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DUALITY IN POLYNOMIAL MODELS WITH SOME APPLICATIONS TO GEOMETRIC CONTROL THEORY

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Duality in polynomial models with some applications to geometric control theory *)

by

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ABSTRACT

Duality is studied in the context of polynomial models for linear systems. The output injection group, the dual of the feedback group, is studied and a polynomial characterization of (C,A)-invariant subspaces as well as of the maximal reachability subspace contained in Ker C is given.

KEY WORDS & PHRASES: *Linear systems, Polynomial models, duality, (C,A)-invariant subspaces*

*) This report will be submitted for publication elsewhere.

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1. INTRODUCTION

The question of duality in linear system theory has remained so far unclarified and is used mostly by transposing matrices. While this may yield results it is far from satisfactory from a theoretical point of view.

In a series of papers [1-6] there was an attempt to study finite dimensional time invariant systems using the polynomial model approach developed by the author in [2]. The use of polynomial models rather than dealing with matrix representations has the advantage of a richer structure which naturally accomadates any study of zeros, poles and system structure and isomorphism.

Our object in this paper is to study problems of duality in the context of polynomial models and their associated rational models. The advantage of this approach is that the dual space is not defined abstractly but is naturally equipped with a suitable polynomial module structure. Thus the dual of a polynomial model system is again a polynomial model system.

The structure of the paper is as follows. Section 2 is devoted to a general study of duality in polynomial models. In Section 3 we analyse the dual of the feedback group namely the output injection group as well as give a polynomial characterization of (C,A)-invariant subspaces. Section 4 is devoted to a polynomial characterization of the maximal reachability subspace in KerC.

The results on duality owe much to many discussions on this subject with Sanjoy K. Mitter. Some of the results on (C,A)-invariant subspaces have been independently discovered by M. Kaashoek.

2. DUALITY IN POLYNOMIAL MODELS

Let F be an arbitrary field, $F[\lambda]$ the ring of polynomials. An m-dimensional vector space over F will be generally identified with F^m. $F^m((\lambda^{-1}))$ is the $F[\lambda]$-module of truncated Laurent series with coefficients in F^m, i.e. the set of series of the form $f(x) = \Sigma_{j=-\infty}^{n_f} f_j\lambda^j$. The quotient module $F^m((\lambda^{-1}))/F^m[\lambda]$ will be identified with $\lambda^{-1}F^m[[\lambda^{-1}]]$ the space of formal power series in λ^{-1} with coefficients in F^m and vanishing constant term. As usual π_+ and π_- will denote the projections of $F^m((\lambda^{-1}))$ on $F^m[\lambda]$ and

$\lambda^{-1}F^m[[\lambda^{-1}]]$ respectively. Given a column vector $\xi \in F^m$ then $\tilde{\xi}$ will denote its transpose. If we define

$$(2.1)\qquad [\xi,\eta] = \tilde{\eta}\xi$$

then F^m is identified with its dual space. Given a polynomial matrix $P \in F^{p\times m}[\lambda]$, with $P(\lambda) = \Sigma_{j=0}^{n} P_j\lambda^j$, we define $\tilde{P} \in F^{m\times p}[\lambda]$ by

$$\tilde{P}(\lambda) = \tilde{P}(\lambda) = \sum_{j=0}^{n} \tilde{P}_j\lambda^j.$$

Next we define a pairing between elements of $F^m((\lambda^{-1}))$. To this end let $f,g \in F^m((\lambda^{-1}))$ be given by $f(\lambda) = \Sigma_{j=-\infty}^{n_f} f_j\lambda^j$ and $g(\lambda) = \Sigma_{j=-\infty}^{n_g} g_j\lambda^j$. We define $[f,g]$ by

$$(2.2)\qquad [f,g] = \sum_{j=-\infty}^{\infty} \tilde{g}_j f_{-j-1}.$$

It is clear that $[f,g]$ is a bilinear form on $F^m((\lambda^{-1}))$. That $[f,g]$ is well defined follows from the fact that the sum in (2.2) has always at most a finite number of nonzero terms. We also note that $[f,g] = 0$ for all $g \in F^m((\lambda^{-1}))$ if and only if $f = 0$.

Given a subset M of $F^m((\lambda^{-1}))$ we define $M^\perp \subset F^m((\lambda^{-1}))$ by

$$(2.3)\qquad M^\perp = \{g \in F^m((\lambda^{-1})) \mid [f,g] = 0 \text{ for all } f \in M\}.$$

In particular we have the following simple result,

$$(2.4)\qquad (F^m[\lambda])^\perp = F^m[\lambda].$$

The dual space of $F^m[\lambda]$ i.e. the space of F-linear functionals is easily characterized.

<u>THEOREM 2.1</u>. *The dual space of* $F^m[\lambda]$ *is isomorphic to* $\lambda^{-1}F^m[[\lambda^{-1}]]$.

<u>PROOF</u>. Clearly given $h \in \lambda^{-1}F^m[[\lambda^{-1}]]$ then the pairing $[f,h]$ of (2.2) defines a linear functional on $F^m[\lambda]$. Conversely if $\phi: F^m[\lambda] \to F$ is a linear functional then ϕ is uniquely determined by its action on elements of the form

$\xi\lambda^n$. As $\phi(\xi\lambda^n)$ is, with n fixed, a linear functional on F^m we have the existence of η_n such that $\phi(\xi\lambda^n) = \tilde{\eta}_n\xi$. It is now easily checked that

$$\phi(f) = [f,h] \tag{2.5}$$

with $h(\lambda) = \Sigma_{j=1}^{\infty} \eta_j\lambda^{-j}$.

Consider how the two shift operators S_+ and S_- acting in $F^m[\lambda]$ and $\lambda^{-1}F^m[[\lambda^{-1}]]$ respectively and defined by

$$(S_+f)(\lambda) = \lambda f(\lambda) \qquad \text{for } f \in F^m[\lambda] \tag{2.6}$$

and

$$S_-h = \pi_-(\lambda h) \qquad \text{for } h \in \lambda^{-1}F^m[[\lambda^{-1}]]. \tag{2.7}$$

Given a linear transformation $A: F^m[\lambda] \to F^p[\lambda]$ its dual or adjoint, denoted by A^*, is the unique transformation $A^*: \lambda^{-1}F^p[[\lambda^{-1}]] \to \lambda^{-1}F^m[[\lambda^{-1}]]$ that satisfies

$$[Af,h] = [f,A^*h] \tag{2.6}$$

for all $f \in F^m[\lambda]$ and $h \in \lambda^{-1}F^p[[\lambda^{-1}]]$.

<u>LEMMA 2.2</u>. *The dual of* S_+ *is* S_-.

