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THE P-MEDIAN PROBLEM WITH MUTUAL COMMUNICATION ON A TREE

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The $p$-median problem with mutual communication on a tree

by

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ABSTRACT

This paper considers the problem of locating $p$ new facilities on a tree, where each vertex represents an existing facility, in order to minimize the total weighted sum of distances between all pairs of new and existing facilities and between all pairs of new facilities.

We present a polynomial-time algorithm for its solution. This algorithm generalizes the well-known algorithm for the $1$-median problem on a tree.

KEY WORDS & PHRASES: $p$-median problem (with mutual communication), tree, location theory.

*) This report will be submitted for publication elsewhere.
1. INTRODUCTION

Let \( T \) be a tree with vertex set \( V = \{v_1, v_2, \ldots, v_n\} \) and edge set \( E = \{e_1, e_2, \ldots, e_{n-1}\} \). Each edge \((v_i, v_j) \in E\) has a nonnegative length \( \ell(i,j) \). A point \( x \) on the tree can be a vertex or a point anywhere along an edge. The length of the shortest path between the points \( x \) and \( y \) on \( T \) is denoted by \( d(x,y) \).

The p-median problem with mutual communication is to find \( p \) new facility locations \( x_1, x_2, \ldots, x_p \) such that

\[
\sum_{i=1}^{n} \sum_{j=1}^{p} a_{ij} d(v_i, x_j) + \frac{1}{2} \sum_{j=1}^{p} \sum_{k=1}^{p} \beta_{jk} d(x_j, x_k)
\]

is minimal, where \( a_{ij} \) (\( i=1, \ldots, n \), \( j=1, \ldots, p \)) and \( \beta_{jk} \) (\( j=1, \ldots, p \), \( k=1, \ldots, p \)) are given nonnegative weights. It is well known that an optimal solution exists with \( x_i \in V \) (\( i=1, \ldots, p \)) [3].

We can think of the vertices of the tree as locations of existing facilities. Let \( a_{ij} \) represent the amount of travel between existing facility \( i \) and new facility \( j \), and let \( \beta_{jk} \) represents the amount of travel between new facilities \( j \) and \( k \). The tree \( T \) corresponds to a transportation network and the p-median problem with mutual communication is to find the new facility locations such that the total travelled distance is minimal.

The p-median problem with mutual communication in the plane using rectilinear distances has received much attention. This problem is to find new facility locations \((x_1, y_1), \ldots, (x_p, y_p)\) in the plane such that

\[
\sum_{i=1}^{n} \sum_{j=1}^{p} a_{ij} (|x_j-a_i| + |y_j-b_i|) + \frac{1}{2} \sum_{j=1}^{p} \sum_{k=1}^{p} \beta_{jk} (|x_j-x_k| + |y_j-y_k|)
\]

is minimal, where \((a_i, b_i)\) (\( i=1, \ldots, n \)) are the existing facility locations. This problem can be decomposed into two independent problems on the line.

We mention the following references for the p-median problem with mutual communication on a line: PRITSKER & SHARE [9], RAO [10], JUEL & LOVE [5], SHERALI & SHETTY [11], CAEOT, FRANCIS & STARY [1], WESOLOWSKY & LOVE [12,13], P-CARD & RATTLIFF [8], and KOLEN [7].

The only algorithm we know of which solves the p-median problem with
mutual communication on a tree is due to PICARD & RATLIFF [8]. In Section 2 we will prove a theorem which characterizes the optimum solution value to this problem. This theorem was already known to FRANCES [2] for the very easy case that $p = 1$. It provides the basis for our algorithm to solve the problem, presented in Section 3. This algorithm differs from the algorithm by PICARD & RATLIFF [8] but has the same time complexity. In the case that $p = 1$, our algorithm reduces to the well-known algorithm by GOLDMAN [4] to solve the 1-median problem on a tree.

