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THE P-MEDIAN PROBLEM WITH MUTUAL COMMUNICATION ON A TREE

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The p-median problem with mutual communication on a tree *)

by

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ABSTRACT

This paper considers the problem of locating p new facilities on a tree, where each vertex represents an existing facility, in order to minimize the total weighted sum of distances between all pairs of new and existing facilities and between all pairs of new facilities.

We present a polynomial-time algorithm for its solution. This algorithm generalizes the well-known algorithm for the 1-median problem on a tree.

KEY WORDS & PHRASES: *p-median problem (with mutual communication), tree, location theory.*

*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

Let T be a tree with vertex set $V = \{v_1,v_2,..,v_n\}$ and edge set $E = \{e_1,e_2,..,e_{n-1}\}$. Each edge $(v_i,v_j) \in E$ has a nonnegative length $\ell(i,j)$. A point x on the tree can be a vertex or a point anywhere along an edge. The length of the shortest path between the points x and y on T is denoted by $d(x,y)$.

The p-median problem with mutual communication is to find p new facility locations $x_1,x_2,..,x_p$ such that

$$\sum_{i=1}^{n} \sum_{j=1}^{p} \alpha_{ij}\, d(v_i,x_j) + \frac{1}{2} \sum_{j=1}^{p} \sum_{k=1}^{p} \beta_{jk}\, d(x_j,x_k)$$

is minimal, where $\alpha_{ij}(i=1,..,n,j=1,..,p)$ and $\beta_{jk}(=\beta_{kj})(j=1,..,p,k=1..,p)$ are given nonnegative weights. It is well known that an optimal solution exists with $x_i \in V$ $(i=1,..,p)$ [3].

We can think of the vertices of the tree as locations of existing facilities. Let α_{ij} represent the amount of travel between existing facility i and new facility j, and let β_{jk} represents the amount of travel between new facilities j and k. The tree T corresponds to a transportation network and the p-median problem with mutual communication is to find the new facility locations such that the total travelled distance is minimal.

The p-median problem with mutual communication in the plane using rectilinear distances has received much attention. This problem is to find new facilitiy locations $(x_1,y_1),..,(x_p,y_p)$ in the plane such that

$$\sum_{i=1}^{n} \sum_{j=1}^{p} \alpha_{ij}\{|x_j-a_i| + |y_j-b_i|\} + \frac{1}{2} \sum_{j=1}^{p} \sum_{k=1}^{p} \beta_{jk}\{|x_j-x_k| + |y_j-y_k|\}$$

is minimal, where $(a_i,b_i)(i=1,..,n)$ are the existing facility locations. This problem can be decomposed into two independent problems on the line. We mention the following references for the p-median problem with mutual communication on a line: PRITSKER & GHARE [9], RAO [10], JUEL & LOVE [5], SHERALI & SHETTY [11], CABOT, FRANCIS & STARY [1], WESOLOWSKY & LOVE [12,13], PICARD & RATLIFF [8], and KOLEN [7].

The only algorithm we know of which solves the p-median problem with mutual communication on a tree is due to PICARD & RATLIFF [8]. In Section 2 we will prove a theorem which characterizes the optimum solution value to this problem. This theorem was already known to FRANCES [2] for the very easy case that $p = 1$. It provides the basis for our algorithm to solve the problem, presented in Section 3. This algorithm differs from the algorithm by PICARD & RATLIFF [8] but has the same time complexity. In the case that $p = 1$, our algorithm reduces to the well-known algorithm by GOLDMAN [4] to solve the 1-median problem on a tree.

2. CHARACTERIZING THE OPTIMUM VALUE

Let us start with the simple case that the tree has only a single edge (v_1,v_2). Let P denote the set $\{1,2,..,p\}$, let $X \subseteq P$ be the index set of the facilities located at v_1, and let $\bar{X} = P \backslash X$. In this case, the objective function is given by

$$\sum_{j\in X} \alpha_{2j}\, d(v_1,v_2) + \sum_{j\in \bar{X}} \alpha_{1j}\, d(v_1,v_2) + \sum_{j\in X} \sum_{k\in \bar{X}} \beta_{jk}\, d(v_1,v_2) =$$

$$\ell(1,2) W_{12}(X),$$

where

$$W_{12}(X) = \sum_{j\in X} \alpha_{2j} + \sum_{j\in \bar{X}} \alpha_{1j} + \sum_{j\in X} \sum_{k\in \bar{X}} \beta_{jk}$$