<u>PROOF</u>. Follows from the easily checked fact that

$$[S_+f,h] = [f,S_-h] \tag{2.7}$$

holds for all $f \in F^m[\lambda]$ and $h \in \lambda^{-1}F^m[[\lambda^{-1}]]$.

The way we identified $F^m[\lambda]^*$ is compatible with the $F[\lambda]$-module structures on $F^m[\lambda]$ and $\lambda^{-1}F^m[[\lambda^{-1}]]$.

<u>LEMMA 2.3</u>. *Let* $V \subset F^m[\lambda]$ *be an* $F[\lambda]$*-submodule then* $V^\perp \subset \lambda^{-1}F^m[[\lambda^{-1}]]$ *is also a submodule.*

PROOF. Follows from (2.7).

The next two lemmas provide simple computational rules.

LEMMA 2.4. *Given the projections* π_+ *and* π_- *we have for all* $f,g \in F^m((\lambda^{-1}))$ *that*

$$[\pi_+ f,g] = [f,\pi_- g]. \tag{2.8}$$

LEMMA 2.5. *Given* $A \in F^{p\times m}[\lambda]$, $f \in F^m[\lambda]$ *and* $h \in \lambda^{-1}F^p[[\lambda^{-1}]]$ *then*

$$[Af,h] = [f,\pi_- \tilde{A}h]. \tag{2.9}$$

Since multiplication by elements of $F^{p\times m}[\lambda]$ represent all $F[\lambda]$-module homomorphisms from $F^m[\lambda]$ into $F^p[\lambda]$ then Lemma 2.5 describes a class of $F[\lambda]$-module homomorphisms from $\lambda^{-1}F^p[[\lambda^{-1}]]$ into $\lambda^{-1}F^m[[\lambda^{-1}]]$. For some results related to this one can refer to [4].

In some cases, given a submodule $V \subset F^m[\lambda]$ the submodule $V^\perp$ of $\lambda^{-1}F^m[[\lambda^{-1}]]$ can be identified. To this end we recall that a submodule V of $F^m[\lambda]$ is called a full submodule if $F^m[\lambda]/V$ is a torsion module or equivalently if V has a representation

$$V = DF^m[\lambda] \tag{2.10}$$

with $D \in F^{m\times m}[\lambda]$ a nonsingular polynomial matrix. Next we recall [2,4,6] that given a nonsingular $D \in F^{m\times m}[\lambda]$ we can define two projections π_D: $F^m[\lambda] \to F^m[\lambda]$ and π^D: $\lambda^{-1}F^m[[\lambda^{-1}]] \to \lambda^{-1}F^m[[\lambda^{-1}]]$ by

$$\pi_D f = D\pi_- D^{-1} f \qquad \text{for } f \in F^m[\lambda] \tag{2.11}$$

and

$$\pi^D h = \pi_- D^{-1}\pi_+ Dh \qquad \text{for } h \in \lambda^{-1}F^m[[\lambda^{-1}]] \tag{2.12}$$

We denote by K_D and L_D the ranges of π_D and π^D respectively and note the equality

(2.13) $$D^{-1}K_D = L_D.$$

THEOREM 2.6. *Let* $V = DF^m[\lambda]$ *with* D *nonsingular in* $F^{m\times m}[\lambda]$. *Then*

(2.14) $$V^{\perp} = L_{\tilde{D}}.$$

PROOF. Let $f \in F^m[\lambda]$ and $h \in V^{\perp}$ then $0 = [Df,h] = [f,\tilde{D}h] = [f,\pi_-\tilde{D}h]$

But this implies $h \in L_{\tilde{D}}$. The converse follows from the same formulas.

Next we compute the adjoint of the projection π_D.

THEOREM 2.7. *The adjoint of the projection* π_D *is* $\pi^{\tilde{D}}$.

PROOF. Let $f \in F^m[\lambda]$ and $h \in \lambda^{-1}F^m[[\lambda^{-1}]]$ then

$$\begin{aligned}[\pi_D f,h] &= [D\pi_- D^{-1}f,h] = [\pi_- D^{-1}f,\tilde{D}h] \\ &= [D^{-1}f,\pi_+\tilde{D}h] = [f,\tilde{D}^{-1}\pi_+\tilde{D}h] \\ &= [\pi_+ f,\tilde{D}^{-1}\pi_+\tilde{D}h] = [f,\pi_-\tilde{D}^{-1}\pi_+\tilde{D}h] \\ &= [f,\pi^{\tilde{D}}h].\end{aligned}$$

Our main interest is to get a convenient and useful representation for K_D^*. To this end we note that in general given a linear space X and a subspace M then if X^* is the dual space of X then we have the isomorphism

(2.15) $$(X/M)^* = M^{\perp}.$$

Recall also [4] that S^D: $L_D \to L_D$ is defined by

(2.16) $$S^D = S_-|L_D.$$

THEOREM 2.8. *Let* $D \in F^{m\times m}[\lambda]$ *be nonsingular, then*

(2.17) $$K_D^* = L_{\tilde{D}}$$

and

(2.18) $\qquad S_D^* = S^{\tilde{D}}$.

PROOF. Since K_D is isomorphic to $F^m[\lambda]/DF^m[\lambda]$ then K_D^* is isomorphic to $(F^m[\lambda]/DF^m[\lambda])$ which by the previous remark is ismorphic to $(DF^m[\lambda])^{\perp}$. By Theorem 2.6 this is equal to $L_{\tilde{D}}$. It is now easily checked that under the pairing (2.2) we actually have (2.17).

Finally let $f \in K_D$ and $h \in L_{\tilde{D}}$ then

$$[S_D f,h] = [\pi_D \lambda f,h] = [\lambda f,\pi^{\tilde{D}} h] = [\lambda f,h] = [f,\lambda h]$$

$$= [\pi_+ f,\lambda h] = [f,\pi_- \lambda h] = [f,S^{\tilde{D}} h].$$

Now the $F[\lambda]$-module L_D is isomorphic to $K_{\tilde{D}}$ hence we can identify K_D^* with $K_{\tilde{D}}$ by defining for all $f \in K_D$ and all $g \in K_{\tilde{D}}$

(2.19) $\qquad \langle f,g\rangle = [D^{-1}f,g] = [f,\tilde{D}^{-1}g].$

As a direct corollary of Theorem 2.8 we have the following

THEOREM 2.8. *The dual space of* K_D *can be identified, under the pairing* (2.19), *with* $K_{\tilde{D}}$. *Moreover we have*

(2.20) $\qquad S_D^* = S_{\tilde{D}},$

i.e.

(2.21) $\qquad \langle S_D f,g\rangle = \langle f,S_{\tilde{D}} g\rangle$

for all $f \in K_D$ *and* $g \in K_{\tilde{D}}$.