2. CHARACTERIZING THE OPTIMUM VALUE

Let us start with the simple case that the tree has only a single edge $(v_1, v_2)$. Let $P$ denote the set $\{1, 2, \ldots, p\}$, let $X \subseteq P$ be the index set of the facilities located at $v_1$, and let $\overline{X} = P \setminus X$. In this case, the objective function is given by

$$
\sum_{j \in X} a_{2j} d(v_1, v_2) + \sum_{j \in \overline{X}} a_{1j} d(v_1, v_2) + \sum_{j \in X} \sum_{k \in \overline{X}} \beta_{jk} d(v_1, v_2) = \\
\ell(1, 2) w_{12}(X),
$$

where

$$
w_{12}(X) = \sum_{j \in X} a_{2j} + \sum_{j \in \overline{X}} a_{1j} + \sum_{j \in X} \sum_{k \in \overline{X}} \beta_{jk}
$$

Hence the value of the objective function is minimal for that subset of $P$ which minimizes $w_{12}(X)$ over all subsets $X \subseteq P$. This leads to the following characterization of the optimum value: the minimum value of the objective function is equal to

$$
\ell(1, 2) \min_{X \subseteq P} w_{12}(X).
$$

We now generalize this result to an arbitrary tree $T$. Let $(v_s, v_t)$ be an arbitrary edge of $T$. Deletion of $(v_s, v_t)$ with the exception of $v_s$
and \( v_t \) from \( T \) results in two subtrees. Let \( T_s(T_t) \) be the subtree containing \( v_s(v_t) \). We define \( W_{st}(X) \) as the total amount of travel between facilities in \( T_s \) and facilities in \( T_t \) if \( X \) denotes the index set of new facilities in \( T_s \):

\[
W_{st}(X) = \sum_{j \in X} \sum_{i \in T_t} \alpha_{ij} + \sum_{j \in X} \sum_{i \in T_s} \alpha_{ij} + \sum_{j \in X} \sum_{k \in X} \beta_{jk}.
\]

We now state our characterization of the optimum value in Theorem 1.

**THEOREM 1.**

\[
\min_{x_1, \ldots, x_p} \sum_{i=1}^{n} \sum_{j \in P} \alpha_{ij} d(i, j) + \frac{1}{2} \sum_{j \in P} \sum_{k \in P} \beta_{jk} d(j, k) = \sum_{(v_s, v_t) \in E} \ell(s, t) \min_{x \subseteq P} W_{st}(X).
\]

Before proving this theorem we prove the following lemma.

**LEMMA 1.** Let \( v_1 \) be a tip of the tree \( T \), let \( v_2 \) be the vertex adjacent to \( v_1 \), and let \((v_s, v_t)\) be an edge of \( T \) not equal to \((v_1, v_2)\) and such that \( v_1 \) and \( v_2 \) are contained in \( T_s \). Let \( Q \) be such that \( W_{12}(Q) = \min_{X \subseteq P} W_{12}(X) \). Then there is a set \( R \) such that \( Q \subseteq R \) and \( \min_{X \subseteq P} W_{st}(X) = \min_{X \subseteq P} W_{12}(X) \).

**PROOF.** Let \( R \) be a set such that \( Q \cap R = S_1 \neq \emptyset \), i.e., there are subsets \( S_2(S_2 \neq \emptyset) \) and \( S_3 \) such that \( Q = S_1 \cup S_2 \) and \( R = S_1 \cup S_3 \). Since \( W_{12}(Q) = \min_{X \subseteq P} W_{12}(X) \) we know that \( W_{12}(S_1 \cup S_2) - W_{12}(S_1) \leq 0 \). We shall prove below that \( W_{st}(S_1 \cup S_2, S_3) - W_{st}(S_1, S_3) \leq W_{12}(S_1 \cup S_2) - W_{12}(S_1) \). This implies that \( W_{st}(S_1 \cup S_2, S_3) - W_{st}(S_1, S_3) \leq 0 \). Therefore if \( W_{st}(R) = \min_{X \subseteq P} W_{st}(X) \), then \( W_{st}(Q \cup R) = W_{st}(R) \), i.e., without loss of generality we may assume that \( Q \subseteq R \). Since

\[
W_{st}(S_1 \cup S_2 \cup S_3) = \sum_{j \in S_2} \sum_{i \in T_t} \alpha_{ij} + \sum_{j \in S_2} \sum_{i \in T_s} \alpha_{ij} + \sum_{j \in S_1 \cup S_2 \cup S_3} \sum_{k \in X} \beta_{jk}.
\]
\[ \sum_{j \in S_1 \cup S_2 \cup S_3} \sum_{k \in (S_1 \cup S_2)} \beta_{jk} \]

\[ W_{12}(S_1 \cup S_2) - W_{12}(S_1) = \]

\[ \sum_{j \in S_2} \sum_{v_1 \in T_2} \alpha_{ij} - \sum_{j \in S_1 \cup S_2 \cup S_3} \sum_{k \in (S_1 \cup S_2 \cup S_3)} \beta_{jk} - \sum_{j \in S_1} \sum_{k \in S_1} \beta_{jk} \]

