the approximation of Case B (first number in box), the approximation of Case C (second number in box), the approximation of Case D (third number in box) and the exact value (fourth number in box). The exact values were found by solving the system of linear equations given by Crommelin (1932). It is by no means a simple matter to solve these equations in particular when c is large and ρ is very close to 1 (cf. also Kühn (1976)) whereas our various recursion schemes can be easily applied for any ρ and c. Our numerical results reveal that Case C gives the best approximation for $\Sigma_{i=0}^{n} p_i$ for all $n \ge c-1$ when $c < 20$ whereas for $c \ge 20$ Case C gives the best approximations for $\Sigma_{i=0}^{n} p_i$ with $c-1 \le n \le c+2$ and Case B gives the best approximations for $\Sigma_{i=0}^{n} p_i$ with $n > c+2$. These best approximations for $\Sigma_{i=0}^{n} p_i$, $n \ge c-1$ are very accurate with errors below 3% when $c \le 50$. Further, our numerical results show that for the M/D/c queue L_q^B(appr) gives a better approximation for the mean queue size than L_q^D(appr) for all values of ρ when $c > 20$ and for lower values of ρ when $c \le 20$.

The tables 5.3 and 5.4 concern the $M/E_k/c$ queue where we have chosen the arrival rate $\lambda=1$. Exact numerical values for the steady-state probabilities have been obtained in Hillier and Lo (1971) for 19 (k,c) combinations. For the delay probability table 5.3 compares the Erlang delay probability approximation (first number in box), the approximation P_d^C(appr) (second number in box) and the exact value (third number in box) for a number of these (k,c) combinations and several values of ρ. For all the (k,c) combinations considered in Hillier and Lo (1971), both the Erlang delay probability approximation and the approximation P_d^C(appr) turned out to be extremely accurate for all values of ρ with errors typically below 3% and in many cases within 0-1%. For the case of k=2 the Erlang delay probability approximation is slightly better than P_d^C(appr) and for $k \ge 3$ the approximation P_d^C(appr) improves in most cases the Erlang delay probability approximation. For the cumulative steady-state probabilities $\Sigma_{i=0}^{n} p_i$ with $n \geq c-1$ table 5.4 compares the approximation of Case B (first number in box), the approximation of Case D (second number in box) and the exact value (third number in box). For all the (k,c) combinations considered in Hillier and Lo (1971), both the approximations of Case B and those of Case D turned out to be very accurate with errors typically below 3% and in many cases within 0-1%. We found for the (k,c) combinations considered that Case D gives slightly better approximations for $\Sigma_{i=0}^{n} p_i$ with $c-1 \leq n \leq c+2$ than Case B and that for $\Sigma_{i=0}^{n} p_i$ with $n > c+2$ the average of the approximation of the Cases B and D is practically equal to the exact value. Further, we believe that for c not too small ($c \geq 20$) Case B gives the best approximations for $\Sigma_{i=0}^{n} p_i$ with $n > c+2$. This is supported by the observation that our numerical results indicate that L_q^B(appr) gives a better approximation for the mean queue size than L_q^D(appr) for all values of ρ when c is not too small (i.e. $c \geq 10$ when $k=2$) and for lower values of ρ when c is small. In view of the accuracy of the approximations of the Cases B and D, and the fact that for $\Sigma_{i=0}^{n} p_i$, $n \geq c$ the evaluation of an approximation requires in Case C more computational work than in the Cases B and D, it will suffice in many cases to evaluate only the approximations of the Cases B and D. We emphasize that for the $M/E_k/c$ queue the various recursion schemes are computationally feasible for any k and c whereas the computational approach used in Hillier and Lo (1971) proved only to be computationally feasible for a restricted number of (k,c) combinations.

For service time distributions with variation coefficient larger than 1 no exact numerical results for the steady-state probabilities seem to be known, However, limited simulation results for the hyperexponential service time distribution indicate that for this distribution the various approximations are also accurate.

We conclude by remarking that future plans concern an extensive computational project for approximations for steady-state probabilities in multi-server $E_k/G/c$ queueing systems with Erlangian arrival times.

Table 5.1. Delay probabilities (Erlang, Case C, Exact) for the M/D/c queue.