Submodules of K_D are associated with factorization of D. In fact a subspace $V \subset K_D$ is a submodule if and only if $V = EK_F$ for some factorization $D = EF$ into nonsingular factors [6]. One is naturally interested in the corresponding representation of $V^{\perp} \subset K_{\tilde{D}}$.

THEOREM 2.10. *Let* $V \subset K_D$ *be a submodule with the representation* $V = EK_F$. *Then* $V^{\perp} \subset K_{\tilde{D}}$ *is also a submodule and is given by* $V^{\perp} = \tilde{F}K_{\tilde{E}}$.

PROOF. That $V^{\perp}$ is a submodule, or equivalently $S_{\tilde{D}}$-invariant follows from (2.21). Let now $f \in V^{\perp}$ then for every $g \in K_F$ we have

$$0 = \langle Eg,f\rangle = [D^{-1}Eg,f] = [F^{-1}g,f] = [g,\tilde{F}^{-1}f]$$

or $\tilde{F}^{-1}f \in K_F^{\perp}$. But clearly $\tilde{F}^{-1}f \in [(F\cdot F^m[\lambda])^{\perp}$ as for any $g \in F^m[\lambda]$

$$[Fg,\tilde{F}^{-1}f] = [g,f] = 0$$

The two identities imply $\pi_-\tilde{F}^{-1}f = 0$ or $f = \tilde{F}\cdot f_1$ with $f_1 \in F^m(\lambda)$. Now $f \in K_{\tilde{D}}$ implying $\pi_+\tilde{D}^{-1}f = 0$. Hence $\pi_+\tilde{E}^{-1}f_1 = 0$ or $f_1 \in K_{\tilde{E}}$, and consequently $f \in \tilde{F}K_{\tilde{E}}$. Conversely if $f \in \tilde{F}K_{\tilde{E}}$ and $g \in EK_F$ then $f = \tilde{F}f_1$, $g = Eg$, with $f_1 \in K_{\tilde{E}}$ and $g_1 \in K_F$. Then

$$\langle g,f\rangle = [D^{-1}Eg_1,\tilde{F}f_1] = [g_1,f_1] = 0$$

It may be noted that dim V = deg det F, dim $V^{\perp}$ = deg det $\tilde{E}$ = deg det E and so dim V + dim $V^{\perp}$ = deg det E + deg det F = deg det D = dim K_D.

So far our considerations were purely module theoretic. Our next step is to relate these concepts of duality to the study of systems. Suppose we are given a strictly proper p×m transfer function G which we assume to have a representation of the form

$$G(\lambda) = N(\lambda)D(\lambda)^{-1}M(\lambda) + P(\lambda) \tag{2.22}$$

with N, M, D and P polynomial matrices of appropriate sizes. As in [3] we associate with this representation of G a realization (A,B,C) in the following way. We let K_D be our state space and define the operators A, B, C by

$$A = S_D, \tag{2.23}$$

$$B\xi = \pi_D M\xi \qquad \text{for } \xi \in F^m \tag{2.24}$$

and

$$(2.25)\qquad Cf = (ND^{-1}f)_{-1} \qquad \text{for } f \in K_D.$$

We call this the realization associated with the representation (2.22). That it is indeed a realization is easily checked, the proof being given in [3].

It is of interest to compute the adjoints of the maps A, B and C. For A the answer is given by Theorem 2.9.

Next we compute $B^*: K_{\tilde{D}} \to F^m$. Let $g \in K_{\tilde{D}}$ and $\xi \in F^m$. Then

$$\langle B\xi,g\rangle = [D^{-1}\pi_D M\xi,g] = [D^{-1}D\pi_- D^{-1}M\xi,g] = [\xi,\tilde{M}\tilde{D}^{-1}g] = \tilde{\xi}(\tilde{M}\tilde{D}^{-1}g)_{-1}.$$

Thus we proved

$$(2.26)\qquad B^*g = (\tilde{M}\tilde{D}^{-1}g)_{-1}.$$

Finally we note that with $\eta \in F^p$ and $f \in K_D$ we have

$$\tilde{\eta}Cf = \tilde{\eta}(ND^{-1}f)_{-1} = [ND^{-1}f,\eta] = [f,\tilde{D}^{-1}\tilde{N}\eta] = [f,\pi_-\tilde{D}^{-1}\tilde{N}\eta)$$

$$= [D^{-1}f,\tilde{D}\pi_-\tilde{D}^{-1}N\eta] = \langle f,\pi_D\tilde{N}\eta\rangle$$

or

$$(2.27)\qquad C^*\eta = \pi_{\tilde{D}}\tilde{N}\eta.$$

Combining these results can be summarized by the following.

<u>THEOREM 2.11</u>. *The adjoint of the realization of the transfer function* G *associated with the representation* $G = ND^{-1}M + P$ *is the realization of* $\tilde{G}$ *associated with the representation* $\tilde{G} = \tilde{M}\tilde{D}^{-1}\tilde{N} + \tilde{P}$.

In particular this implies that the two associated polynomial system matrices are related by transposition.

One can look also at duality from the input/output point of view. To this end let $f: F^m[\lambda] \to \lambda^{-1}F^p[[\lambda^{-1}]]$ be a restricted input/output map, that is an $F[\lambda]$-homomorphism. There exists a dual map $f^*: (\lambda^{-1}F^p[[\lambda^{-1}]])^* \to (F^m[\lambda])^*$. We already identified $(F^m[\lambda])^*$ with $\lambda^{-1}F^m[[\lambda^{-1}]]$. Now $(\lambda^{-1}F^p[[\lambda^{-1}]])^*$ is generally too big. However it contains a copy of $F^p[\lambda]$ as each space is embedded in its double dual. If we restrict f^* to $F^p[\lambda]$ we obtain a module homomorphism from $F^p[\lambda]$ into $\lambda^{-1}F^m[[\lambda^{-1}]]$ which we still denote by f^*. This way will be called the dual input/output map.

If we assume the input/output map to have G as transfer function then

(2.28) $$f(u) = \pi_- Gu \qquad \text{for } u \in F^m[\lambda].$$

Given any $v \in F^p[\lambda]$ and $g \in F^m[\lambda]$ we have $f^*(v) \in (F^m[\lambda])$ and computing

$$[f^*(v),g] = [v,f(g)] = [v,\pi_- Gg] = [\pi_+ v,Gg] = [v,Gg] = [\tilde{G}v,g]$$

$$= [\pi_- \tilde{G}v,g]$$

and to

(2.29) $$f^*(v) = \pi_- \tilde{G}v \qquad \text{for } v \in F^p[\lambda].$$

Hence the transfer function associated with f^* is just $\tilde{G}$.