and moreover \( T_t \subset T_s \) and \( v_1 \in T_s \), we have

\[ (W_{st}(S_1 \cup S_2 \cup S_3) - W_{st}(S_1 \cup S_3)) - (W_{12}(S_1 \cup S_2) - W_{12}(S_1)) = \]

\[ - \sum_{j \in S_2} \sum_{v_1 \in T_2 \setminus T_t} \alpha_{ij} - \sum_{j \in S_2} \sum_{v_1 \in T_s \setminus \{v_1\}} \alpha_{ij} - 2 \sum_{j \in S_2} \sum_{k \in S_3} \beta_{jk} \leq 0. \]

This yields the desired result. \( \Box \).

We shall now give a proof of Theorem 1.

**Proof of Theorem 1.** The proof is by induction on the number of vertices \(|V|\).

For the case \(|V| = 2\), we have shown at the beginning of this section that the theorem is true. Suppose the theorem is true for all trees with \(|V| < n\), and consider a tree \(T\) with \(n\) vertices \((n \geq 3)\). Let \(v_1\) be a tip of the tree and let \(v_2\) be the vertex adjacent to \(v_1\). Assume we locate new facilities indicated by the index set \(X \subseteq P\) at \(v_1\) and we want to minimize the objective function with respect to the remaining new facilities indicated by \(X\), which have to be located on the tree \(\hat{T} = (\hat{V}, \hat{E})\), where \(\hat{T} = T_2\). Deletion of an edge \((v_s, v_t) \in \hat{E}\) results in two subtrees \(\hat{T}_s\) and \(\hat{T}_t\) containing respectively \(v_s\) and \(v_t\). Given the locations \(x_j (j \in X)\), we can write the objective function as

\[ \sum_{i=1}^{n} \sum_{j \in P} \alpha_{ij} d(v_i, x_j) + \frac{1}{2} \sum_{j \in P} \sum_{k \in P} \beta_{jk} d(x_j, x_k) = \]
\[
\sum_{i=2}^{n} \sum_{j \in X} a_{ij} d(v_i, x_j) + \sum_{i=2}^{n} \sum_{j \in X} a_{ij} d(v_1, v_i) + \sum_{j \in X} a_{1j} d(v_1, x_j) + \\
\sum_{j \in X} \sum_{k \in X} \beta_{jk} d(v_1, x_j) + \frac{1}{2} \sum_{j \in X} \sum_{k \in X} \beta_{jk} d(x_j, x_k).
\]

Since \(d(v_i, x_j) = d(v_1, v_2) + d(v_2, v_j)\) for all \(j \in \bar{X}\), we can rewrite the objective function as \(Q_1 + Q_2\), where

\[
Q_1 = \sum_{i=2}^{n} \sum_{j \in X} a_{ij} d(v_i, v_1) + \sum_{j \in X} a_{1j} d(v_1, v_2) + \sum_{j \in X} \sum_{k \in X} \beta_{jk} d(v_1, v_2).
\]

\[
Q_2 = \sum_{i=2}^{n} \sum_{j \in X} \hat{a}_{ij} d(v_i, x_j) + \frac{1}{2} \sum_{j \in X} \sum_{k \in X} \beta_{jk} d(x_j, x_k),
\]

\[
\hat{a}_{2j} = a_{2j} + a_{1j} + \sum_{k \in X} \beta_{jk} \text{ for all } j \in \bar{X}, \text{ and}
\]

\[
\hat{a}_{ij} = a_{ij} \text{ for } i = 3, \ldots, n, j \in \bar{X}.
\]

Note that \(Q_1\) does not depend on the locations of the new facilities \(j \in \bar{X}\). If we want to minimize \(Q_1 + Q_2\) with respect to the new facility locations \(x_j \in \bar{X}\), then we have to minimize \(Q_2\). The induction hypothesis implies that the minimum value of \(Q_2\) is equal to

\[
\sum_{(v_s, v_t) \in E} \ell(s, t) \min_{y \in \bar{X}} \hat{W}_{st}(Y),
\]

where

\[
\hat{W}_{st}(Y) = \sum_{j \in Y} \sum_{i \in T_t} \hat{a}_{ij} + \sum_{j \in Y} \sum_{i \in T_s} \hat{a}_{ij} + \sum_{j \in Y} \sum_{k \in \bar{Y}} \beta_{jk},
\]

and
\[ \bar{y} = \bar{x} \setminus y. \]