Hence the value of the objective function is minimal for that subset of P which minimizes $W_{12}(X)$ over all subsets $X \subseteq P$. This leads to the following characterization of the optimum value: the minimum value of the objective function is equal to

$$\ell(1,2) \min_{X\subseteq P} W_{12}(X).$$

We now generalize this result to an arbitrary tree T. Let (v_s,v_t) be an arbitrary edge of T. Deletion of (v_s,v_t) with the exception of v_s

and v_t from T results in two subtrees. Let $T_s(T_t)$ be the subtree containing $v_s(v_t)$. We define $W_{st}(X)$ as the total amount of travel between facilities in T_s and facilities in T_t if X denotes the index set of new facilities in T_s:

$$W_{st}(X) = \sum_{j\in X} \sum_{v_i\in T_t} \alpha_{ij} + \sum_{j\in \bar{X}} \sum_{v_i\in T_s} \alpha_{ij} + \sum_{j\in X} \sum_{k\in \bar{X}} \beta_{jk}.$$

We now state our characterization of the optimum value in Theorem 1.

THEOREM 1.

$$\min_{x_1,\ldots,x_p} \sum_{i=1}^{n} \sum_{j\in P} \alpha_{ij}\, d(v_i,v_j) + \frac{1}{2} \sum_{j\in P} \sum_{k\in P} \beta_{jk}\, d(x_j,x_k) =$$

$$\sum_{(v_s,v_t)\in E} \ell(s,t) \min_{X\subseteq P} W_{st}(X).$$

Before proving this theorem we prove the following lemma.

LEMMA 1. *Let* v_1 *be a tip of the tree* T, *let* v_2 *be the vertex adjacent to* v_1, *and let* (v_s,v_t) *be an edge of* T *not equal to* (v_1,v_2) *and such that* v_1 *and* v_2 *are contained in* T_s. *Let* Q *be such that* $W_{12}(Q) = \min_{X\subseteq P} W_{12}(X)$. *Then there is a set* R *such that* $Q \subseteq R$ *and* $W_{st}(R) = \min_{X\subseteq P} W_{st}(X)$.

PROOF. Let R be a set such that $Q \cap R = S_1 \neq Q$, i.e., there are subsets $S_2 (S_2 \neq \phi)$ and S_3 such that $Q = S_1 \cup S_2$ and $R = S_1 \cup S_3$. Since $W_{12}(Q) = \min_{X\subseteq P} W_{12}(X)$ we know that $W_{12}(S_1 \cup S_2) - W_{12}(S_1) \leq 0$. We shall prove below that $W_{st}(S_1 \cup S_2 \cup S_3) - W_{st}(S_1 \cup S_3) \leq W_{12}(S_1\ S_2) - W_{12}(S_1)$. This implies that $W_{st}(S_1 \cup S_2 \cup S_3) - W_{st}(S_1 \cup S_3) \leq 0$. Therefore if $W_{st}(R) = \min_{X\subseteq P} W_{st}(X)$, then $W_{st}(Q\cup R) = W_{st}(R)$, i.e., without loss of generality we may assume that $Q \subseteq R$. Since

$$W_{st}(S_1 \cup S_2 \cup S_3) - W_{st}(S_1 \cup S_3) =$$

$$\sum_{j\in S_2} \sum_{v_i\in T_t} \alpha_{ij} - \sum_{j\in S_2} \sum_{v_i\in T_2} \alpha_{ij} + \sum_{j\in S_1\cup S_2\cup S_3} \sum_{k\in \overline{(S_1\cup S_2\cup S_3)}} \beta_{jk} -$$

$$\sum_{j\in S_1\cup S_3} \sum_{k\in \overline{(S_1\cup S_3)}} \beta_{jk},$$

$$W_{12}(S_1 \cup S_2) - W_{12}(S_1) =$$

$$\sum_{j\in S_2} \sum_{v_i\in T_2} \alpha_{ij} - \sum_{j\in S_2} \alpha_{1j} + \sum_{j\in S_1\cup S_2} \sum_{k\in \overline{(S_1\cup S_2)}} \beta_{jk} - \sum_{j\in S_1} \sum_{k\in \bar{S}_1} \beta_{jk}$$

and moreover $T_t \subseteq T_2$ and $v_1 \in T_s$ we have

$$(W_{st}(S_1\cup S_2\cup S_3) - W_{st}(S_1\cup S_3)) - (W_{12}(S_1\cup S_2) - W_{12}(S_1)) =$$

$$- \sum_{j\in S_2} \sum_{v_i\in T_2/T_t} \alpha_{ij} - \sum_{j\in S_2} \sum_{v_i\in T_s/\{v_1\}} \alpha_{ij} - 2\sum_{j\in S_2} \sum_{k\in S_3} \beta_{jk} \le 0.$$

This yields the desired result. Q.E.D.