To conclude this section we establish how Toeplitz operators, playing such a prominent role in the study of feedback [5], transform by duality.

Here we have two options. First given $A \in F^{p\times m}((\lambda^{-1}))$ we define the induced Toeplitz operator $T_A: F^m[\lambda] \to F^p[\lambda]$ by

(2.30) $$T_A f = \pi_+ Af \qquad \text{for } f \in F^m[\lambda].$$

The adjoint map $T_A: \lambda^{-1}F^p[[\lambda^{-1}]] \to \lambda^{-1}F^m[[\lambda^{-1}]]$ is given by

(2.31) $$T_A^* h = \pi_- \tilde{A}h$$

which operator we also denote by $T^{\tilde{A}}$. This is a direct consequence of the equality

$$[T_A f,h] = [\pi_+ Af,h] = [Af,h] = [f,\tilde{A}h] = [f,\pi_-\tilde{A}h].$$

The second approach is to study the Toeplitz map from K_D into K_{D_1}. We deal only with the case that $\Gamma = D_1 D^{-1}$ is a bicausal isomorphism. In that case we know that actually $T_{DD_1^{-1}}$ is an invertible map from K_{D_1} onto K_D [5, Theorem 4.3].

<u>THEOREM 2.12</u>. *The dual map* $T^*_{DD_1^{-1}}$ *of* $T_{DD_1^{-1}}$ *is the map from* $K_{\tilde{D}}$ *onto* $K_{\tilde{D}_1}$ *given by*

$$(2.32) \qquad T^*_{DD_1^{-1}} f = f \qquad \text{for all } f \in K_{\tilde{D}_1}.$$

<u>PROOF</u>. First we note that the map X: $K_{\tilde{D}_1} \to K_{\tilde{D}}$ given by Xf = f is well defined. This is a consequence of the part [6, Lemma 5.5] that if $T_1^{-1}T$ is a bicausal isomorphism then K_T and K_{T_1} contain the same elements (but differ in their module structure).

To prove (2.32) let g and f be arbitrary elements of $K_{\tilde{D}}$ and $K_{\tilde{D}_1}$ respectively. Then

$$\begin{aligned} \langle f,T^*_{DD_1^{-1}}g\rangle &= \langle T_{DD_1^{-1}}f,g\rangle = [D^{-1}\pi_+ DD_1^{-1}f,g] = [\pi_+ DD_1^{-1}f,\tilde{D}^{-1}g] \\ &= [DD_1^{-1}f,\tilde{D}^{-1}g] = [D_1^{-1}f,g] = \langle f,g\rangle, \end{aligned}$$

which proves the theorem.

This result indicates already that the study of the dual of the feedback groups and hence also the study of (C,A)-invariant subspaces may be substantially simpler than the study of feedback itself. This will be taken up in the next section.

3. THE OUTPUT INJECTION GROUP AND (C,A)-INVARIANT SUBSPACES

Suppose (A,B,C) is an observable realization of a p×m transfer function G, i.e. $G(\lambda) = C(\lambda I-A)^{-1}B$. Since C and $(\lambda I-A)$ are right coprime it follows that G can be written as $G(\lambda) = T(\lambda)^{-1}U(\lambda)$ and the realization associated with this representation in the state space K_T is isomorphic to the original system. We define the output injection geoup as the goup which acts on triples by $(A,B,C) \to (R^{-1}(A+HC)R,\ R^{-1}B,\ PCR)$ with P and R invertible. This is clearly the dual to the feedback group. Our main interest is to study the changes in the transfer function G by application of a group element.

The result that follows is a reformulation of a theorem of HAUTUS and HEYMANN [8, 5] in this context. Thus one approach to prove the theorem is to dualize the corresponding feedback result. Since however a direct proof for the output injection case is easier than that of the feedback case it is of interest to give an independent derivation with the option of getting the Hautus-Heymann theorem by duality considerations. This we proceed to do adapting the argument in [5]. First we note the following standard result in linear algebra.

LEMMA 3.1. *Let* V_0, V_1, V_2 *be finite dimensional linear spaces over a field* F *and let* $D: V_0 \to V_2$ *and* $C: V_0 \to V_1$ *be linear transformations. Then there exists a linear transformation* $H: V_1 \to V_2$ *such that*

$$D = HC \tag{3.1}$$

if and only if

$$\mathrm{Ker} D \supset \mathrm{Ker} C. \tag{3.2}$$

THEOREM 3.2. *Let* (A,B,C) *be an observable realization of the transfer function* $G(\lambda) = T(\lambda)^{-1}U(\lambda)$. *Then* $G_1(\lambda)$ *is the transfer function of a system* (A_1,B_1,C_1) *output injection equivalent to* (A,B,C) *if and only if* $G_1(\lambda) = T_1(\lambda)^{-1}U(\lambda)$ *and* $T_1(\lambda)^{-1}T(\lambda)$ *is a bicausal isomorphism.*

PROOF. Clearly similarity transformations do not change the transfer function and a change of basis tranformation in the output space changes the transfer function only by left multiplication by the invertible map. Thus we assume without loss of generality that $A_1 = A + HC$, $B_1 = B$ and $C_1 = C$. Then

$$C_1(\lambda I-A_1)^{-1} = C(\lambda C-HC)^{-1} = C[(I-HC(\lambda I-A)^{-1})(\lambda C-A)]^{-1}$$

$$= C(\lambda C-A)^{-1}(I-HC(\lambda C-A)^{-1})^{-1}$$

$$= (I-C(\lambda I-A)^{-1}H)^{-1}C(\lambda C-A)^{-1}$$

which in turn implies that

$$G_1(\lambda) = C_1(\lambda C-A_1)^{-1}B_1 = \Gamma(\lambda)^{-1}G(\lambda) = \Gamma(\lambda)^{-1}T(\lambda)^{-1}U(\lambda),$$

where $\Gamma(\lambda) = (I-C(\lambda C-A)^{-1}H)$ is a bicausal isomorphism. Moreover

$$T_1(\lambda) = T(\lambda)\Gamma(\lambda) = T(\lambda) + T(\lambda)C(\lambda C-A)^{-1}H = T(\lambda) + Q(\lambda)$$

where $Q(\lambda)$ is a polynomial matrix such that $T(\lambda)^{-1}Q(\lambda)$ is strictly proper.