Without loss of generality we may assume that \( v_2 \in \hat{T}_s \). Then the vertex set of \( \hat{T}_s \) is equal to the vertex set of \( T_s \) and if we add \( v_1 \) to the vertex set of \( \hat{T}_s \) we get the vertex set of \( T_s \). Substitution of the value of \( \hat{a}_{ij} \) in \( \bar{w}_{st}(Y) \) gives

\[ \bar{w}_{st}(Y) = \sum_{j \in Y} \sum_{v_i \in T_t} a_{ij} + \sum_{j \in Y} \sum_{v_i \in T_S} a_{ij} + \sum_{j \in (X \cup Y)} \sum_{k \in Y} b_{jk}. \]

Taking a closer look at \( Q_1 \), we find that

\[ Q_1 = \left[ \sum_{i=2}^{n} \sum_{j \in X} a_{ij} + \sum_{j \in X} a_{1j} + \sum_{j \in X} \sum_{k \in X} b_{jk} \right] d(v_1, v_2) + \]

\[ \sum_{i=2}^{n} \sum_{j \in X} a_{ij} d(v_2, v_i) \]

\[ = \ell(1,2) W_{12}(X) + \sum_{(v_s, v_t) \in E} \ell(s,t) \sum_{j \in X} \sum_{v_i \in T_t} a_{ij}. \]

The minimum value of the objective function with respect to new facilities \( j \in \bar{x} \) is now equal to

\[ \ell(1,2) W_{12}(X) + \sum_{(v_s, v_t) \in E} \ell(s,t) \min_{Y \subseteq X} \bar{w}_{st}(X \cup Y) = \]

\[ \ell(1,2) W_{12}(X) + \sum_{(v_s, v_t) \in E} \ell(s,t) \min_{2 \subseteq P} W_{st}(Z). \]

Since

\[ (2.1) \quad W_{12}(X) \geq \min_{Y \subseteq P} W_{12}(Y) \]
and

\[(2.2) \quad \min_{Z \subseteq P} W_{st}(Z) \geq \min_{Y \subseteq P} W_{st}(Y) \]

it follows that the right-hand side of Theorem 1 is less than or equal to the left-hand side. We know from Lemma 1 that equality holds in (2.1) and (2.2) if we choose \( X \) such that \( W_{12}(X) = \min_{Y \subseteq P} W_{12}(Y) \). Thus we have shown that the left-hand and right-hand sides of Theorem 1 are equal. \( \Box \).

**Corollary 1.** Let \( X \) be such that \( W_{12}(X) = \min_{Y \subseteq P} W_{12}(Y) \). Then an optimal solution to the p-median problem with mutual communication can be found by locating new facilities \( j(\bar{x}_X) \) at \( v_1 \) and subsequently finding new facility locations \( x_j(j \in \bar{x}) \) in \( \tau_2 \) such that

\[
\sum_{i=2}^{n} \sum_{j \in \bar{x}} \tilde{a}_{1j} d(v_1, x_j) + \frac{1}{2} \sum_{j \in \bar{x}} \sum_{k \in \bar{x}} \beta_{jk} d(x_j, x_k)
\]

is minimal, where

\[\tilde{a}_{2j} = a_{2j} + \sum_{k \in \bar{x}} \beta_{jk} (j \in \bar{x})\]

and

\[\tilde{a}_{1j} = a_{1j} \quad (i = 3, \ldots, n, \ j \in \bar{x}).\]

**Proof.** We have shown in the proof of Theorem 1 that the optimum value of the objective function, if \( X \) is such that \( W_{12}(X) = \min_{Y \subseteq P} W_{12}(Y) \), is equal to \( Q_1 + \min_{x_j(j \in \bar{x})} Q_2 \), where \( Q_1 \) and \( Q_2 \) are defined in Theorem 1. \( \Box \).

We observe that the optimal solution only depends on the weights and is independent of the edge lengths.
3. A POLYNOMIAL-TIME ALGORITHM

Corollary 1 immediately suggests an algorithm in which we look at a
tip 1, adjacent to, say, 2: determine the set X such that \( W_{12}(X) = \min_{Y \subseteq P} W_{12}(Y) \), locate all new facilities \( j \in X \) at 1, add \( \alpha_{1j} + \sum_{k \in X} \beta_{jk} \) to the weight \( \alpha_{2j} \) for all \( j \in X \), delete the edge \((1, 2)\) from the
tree, and repeat the algorithm on the resulting tree with new facilities
\( j \in X \). This algorithm is polynomial bounded if we are able to determine the
set X such that \( W_{12}(X) = \min_Y W_{12}(Y) \) in polynomial time.