ρ\c	2	3	4	5	10	15	20	30	40	60	80	100	200
.1	.01818	.00370	.00079	.00018									
	.01791	.00362	.00077	.00017									
	.01777	.00361	.00078	.00017									
.2	.06667	.02466	.00958	.00383									
	.06489	.02369	.00914	.00364									
	.06449	.02362	.00917	.00368									
.3	.13846	.07003	.03705	.02014	.00116								
	.13359	.06647	.03484	.01883	.00107								
	.13358	.06648	.03495	.01897	.00110								
.4	.22857	.14118	.09070	.05970	.00881	.00149	.00026						
	.21936	.13303	.08461	.05534	.00806	.00135	.00024						
	.22082	.13380	.08504	.05565	.00823	.00140	.00025						
.5	.33333	.23684	.17391	.13037	.03611	.01129	.00373	.00044					
	.31927	.22269	.16188	.12059	.03297	.01026	.00338	.00040					
	.32326	.22527	.16329	.12134	.03315	.01042	.00348	.00042					
.6	.45000	.35474	.28704	.23615	.10130	.04823	.02413	.00653	.00187	.00017			
	.43167	.33447	.26815	.21936	.09301	.04411	.02203	.00595	.00170	.00015			
	.43869	.33983	.27146	.22116	.09255	.04381	.02196	.00601	.00174	.00016			
.7	.57647	.49234	.42865	.37784	.22173	.14115	.09356	.04392	.02168	.00572	.00159	.00046	
	.55579	.46778	.40415	.35457	.20612	.13080	.08657	.04057	.02001	.00527	.00147	.00042	
	.56537	.47609	.40995	.35812	.20432	.12827	.08443	.03951	.01958	.00523	.00148	.00043	
.8	.71111	.64719	.59643	.55411	.40918	.31919	.25608	.17286	.12118	.06339	.03479	.01965	.00138
	.69153	.62260	.57059	.52833	.38754	.30164	.24173	.16300	.11421	.05971	.03276	.01850	.00130
	.70190	.63254	.57828	.53362	.38472	.29554	.23456	.15616	.10864	.05652	.03107	.01762	.00127
.9	.85263	.81706	.78775	.76249	.66873	.60263	.55077	.47141	.41156	.32456	.26307	.21694	.09447
	.83931	.79959	.76862	.74265	.64905	.58422	.53364	.45649	.39843	.31411	.25457	.20991	.09140
	.84711	.80769	.77544	.74783	.64686	.57713	.52327	.44232	.38248	.29738	.23864	.19534	.08368
.95	.92564	.90701	.89142	.87780	.82559	.78696	.75540	.70453	.66364	.59904	.54835	.50646	.36526
	.91798	.89676	.88000	.86577	.81282	.77434	.74307	.69283	.65253	.58893	.53906	.49785	.35903
	–	–	–	–	–	.76949	.73533	.68082	.63749	.56995	.51774	.47512	.33513

Table 5.2. *Cumulative steady-state probabilities (Case B, Case C, Case D, Exact) for the* M/D/c *queue*

c=5	c−1	c	c+1	c+2	c+3	c+4	c+5	c+6	c+7	15	20	25
ρ=.9	.23751	.34132	.44797	.54497	.62791	.69681	.75331	.79941	.83692	.91239	.96890	.98896
	.25735	.37114	.47825	.57178	.65043	.71533	.76844	.81172	.84693	.91777	.97081	.98964
	.23751	.36117	.47777	.57525	.65472	.71933	.77184	.81453	.84923	.91901	.97125	.98980
	.25217	.36352	.47054	.56544	.64565	.71177	.76571	.80957	.84521	.91685	.97048	.98952

c=25	c−1	c	c+1	c+2	c+3	c+4	c+5	c+6	c+7	35	40	45
ρ=.9	.49208	.55849	.62707	.69041	.74545	.79179	.83019	.86172	.88749	.93951	.97853	.99238
	.50804	.58355	.65372	.71496	.76669	.80962	.84492	.87380	.89735	.94483	.98041	.99305
	.49208	.57445	.65213	.71706	.77000	.81303	.84801	.87645	.89957	.94605	.98085	.99320
	.52066	.58709	.64875	.70450	.75373	.79633	.83258	.86299	.88823	.93987	.97869	.99244

c=50	c−1	c	c+1	c+2	c+3	c+4	c+5	c+6	c+7	60	65	70
ρ=.9	.63614	.68347	.73238	.77761	.81699	.85021	.87777	.90042	.91896	.95642	.98453	.99451
	.64781	.70189	.75207	.79585	.83284	.86356	.88883	.90952	.92639	.96043	.98595	.99501
	.63614	.69515	.75079	.79731	.83523	.86606	.89112	.91149	.92805	.96135	.98628	.99513
	.66446	.70938	.75051	.78758	.82054	.84945	.87450	.89597	.91418	.95284	.98321	.99406

Table 5.3. *Delay probabilities (Erlang, Case C, Exact) for the* $M/E_k/c$ *queue*