Conversely assume $T_1(\lambda) = T(\lambda) + Q(\lambda)$ with $T^{-1}Q$ strictly proper. Then $\Gamma = T_1^{-1}T$ is a bicausal isomorphism with the constant term equal to the identity. By Lemma 5.5 in [6] K_T and K_{T_1} are equal as sets. Let (A,C) and (A_1,C_1) be the transformations arising out of the factorizations $T^{-1}U$ and $T_1^{-1}U$ as given by formula (2.23) and (2.25). As the constant term of $T_1^{-1}T$ is the identity it follows that for $f \in K_T = K_{T_1}$

$$Cf = (T^{-1}f)_{-1} = (T_1^{-1}TT^{-1}f)_{-1} = (T_1^{-1}f)_{-1} = C_1f$$

of $C = C_1$.

To complete the proof it suffices to show the existence of maps $X\colon K_{T_1} \to K_T$ and $H\colon F^p[\lambda] \to K_T$ such that

$$XA_1 - AX = HC. \tag{3.3}$$

We will prove (3.3) for the map X given by $Xf = f$. Thus, using Lemma 3.1

it suffices to show that $\mathrm{Ker}(A_1-A) \supset \mathrm{Ker}\, C$. To this end let $f \in \mathrm{Ker}C = \{f \in K_T \mid (T^{-1}f)_{-1} = 0\}$. Computing $S_T f$ we find

$$S_T f = \pi_T \lambda f = T\pi_- T^{-1}\lambda f = T\cdot T^{-1}\lambda f = \lambda f$$

as by our assumption $\lambda T^{-1} f$ is strictly proper. As the same is true for S_{T_1} it follows that $(S_T - S_{T_1})f = 0$ for every $f \in \mathrm{Ker}C$. This poses the theorem.

We pass onto the characterization of (C,A)-invariant subspaces in polynomial terms. A subspace V of the state space X is called (C,A)-invariant if there exists a linear transformation H such that $(A+HC)V \subset V$. It has been shown in [11] that V is (C,A)-invariant if and only if $A(V \cap \mathrm{Ker}C) \subset V$.

<u>THEOREM 3.3</u>. *Let* (A,B,C) *be the observable realization associated with the transfer function* $G(\lambda) = T(\lambda)^{-1}U(\lambda)$. *Then a subspace* $V \subset K_T$ *is a* (C,A)-*invariant subspace if and only if*

(3.4) $$V = E_1 K_{F_1}$$

where $T_1 = E_1 F_1$ *is such that* $T_1^{-1}T$ *is a bicausal isomorphism.*

We will give two proofs of the theorem.

<u>PROOF I</u>. V is (C,A)-invariant if and only if it is invariant for $A_1 = A+HC$. In the case of the pair (A,C) arising out of $G = T^{-1}U$ (A_1,C) will be associated, by Theorem 3.2, with $T_1^{-1}U$ where $T_1^{-1}T$ is a bicausal isomorphism. Thus, since K_T and K_{T_1} are equal as sets, V is an S_{T_1}-invariant subspace of K_{T_1}. Those are, by Theorem 2.9 of [6], of the form $V = E_1 K_{F_1}$ with $T_1 = E_1 F_1$.

<u>PROOF II</u>. In this proof we use duality and the results of [6]. The subspace V of K_T is (C,A)-invariant if and only if $V^\perp \subset K_{\tilde{T}}$ is (A^*,C^*)-invariant, i.e. an $(S_{\tilde{T}},\pi_{\tilde{T}})$-invariant subspace. By Theorem 4.2 of [6] there exists a $T_1 \in F^{p\times p}[\lambda]$ such that TT_1^{-1} is a bicausal isomorphism and

$$V^\perp = T_{\tilde{T}\tilde{T}_1^{-1}}(\tilde{F}_1 K_{\tilde{E}_1})$$

where $T_1 = E_1 F_1$ (hence also $\tilde{T}_1 = \tilde{F}_1 \tilde{E}_1$). By elementary properties of dual maps we have

$$T^{*}_{\widetilde{T}\widetilde{T}_1^{-1}} V = V_1 \subset K_{T_1}$$

and $V_1^{\perp} = \widetilde{F}_1 K_{\widetilde{E}_1}$. By Theorem 2.10 we have $V_1 = E_1 K_{F_1}$ and since

$$T^{*}_{\widetilde{T}\widetilde{T}_1^{-1}} : K_T \rightarrow K_{T_1}$$

acts as the identity map it follows that $V = E_1 K_{F_1}$.

<u>COROLLARY 3.4</u>. *If a* (C,A)*-invariant subspace of* K_T *of the form* $E_1 K_{F_1}$ *contains* $\mathcal{B}$ = Range B = $\{U_\xi \mid \xi \in F^m\}$ *then there exists a* $U_1 \in F^{p\times m}[\lambda]$ *such that* $U = E_1 U_1$.

<u>PROOF</u>. For each $\xi \in F^m$, $U_\xi \in E_1 K_{F_1}$ so $U_\xi = E_1 f_\xi$, from which the result follows.

<u>LEMMA 3.5</u>. *Let* $V \subset K_T$ *be a* (C,A)*-invariant subspace, having the representation* $V = E_1 K_{F_1}$ *of Theorem* 3.3. *Then* $f \in K_T$ *is in* V *if and only if* $f = E_1 g$ *for some* $g \in F^p[\lambda]$.

<u>PROOF</u>. If $f \in E_1 K_{F_1}$ then clearly $f = E_1 g$ for some $g \in K_{F_1} \subset F^p[\lambda]$. Suppose conversely that $f \in K_T$ and $f = E_1 g$. Since $f \in K_T$, and as K_T and K_{T_1} are equal, by Lemma 5.5 in [6], as sets we have $f \in K_{T_1}$. Hence $f = T_1 h = E_1 F_1 h$ for some $h \in \lambda^{-1} F^p[[\lambda^{-1}]]$. From $E_1 F_1 h = E_1 g$ and the nonsingularity of E_1 it follows that $g = F_1 h$ or $g \in K_{F_1}$ and the proof is complete.

Next we characterize the left factors $E_1 \in F^{p\times p}[\lambda]$ that can be right multiplied to yield a polynomial matrix $T_1 = E_1 F_1$ for which $T_1^{-1} T$ is a bicausal isomorphism. This is the dual result to Theorem 4.4 in [6].

<u>THEOREM 3.6</u>. *Let* $T, E_1 \in F^{p\times p}[\lambda]$ *be nonsingular. Then there exists* $F_1 \in F^{p\times p}[\lambda]$ *such that*

(i) $T_1 = E_1 F_1$

(ii) $T_1^{-1} T$ *is a bicausal isomorphism*

if and only if all the right Wiener-Hopf factorization indices at infinity of $E_1^{-1} T$ *are nonnegative.*

PROOF. The proof is as of Theorem 4.4 in [6] or follows from that theorem by duality.