Consider the following network (Fig. 1), where we have two nodes
1 and 2 and a node \( j \) for each new facility \( j \) (\( j = 1, 2, \ldots, m \)). We have
arcs from 1 to all nodes \( j \) of capacity \( a_{1j} \) (\( j = 1, 2, \ldots, m \)), arcs
from all nodes \( j \) to 2 of capacity \( a_{1j} \) (\( j = 1, 2, \ldots, m \)) and arcs from node
\( j \) to node \( k \) of capacity \( \beta_{jk} \) (\( j, k, k = 1, 2, \ldots, m \)).

A cut in the network given by Fig. 1 is defined to be a set of arcs between
\( A \) (\( v_1 \in A \)) and \( \bar{A} \) (\( v_2 \in \bar{A} \)) such that every path from 1 to 2 in the network
contains at least one arc from this set. The capacity of a cut is the
sum of the capacities of the arcs contained in the cut. Since the maximum
flow from 1 to 2 in the network is equal to the minimum capacity of
a cut, we can determine a minimum cut in \( O(m^3) \) times [6]. We will row
show that a minimum cut in the network defines the set X such that
\( W_{12}(X) = \min_Y W_{12}(Y) \); this observation is originally due to PICARD &
RATCLIFF [8].
The arcs between the sets \( v_2 \cup X \) and \( v_1 \cup \bar{X} \) form a cut for every \( X \subseteq \{1,2,\ldots,m\} \). Conversely, any cut determines a subset \( X \subseteq \{1,2,\ldots,m\} \) such that the cut contains all arcs between the set \( v_2 \cup X \) and \( v_1 \cup \bar{X} \). By an arc between \( a \) and \( b \) we mean that the arc goes from \( a \) to \( b \) or from \( b \) to \( a \). The capacity of the cut between \( v_2 \cup X \) and \( v_1 \cup \bar{X} \) is equal to

\[
\sum_{j \in X} \sum_{i \in T_2} a_{1j} + \sum_{j \not\in X} a_{1j} + \sum_{j \in X} \sum_{k \in \bar{X}} \beta_{jk} = W_{12}(X).
\]

The first term is the sum of the capacities of arcs from \( v_1 \) to the set \( X \). The second term is the sum of the capacities of arcs from \( \bar{X} \) to \( v_2 \). The third term is the sum of the capacities of arcs between \( X \) and \( \bar{X} \). Therefore determining the minimum of \( W_{12}(X) \) over all subsets \( X \subseteq \{1,2,\ldots,m\} \) is equal to determining a minimum cut in a corresponding network. It follows that the \( p \)-median problem with mutual communication can be solved as a sequence of at most \( n-1 \) minimum cut problems on a network with at most \( p+2 \) vertices. The running time of the algorithm is \( O(np^3) \).

We summarize the algorithm below.

**ALGORITHM.**

Initialize: Set \( m = p \), \( \ell = n \).

Iterate: If \( \ell = 1 \), then locate all remaining facilities at the remaining vertex: stop.

Otherwise, choose a tip \( v_1 \) of the tree; let \( v_2 \) be the vertex adjacent to \( v_1 \).

Determine \( X \subseteq \{1,2,\ldots,m\} \) such that \( W_{12}(X) = \min_Y W_{12}(Y) \).

Locate all facilities \( j(\in X) \) at \( v_1 \).

Add \( a_{1j} + \sum_{k \in X} \beta_{jk} \) to \( a_{2j} \) for all \( j \in \bar{X} \).

Set \( m = m - |X| \).

If \( m = 0 \), then all facilities are located: stop.

Otherwise, renumber the new facilities in \( \bar{X} \) from 1 up to \( m \).

Delete \( (v_1,v_2) \) from the tree, set \( \ell = \ell - 1 \), and iterate.

In the case that \( p = 1 \), we only have to compute \( W_{12}(\emptyset) = a_{11} \) and \( W_{12}(\{1\}) = \)
\[ \sum_{i=2}^{n} a_{ij}. \]

It is easily seen that in this case our algorithm corresponds to the well-known algorithm due to Goldman [4].

REFERENCES