We shall now give a proof of Theorem 1.

PROOF OF THEOREM 1. The proof is by induction on the number of vertices $|V|$. For the case $|V| = 2$, we have shown at the beginning of this section that the theorem is true. Suppose the theorem is true for all trees with $|V| < n$, and consider a tree T with n vertices $(n \geq 3)$. Let v_1 be a tip of the tree and let v_2 be the vertex adjacent to v_1. Assume we locate new facilities indicated by the index set $X \subseteq P$ at v_1 and we want to minimize the objective function with respect to the remaining new facilities indicated by $\bar{X}$, which have to be located on the tree $\hat{T} = (\hat{V},\hat{E})$, where $\hat{T} = T_2$. Deletion of an edge $(v_s,v_t) \in \hat{E}$ results in two subtrees $\hat{T}_s$ and $\hat{T}_t$ containing respectively v_s and v_t. Given the locations $x_j (j \in \bar{X})$ we can write the objective function as

$$\sum_{i=1}^{n} \sum_{j\in P} \alpha_{ij}\, d(v_i,x_j) + \frac{1}{2} \sum_{j\in P} \sum_{k\in P} \beta_{jk}\, d(x_j,x_k) =$$

$$\sum_{i=2}^{n} \sum_{j\in\bar{X}} \alpha_{ij}\, d(v_i,x_j) + \sum_{i=2}^{n} \sum_{j\in X} \alpha_{ij} d(v_i,v_1) + \sum_{j\in\bar{X}} \alpha_{1j}\, d(v_1,x_j) +$$

$$\sum_{j\in\bar{X}} \sum_{k\in X} \beta_{jk}\, d(v_1,x_j) + \frac{1}{2} \sum_{j\in\bar{X}} \sum_{k\in\bar{X}} \beta_{jk}\, d(x_j,x_k).$$

Since $d(v_1,x_j) = d(v_1,v_2) + d(v_2,v_j)$ for all $j \in \bar{X}$, we can rewrite the objective function as $Q_1 + Q_2$, where

$$Q_1 = \sum_{i=2}^{n} \sum_{j\in X} \alpha_{ij}\, d(v_i,v_1) + \sum_{j\in\bar{X}} \alpha_{1j} d(v_1,v_2) + \sum_{j\in\bar{X}} \sum_{k\in X} \beta_{jk}\, d(v_1,v_2).$$

$$Q_2 = \sum_{i=2}^{n} \sum_{j\in\bar{X}} \hat{\alpha}_{ij}\, d(v_i,x_j) + \frac{1}{2} \sum_{j\in\bar{X}} \sum_{k\in\bar{X}} \beta_{jk}\, d(x_j,x_k),$$

$$\hat{\alpha}_{2j} = \alpha_{2j} + \alpha_{1j} + \sum_{k\in X} \beta_{jk} \text{ for all } j \in \bar{X}, \text{ and}$$

$$\hat{\alpha}_{ij} = \alpha_{ij} \text{ for } i = 3,..,n,\ j \in \bar{X}.$$

Note that Q_1 does not depend on the locations of the new facilities $j(j \in \bar{X})$. If we want to minimize $Q_1 + Q_2$ with respect to the new facility locations $x_j(j \in \bar{X})$, then we have to minimize Q_2. The induction hypothesis implies that the minimum value of Q_2 is equal to

$$\sum_{(v_s,v_t)\in\hat{E}} \ell(s,t) \min_{Y\subseteq\bar{X}} \hat{W}_{st}(Y) ,$$

where

$$\hat{W}_{st}(Y) = \sum_{j\in Y} \sum_{v_i\in\hat{T}_t} \hat{\alpha}_{ij} + \sum_{j\in\bar{Y}} \sum_{v_i\in\hat{T}_s} \hat{\alpha}_{ij} + \sum_{j\in Y} \sum_{k\in\bar{Y}} \beta_{jk} ,$$

and

$$\bar{Y} = \bar{X}\backslash Y.$$

Without loss of generality we may assume that $v_2 \in \hat{T}_s$. Then the vertex set of $\hat{T}_t$ is equal to the vertex set of T_t and if we add v_1 to the vertex set of $\hat{T}_s$ we get the vertex set of T_s. Substitution of the value of $\hat{\alpha}_{ij}$ in $\hat{W}_{st}(Y)$ gives

$$\hat{W}_{st}(Y) = \sum_{j\in Y} \sum_{v_i\in T_t} \alpha_{ij} + \sum_{j\in\bar{Y}} \sum_{v_i\in T_s} \alpha_{ij} + \sum_{j\in(X\cup Y)} \sum_{k\in\bar{Y}} \beta_{jk}.$$