THEOREM 3.7. *Let* $G(\lambda) = T(\lambda)^{-1}U(\lambda)$ *be a strictly proper* $p\times m$ *rational function of full row rank and assume the factorization is left coprime. Let* (A,B,C) *be the realization associated with this factorization in the state space* K_T. *Let* $E_\rho \in F^{p\times p}[\lambda]$ *be such that* $E_\rho F^p[\lambda] = UF^m[\lambda]$, *i.e.*

$$(3.5)\qquad U = E_\rho U_\rho$$

and U_ρ *is right unimodular (right invertible element of* $F^{p\times m}[\lambda]$*). Then* $V \subset K_T$ *is a* (C,A)*-invariant subspace that contains* $\mathcal{B}$ = Range B *if and only if*

$$(3.6)\qquad V = E_\sigma K_{F_\sigma}$$

where $T_\sigma = E_\sigma F_\sigma$, $T_\sigma^{-1}T$ *is a bicausal isomorphism and*

$$(3.7)\qquad E_\rho = E_\sigma \cdot H$$

for some $H \in F^{p\times p}[\lambda]$.

PROOF. If $V \subset K_T$ has the representation (3.6) with $T_\sigma = E_\sigma F_\sigma$, $T_\sigma^{-1}T$ a bicausal isomorphism and (3.7) holds, then V is (C,A)-invariant by Theorem 3.3. By Lemma 3.4 $V = \{f \in K_T \mid f = E_\rho g,\ g \in F^p[\lambda]\}$. Now $\mathcal{B} = \{U(\lambda)\xi \mid \xi \in F^m\} = \{E_\rho(\lambda)U_\rho(\lambda)\xi) \mid \xi \in F^m\} = \{E_\sigma(HU_\rho(\lambda)\xi) \mid \xi \in F^m\} \subset V$.

To prove the converse we show first that there exists $F_\rho \in F^{p\times p}[\lambda]$ such that $T_\rho = E_\rho F_\rho$ and $T_\rho^{-1}T$ is a bicausal isomorphism.

To this end we show that all the right Wiener-Hopf factorization indices at infinity of $T^{-1}E_\rho$ are nonpositive. $T^{-1}U$ and $T^{-1}E_\rho$ have the same right factorization indices at infinity. To see this let $\begin{pmatrix} U_\rho \\ U_\tau \end{pmatrix}$ be any completion of U_ρ to a unimodular matrix in $F^{m\times m}[\lambda]$ and let $T^{-1}E_\rho = \Omega\Delta W$ be a right Wiener-Hopf factorization. Thus Ω is a bicausal isomorphism, W unimodular and $\Delta(\lambda) = \mathrm{diag}(\lambda^{\alpha_1},\dots,\lambda^{\alpha_p})$. Now $T^{-1}U = T^{-1}E_\rho U_\rho = \Omega\Delta WU_\rho = \Omega(\Delta\ 0)\begin{pmatrix} WU_\rho \\ U_\tau \end{pmatrix}$. $T^{-1}U$, being strictly proper, all its right factorization indices α_i are nonpositive [7]. The existence of F_ρ

follows from Theorem 3.6.

We proceed to show that the inclusion relation

(3.8) $$E_\rho K_{F_\rho} \supset E_\rho K_{U_\rho}$$

holds. In fact, since $T_\rho = E_\rho F_\rho = T\Gamma$ where Γ is a bicausal isomorphism, it follows that $T_\rho^{-1}U = \Gamma^{-1}T^{-1}U = \Gamma^{-1}F_\rho^{-1}E_\rho E_\rho U_\rho = \Gamma^{-1}F_\rho^{-1}U_\rho$ or $F_\rho^{-1}U_\rho$ is strictly proper. This implies

(3.9) $$K_{F_\rho} \supset K_{U_\rho}$$

and hence (3.8) follows too. We already saw at the beginning of the proof that $E_\rho K_{F_\rho} \supset \mathcal{B}$.

Let now $V \subset K_T$ be (C,A)-invariant and assume $V \supset \mathcal{B}$. By Theorem 3.3 $V = E_\alpha K_{F_\alpha}$. Now $F^p[\lambda] \supset K_{F_\alpha} \supset E_\alpha^{-1}\mathcal{B} = \{E_\alpha^{-1}U\xi \mid \xi \in F^m\}$.

It follows that $F^p[\lambda] \supset E_\alpha^{-1}E_\rho F^p[\lambda]$ and so $H = E_\alpha^{-1}E_\rho \in F^{p\times p}[\lambda]$ or (3.7) follows.

We point out that another proof of this theorem can be obtained from Theorem 5.3 in [6] by duality considerations. The details are simple and omitted.

COROLLARY 3.8. *Under the assumptions of Theorem* 3.6 *the minimal* (C,A)-*invariant subspace containing* $\mathcal{B}$, *denoted by* $\mathcal{V}_*(\mathcal{B})$, *is given by*

(3.10) $$\mathcal{V}_*(\mathcal{B}) = E_\rho K_{F_\rho} .$$

4. ON THE MAXIMAL REACHABILITY SUBSPACE IN Ker C

Let G be a p×m strictly proper transfer function and let

(4.1) $$G(\lambda) = T(\lambda)^{-1}U(\lambda)$$

be a left coprime factorization of G. With this factorization is associated a state space realization in K_T as described in Section 2.

It has been shown in [6] that relative to this realization of G, every (A,B)-invariant subspace V of K_T which is included in Ker C is of the form

(4.2) $$V = U_0 K_{E_0}$$

where

(4.3) $$U = U_0 E_0$$

is a factorization of U with E_0 nonsingular, and every such subspace has such a representation. On the other hand it was also shown in [6] that subspaces of the form

(4.4) $$V = E_1 K_{U_1}$$

where

(4.5) $$U = E_1 U_1$$

is a factorization of U, with $E_1 \in F^{p\times p}[\lambda]$ nonsingular, is also an (A,B)-invariant subspace contained in Ker C, but not all such subspaces have a representation of the second kind. One naturally looks for an intrinsic characterization of the second class of subspaces and it may not come as a surprise that the problem has to do with reachability subspaces.

For the analysis that follows we will assume that the transfer function G, as a matrix over the field of rational functions, has full row rank. Thus in a left coprime factorization (2.1) the numerator matrix $U \in F^{p\times m}[\lambda]$ has full row rank over $F[\lambda]$. This assumption is not really necessary and with some obvious modifications the theorems and proofs can be adapted to the general case. Thus, since the factors in a left coprime factorization are determined only up to a common left unimodular factor, this factor can be chosen so that U is of the form

$$U(\ ) = \begin{pmatrix} U'(\lambda) \\ 0 \end{pmatrix}$$

with U' of full row rank. The main results characterizing $\mathcal{R}^*(\text{Ker } C)$ the maximal reachability subspace in Ker C, closely resembles the work of

KHARGONEKAR & EMRE [9] but the final form seems to be more satisfactory

As in the previous section we let

$$U = E_\rho U_\rho \tag{4.6}$$

with U_ρ right unimodular. This is possible by Theorem 3.7 in [6].