(k,c)	(2,2)	(2,4)	(2,6)	(2,8)	(2,10)	(3,3)	(3,4)	(3,5)	(4,2)	(4,3)	(6,2)	(8,2)
ρ=.5	.33333	.17391	.09914	.05904	.03611	.23684	.17391	.13037	.33333	.23684	.33333	.33333
	.33239	.17378	.09929	.05921	.03624	.23567	.17345	.13028	.33009	.23525	.32925	.32883
	.33083	.17105	.09703	.05762	.03518	.23213	.16946	.12650	.32842	.23212	.32723	.32651
ρ=.9	.85263	.78775	.74013	.70153	.66873	.81706	.78775	.76249	.85263	.81706	.85263	.85263
	.85269	.78782	.74126	.70362	.67163	.81588	.78745	.76311	.85034	.81538	.84933	.84880
	.85117	.78435	.73546	.69596	.66248	.81309	.78248	.75617	.84981	.81309	.84916	.84877
ρ=.99	.98503	.97791	.97242	.96780	.96374	.98117	.97791	.97503	.98503	.98117	.98503	.98503
	.98504	.97790	.97257	.96811	.96419	.98100	.97784	.97509	.98474	.98093	.98461	.98454
	.98486	.97748	.97179	.96700	.96279	.98069	.97724	.97420	.98470	.98069	.98462	.98458

Table 5.4. Cumulative steady-state probabilities (Case B, Case D, Exact) for the $M/E_k/c$ *queue.*

ρ=.5	c−1	c	c+1	c+2	c+3	c+4	c+5	c+6	c+7	c+8
c=2	.66667	.86442	.95114	.98349	.99459	.99825	.99944	.99982	.99994	.99998
k=8	.66667	.87472	.95787	.98635	.99562	.99860	.99955	.99986	.99995	.99999
	.67349	.87077	.95440	.98483	.99508	.99842	.99949	.99984	.99995	.99998
c=5	.86963	.94127	.97538	.99014	.99616	.99853	.99944	.99979	.99992	.99997
k=3	.86963	.94628	.97928	.99219	.99708	.99891	.99959	.99985	.99994	.99998
	.87350	.94386	.97654	.99058	.99632	.99859	.99946	.99980	.99992	.99997
c=10	.96389	.98304	.99239	.99669	.99859	.99941	.99975	.99990	.99996	.99998
k=2	.96389	.98420	.99337	.99726	.99887	.99954	.99981	.99992	.99997	.99999
	.96482	.98352	.99252	.99668	.99856	.99938	.99974	.99989	.99995	.99998

ρ=.9	c−1	c	c+1	c+2	c+3	c+4	c+5	20	25	30	40	50
c=2	.14737	.26560	.38158	.48386	.57048	.64286	.70311	.97314	.98934	.99577	.99933	.99989
k=8	.14737	.27492	.39438	.49620	.58119	.65186	.71060	.97382	.98961	.99588	.99935	.99990
	.15123	.27375	.39056	.49215	.57764	.64889	.70814	.97359	.98952	.99504	.99934	.99990
c=5	.23751	.32974	.41927	.50054	.57189	.63359	.68660	.93457	.97011	.98634	.99715	.99941
k=3	.23751	.33892	.43286	.51473	.58504	.64521	.69666	.93669	.97108	.98679	.99724	.99942
	.24383	.33904	.42851	.50873	.57892	.63959	.69171	.93563	.97059	.98657	.99720	.99941
c=10	.33127	.40684	.47882	.54443	.60284	.65420	.69911	.85027	.92552	.96295	.99083	.99773
k=2	.33127	.41319	.48846	.55485	.61281	.66327	.70716	.85434	.92755	.96396	.99108	.99779
	.33752	.41452	.48572	.55000	.60717	.65759	.70181	.85139	.92607	.96323	.99090	.99775

ρ=.99	c−1	c	c+1	c+2	c+3	20	30	50	100	150	250	350
c=2	.01497	.02915	.04523	.06181	.07832	.29475	.41000	.58709	.83081	.93067	.98836	.99805
k=8	.01497	.03033	.04708	.06388	.08043	.29637	.41137	.58804	.83120	.93083	.98839	.99805
	.01542	.03024	.04661	.06329	.07982	.29591	.41098	.58777	.83109	.93079	.98838	.99805
c=5	.02497	.03704	.05032	.06405	.07787	.23026	.33789	.51012	.76932	.89138	.97592	.99466
k=3	.02497	.03830	.05239	.06650	.08044	.23250	.33982	.51154	.76999	.89169	.97599	.99468
	.02580	.03844	.05191	.06568	.07948	.23160	.33905	.51097	.76972	.89157	.97596	.99467
c=10	.03626	.04732	.05913	.07125	.08343	.16532	.26995	.44150	.71411	.85366	.96165	.98995
k=2	.03626	.04828	.06075	.07319	.08551	.16735	.27172	.44285	.71480	.85401	.96175	.98998
	.03721	.04866	.06053	.07257	.08465	.16626	.27076	.44212	.71443	.85382	.96170	.98996

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