Taking a closer look at Q_1, we find that

$$Q_1 = \left[\sum_{i=2}^{n} \sum_{j\in X} \alpha_{ij} + \sum_{j\in\bar{X}} \alpha_{1j} + \sum_{j\in\bar{X}} \sum_{k\in X} \beta_{jk}\right] d(v_1,v_2) + \sum_{i=2}^{n} \sum_{j\in X} \alpha_{ij} d(v_2,v_i)$$

$$= \ell(1,2) W_{12}(X) + \sum_{\substack{(v_s,v_t)\in\hat{E} \\ v_2\in T_s}} \ell(s,t) \sum_{j\in X} \sum_{v_i\in T_t} \alpha_{ij}.$$

The minimum value of the objective function with respect to new facilities $j \in \bar{X}$ is now equal to

$$\ell(1,2)\ W_{12}(X) + \sum_{\substack{(v_s,v_t)\in\hat{E} \\ v_2\in T_s}} \ell(s,t) \min_{Y\subseteq\bar{X}} W_{st}(X\cup Y) =$$

$$\ell(1,2)\ W_{12}(X) + \sum_{\substack{(v_s,v_t)\ \hat{E} \\ v_2\subset T_s}} \ell(s,t) \min_{\substack{Z\subseteq P \\ Z\supseteq X}} W_{st}(Z).$$

Since

(2.1) $$W_{12}(X) \geq \min_{Y\subseteq P} W_{12}(Y)$$

and

$$\text{(2.2)} \qquad \min_{\substack{Z \subseteq P \\ Z \supseteq X}} W_{st}(Z) \geq \min_{Y \subseteq P} W_{st}(Y)$$

it follows that the right-hand side of Theorem 1 is less than or equal to the left-hand side. We know from Lemma 1 that equality holds in (2.1) and (2.2) if we choose X such that $W_{12}(X) = \min_{Y \subseteq P} W_{12}(Y)$. Thus we have shown that the left-hand and right-hand sides of Theorem 1 are equal. Q.E.D.

COROLLARY 1. *Let X be such that* $W_{12}(X) = \min_{Y \subseteq P} W_{12}(Y)$. *Then an optimal solution to the p-median problem with mutual communication can be found by locating new facilities* $j(j \in X)$ *at* v_1 *and subsequently finding new facility locations* $x_j (j \in \bar{X})$ *in* T_2 *such that*

$$\sum_{i=2}^{n} \sum_{j \in \bar{X}} \hat{\alpha}_{ij} \, d(v_i, x_j) + \frac{1}{2} \sum_{j \in \bar{X}} \sum_{k \in \bar{X}} \beta_{jk} d(x_j, x_k)$$

is minimal, where

$$\hat{\alpha}_{2j} = \alpha_{2j} + \alpha_{1j} + \sum_{k \in X} \beta_{jk} \quad (j \in \bar{X})$$

and

$$\hat{\alpha}_{ij} = \alpha_{ij} \quad (i = 3, \ldots, n, \; j \in \bar{X}).$$

PROOF. We have shown in the proof of Theorem 1 that the optimum value of the objective function, if X is such that $W_{12}(X) = \min_{Y \subseteq P} W_{12}(Y)$, is equal to $Q_1 + \min_{x_j (j \in \bar{X})} Q_2$, where Q_1 and Q_2 are defined in Theorem 1. Q.E.D.

We observe that the optimal solution only depends on the weights and is independent of the edge lengths.

3. A POLYNOMIAL-TIME ALGORITHM

Corollary 1 immediately suggests an algorithm in which we look at a tip v_1, adjacent to, say, v_2: determine the set X such that $W_{12}(X) = \min_{Y\subseteq P} W_{12}(Y)$, locate all new facilities $j \in X$ at v_1, add $\alpha_{1j} + \Sigma_{k\in X} \beta_{jk}$ to the weight α_{2j} for all $j \in \bar{X}$, delete the edge (v_1,v_2) from the tree, and repeat the algorithm on the resulting tree with new facilities $j \in \bar{X}$. This algorithm is polynomial bounded if we are able to determine the set X such that $W_{12}(X) = \min_Y W_{12}(Y)$ in polynomial time.

Consider the following network (Fig.1), where we have two nodes v_1 and v_2 and a node j for each new facility $j(j = 1,2,..,m)$. We have arcs from v_1 to all nodes j of capacity $\Sigma_{v_i\in T_2} \alpha_{ij}$ $(j = 1,2,..,m)$, arcs from all nodes j to v_2 of capacity α_{1j} $(j = 1,2,..,m)$ and arcs from node j to node k of capacity β_{jk} $(j<k,j,k = 1,2..,m)$.