THEOREM 4.1. *Let* $G = T^{-1}U$ *be strictly proper, the factorization left coprime and* U *assumed of full row rank with* (4.6) *holding and* U_ρ *right unimodular. Then we have*

$$\mathcal{R}^*(\text{Ker } C) = E_\rho K_{U_\rho} . \tag{4.7}$$

PROOF. Let $R = E_\rho K_{U_\rho}$. Then we know from Theorem 5.6 in [6] that R is an (A,B)-invariant subspace included in Ker C. Next we show that $K_U \cap \mathcal{B} \subset R$. In fact if $f \in K_U \cap \mathcal{B}$ and taking into account that $\mathcal{B} = \{U\xi \mid \xi \in F^m\}$ and that $K_U = \{f \in P^p[\lambda] \mid f = Uh,\ h \in \lambda^{-1}F^m[[\lambda]]\}$, it follows that $f = Uh = U\xi$. So $E_\rho U_\rho h = E_\rho U_\rho \xi$ and as E_ρ is nonsingular $U_\rho h = U_\rho \xi$ or $U_\rho h \in K_{U_\rho}$. So $f = E_\rho U_\rho h \in E_\rho K_{U_\rho} = R$. This implies that $\mathcal{R}^*(\text{Ker } C) \subset R$.

To prove the converse it suffices to show that R is a reachability subspace. Since $R = E_\rho K_{U_\rho}$ every element of R has a representation, not necessarily unique, of the form $f(\lambda) = U(\lambda)g(\lambda)$ with $g(\lambda) = \gamma_0 + \gamma_1\lambda + \ldots + \gamma_s\lambda^s \in F^p[\lambda]$. Let $L = \{\xi \in F^m \mid \exists h \in \lambda^{-1}F^m[[\lambda^{-1}]],\ U\xi = Uh\}$. We prove first two lemmas.

LEMMA 4.2. *If* $f(\lambda) = U(\lambda)(\gamma_0+\ldots+\gamma_s\lambda^s) \in K_U$ *then* $\gamma_s \in L$.

PROOF. If $f \in K_U$ then $f = Uh$ for some $h \in \lambda^{-1}F^m[[\lambda^{-1}]]$. Let

$$h(\lambda) = \frac{h_{-1}}{\lambda} + \frac{h_{-2}}{\lambda^2} + \ldots$$

then

$$U(\lambda)\left(\gamma_s\lambda^s + \ldots + \gamma_0 - \frac{h_{-1}}{\lambda} - \ldots\right) = 0.$$

Therefore

$$U(\lambda)\gamma_s = U(\lambda)\left(-\frac{\gamma_{s-1}}{\lambda} - \ldots - \frac{\gamma_0}{\lambda^s} + \frac{h_{-1}}{\lambda^{s+1}} + \ldots\right)$$

and $\gamma_s \in L$.

LEMMA 4.3. *Let* $K: K_T \to F^m$ *be such that* $(S_T + BK)K_U \subset K_U$. *Then given* $\gamma \in L$

$$(S_T + BK)^s U_\gamma = U(\gamma_0 + \ldots + \gamma_s \lambda^s)$$

with $\gamma_s = \gamma$.

PROOF. By induction. For $s = 1$ since $U\gamma \in K_U$ we have $S_T U\gamma = \lambda U\gamma = U(\lambda\gamma)$. Also $BKU\gamma = U\gamma_0$ so $(S_T + BK)U\gamma = U(\gamma_0 + \lambda\gamma_1)$ with $\gamma_1 = \gamma$. Assume the result holds for $s - 1$. Then

$$(S_T + BK)^s U\gamma = (S_T + BK)U(\gamma_0' + \ldots + \gamma_{s-1}'\lambda^{s-1})$$

with $\gamma_{s-1}' = \gamma$. Again $U(\gamma_0' + \ldots + \gamma_{s-1}'\lambda^{s-1}) \in K_U$ and so $S_T U(\gamma_0' + \ldots + \gamma_{s-1}'\lambda^{s-1}) = = \lambda U(\gamma_0' + \ldots + \gamma_{s-1}'\lambda^{s-1}) = U(\gamma_0'\lambda + \ldots + \gamma_{s-1}'\lambda^s)$ whereas $BKU(\gamma_0' + \ldots + \gamma_{s-1}'\lambda^{s-1}) = = U\gamma_0$ for some $\gamma_0 \in F^m$. This proves the lemma.

We complete the proof of Theorem 4.1 by induction. Choose $K: K_T \to F^m$ so that $(S_T + BK)(K_U) \subset K_U$. We will show that if $f \in R$ then $f = \sum_j (S_T + BK)^j B\beta_j$ with $\beta_j \in L$.

If $f = U(\lambda)\xi \in R$ then, since $R \subset K_U$, $\xi \in L$ and we are done. Suppose we proved every $f \in R$ of the form $f(\lambda) = U(\lambda)(\gamma_0 + \ldots + \gamma_{s-1}\lambda^{s-1})$ has such a representation. Let $f(\lambda) = U(\lambda)(\gamma_0 + \ldots + \gamma_s\lambda^s) \in R$. By Lemma 4.2 $\gamma_s \in L$ and by Lemma 4.3 $(S_T + BK)^s U\gamma_s = U(\lambda)(\beta_0 + \ldots + \beta_{s-1}\lambda^{s-1} + \gamma_s\lambda^s)$. Hence $f - (S_T + BK)^s U\gamma_s = U(\lambda)(\gamma_0' + \ldots + \gamma_{s-1}'\lambda^{s-1})$ and we are done by the induction hypothesis.

Given an (A,B)-invariant subspace $V \subset K_T$ we let

$$\underline{F}(V) = \{K: K_T \to F^m \mid (A+BK)V \subset V\}.$$

The following theorem will turn out to be a generalization of Corollary 5.1 in [1].