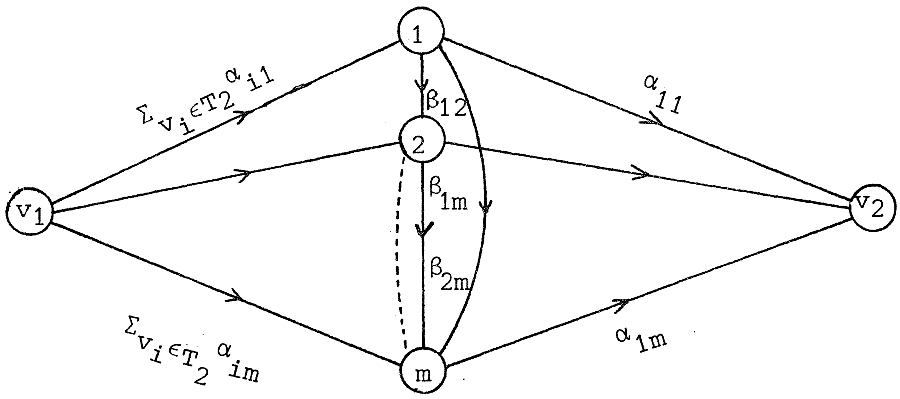

Fig. 1.

A cut in the network given by Fig. 1 is defined to be a set of arcs between A $(v_1\in A)$ and $\bar{A}$ $(v_2\in\bar{A})$ such that every path from v_1 to v_2 in the network contains at least one arc from this set. The capacity of a cut is the sum of the capacities of the arcs contained in the cut. Since the maximum flow from v_1 to v_2 in the network is equal to the minimum capacity of a cut, we can determine a minimum cut in $\mathcal{O}(m^3)$ times [6]. We will now show that a minimum cut in the network defines the set X such that $W_{12}(X) = \min_Y W_{12}(Y)$; this observation is originally due to PICARD & RATCLIFF [8].

The arcs between the sets $v_2 \cup X$ and $v_1 \cup \bar{X}$ form a cut for every $X \subseteq \{1,2,..,m\}$. Conversely, any cut determines a subset $X \subseteq \{1,2,..,m\}$ such that the cut contains all arcs between the set $v_2 \cup X$ and $v_1 \cup \bar{X}$. By an arc between a and b we mean that the arc goes from a to b or from b to a. The capacity of the cut between $v_2 \cup X$ and $v_1 \cup \bar{X}$ is equal to

$$\sum_{j\in X} \sum_{v_i \in T_2} \alpha_{ij} + \sum_{j\in \bar{X}} \alpha_{1j} + \sum_{j\in X} \sum_{k\in \bar{X}} \beta_{jk} = W_{12}(X).$$

The first term is the sum of the capacities of arcs from v_1 to the set X. The second term is the sum of the capacities of arcs from $\bar{X}$ to v_2. The third term is the sum of the capacities of arcs between X and $\bar{X}$. Therefore determining the minimum of $W_{12}(X)$ over all subsets $X \subseteq \{1,2,..,m\}$ is equal to determining a minimum cut in a corresponding network. It follows that the p-median problem with mutual communication can be solved as a sequence of at most n-1 minimum cut problems on a network with at most p+2 vertices. The running time of the algorithm is $\mathcal{O}(np^3)$.

We summarize the algorithm below.

<u>ALGORITHM</u>.

Initialize: Set $m: = p$, $\ell: = n$.

Iterate: If $\ell = 1$, then locate all remaining facilites at the remaining vertex: stop.

Otherwise, choose a tip v_1 of the tree; let v_2 be the vertex adjacent to v_1.

Determine $X \subseteq \{1,2,..,m\}$ such that $W_{12}(X) = \min_Y W_{12}(Y)$.

Locate all facilities $j (j\in X)$ at v_1.

Add $\alpha_{1j} + \Sigma_{k\in X} \beta_{jk}$ to α_{2j} for all $j \in \bar{X}$.

Set $m: = m - |X|$.

If $m = 0$, then all facilities are located: stop.

Otherwise, renumber the new facilities in $\bar{X}$ from 1 up to m.

Delete (v_1,v_2) from the tree, set $\ell: = \ell - 1$, and iterate.

In the case that $p = 1$, we only have to compute $W_{12}(\phi) = \alpha_{11}$ and $W_{12}(\{1\}) =$

$\Sigma_{i\geq 2}\ \alpha_{i1}$. It is easily seen that in this case our algorithm corresponds to the well-known algorithm due to GOLDMAN [4] .

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