THEOREM 4.4. *Let* $K: K_T \to F^m$ *be such that* $K \in \underline{F}(K_U)$. *Then* $K \in \underline{F}(E_\alpha K_{U_\alpha})$ *for every factorization*

(4.8) $$U = E_\alpha U_\alpha$$

with E_α *nonsingular.*

PROOF. Given $f \in K_U$ we have $f = Uh$ for some $h \in \lambda^{-1}F^m[[\lambda^{-1}]]$. Thus $T^{-1}f = T^{-1}Uh$ is the product of two strictly proper functions, hence $\lambda T^{-1}f = T^{-1}(\lambda f)$ is also proper. This implies that for $f \in K_U$

(4.9) $$(S_T f)(\lambda) = \pi_T \lambda f = \lambda f(\lambda).$$

Therefore for $f \in K_U$ we have

$$(S_T + BK)f = \lambda f(\lambda) + U(\lambda)\xi_f$$

where $\xi_f = Kf \in F^m$ and depends linearly on f. If we assume the factorization (4.8) and that $f \in E_\alpha K_{U_\alpha}$ then $f = E_\alpha g$ with $g \in K_{U_\alpha}$ and

$$(S_T + BK)f = \lambda f(\lambda) + U(\lambda)\xi_f = E_\alpha(\lambda)\lambda g(\lambda) + E_\alpha(\lambda)U_\alpha(\lambda)\xi_f =$$

$$= E_\alpha(\lambda)\{\lambda g(\lambda) + U_\alpha(\lambda)\xi_f\}.$$

By Lemma 3.4 $(S_T + BK)f \in E_\alpha K_{U_\alpha}$ or $K \in \underline{F}(E_\alpha K_{U_\alpha})$.

A special case is the following.

COROLLARY 4.5. $K \in \underline{F}(V^*(\text{Ker } C))$ *implies* $K \in \underline{F}(R^*(\text{Ker } C))$.

Given $K \in \underline{F}(K_U)$ then K_U has a naturally induced $F[\lambda]$-module structure namely the one induced by the operator $S_T + BK$ and $E_\rho K_{U_\rho} \simeq R^*(\text{Ker } C)$ is a submodule. The next theorem identifies the quotient module structure.

THEOREM 4.6. *We have the* $F[\lambda]$*-module isomorphism*

(4.10) $$K_U/E_\rho K_{U_\rho} \simeq K_{E_\rho}.$$

PROOF. Choose $K \in \underline{F}(K_U)$ which implies that $K \in \underline{F}(E_\rho K_{U_\rho})$ and $E_\rho K_{U_\rho}$ is a submodule of K_U. Define a map $R: K_U \to K_{E_\rho}$ by

$$(4.11) \qquad Rf = \pi_{E_\rho} f \quad \text{for} \quad f \in K_U.$$

We will show that R is a module homomorphism of K_U onto K_{E_ρ} with $\text{Ker } R = E_\rho K_{U_\rho}$.

Indeed for $f \in K_U$ we have

$$R(S_T + BK)f = R(\lambda f + U\xi_f) = \pi_{E_\rho}(\lambda f + U\xi_f) = \pi_{E_\rho}\lambda f =$$

$$= \pi_{E_\rho}\lambda\pi_{E_\rho} f = S_{E_\rho} Rf$$

or

$$(4.12) \qquad R(S_T + BK) = S_{E_\rho} R$$

which shows that R is a module homomorphism. To show that R is surjective we note that $K_U + E_\rho F^p[\lambda] = K_U + UF^m[\lambda]$.

Now U is assumed to be of full row rank, hence there exists a rational Ω such that $U\Omega = I$. Given $g \in F^p[\lambda]$ we have $g = U\Omega g = Ug_+ + U_{g-}$ with $g_+ = \pi_+\Omega g$ and $g_- = \pi_-\Omega g$. It follows that $U_{g-} = g - Ug_+ \in K_U$ and $Ug_+ \in UF^m[\lambda]$. This implies $K_U + UF^m[\lambda] = F^p[\lambda]$ or

$$(4.13) \qquad K_U + E_\rho F^p[\lambda] = F^p[\lambda].$$

Since $\pi_{E_\rho} F^p[\lambda] = K_{E_\rho}$ the map R is clearly surjective. Finally $f \in \text{Ker } R$ if and only if $f = E_\rho f'$ for some $f' \in F^p[\lambda]$. By Lemma 3.4, this implies the equality $\text{Ker } R = E_\rho K_{U_\rho}$. This completes the proof.

The proof of the surjectivity of the map R is adapted from [9].

The previous theorem gives a very clear representation of the transmission zeroes of $T^{-1}U$. Thus the transmission zeroes are the zeroes of $\det E_\rho$ and for every $K \in \underline{F}(K_U)$, the map $\overline{S_T + BK}$ in $K_U/E_\rho K_{U_\rho}$ induced by $S_T + BK$ is isomorphic to S_{E_ρ} and hence the invariant factors of $\overline{S_T + BK}$ coincide with the invariant factors of E_ρ.

COROLLARY 4.7. *A subspace* $V \subset K_T$ *is an* (A,B)*-invariant subspace contained in* Ker C *and containing* $\mathcal{R}^*(\text{Ker } C) = E_\rho K_{U_\rho}$ *if and only if*

$$V = E_\alpha K_{U_\alpha} \tag{4.14}$$

with

$$U = E_\alpha U_\alpha \tag{4.15}$$

and E_α nonsingular, *and for some* H

$$E_\rho = E_\alpha H. \tag{4.16}$$

PROOF. Assume V is of the form (4.14) with (4.15) and (4.16) satisfied. Then

$$\mathcal{R}^*(\text{Ker } C) = E_\rho K_{U_\rho} = E_\alpha H K_{U_\rho} \subset E_\alpha K_{U_\alpha} = V$$

where $U_\alpha = HU_\rho$.

To prove the converse let V be (A,B)-invariant contained in Ker C and containing $\mathcal{R}^*(\text{Ker } C)$. Since $V \subset K_U = \mathcal{V}^*(\text{Ker } C)$, V and K_U are compatible [6,12] and hence there exists $K \in \underline{F}(V) \cap \underline{F}(K_U)$. By Theorem 4.4 $K \in \underline{F}(E_\rho K_{U_\rho})$. Thus we have the module inclusions $K_U \supset V \supset E_\rho K_{U_\rho}$. Let $R: K_U \to K_{E_\rho}$ be defined by (4.11). $R(V) = \pi_{E_\rho}(V)$ is a submodule of K_{E_ρ} and hence of the form $\pi_{E_\rho}(V) = E_\alpha K_H$ with $E_\rho = E_\alpha H$. Now $f \in K_U$ and $\pi_{E_\rho} f \in E_\alpha K_H$ if and only if $f = E_\alpha g + E_\rho p$ with $g \in K_H$ and $p \in F^p[\lambda]$. Thus $f = E_\alpha(g + H_p)$ and by Lemma 3.4 $f \in E_\alpha K_{U_\alpha}$. Conversely if $f \in E_\alpha K_{U_\alpha}$ then $f = E_\alpha g$ and

$$\pi_{E_\rho} f = E_\alpha H \pi_{-} H^{-1} E_\alpha^{-1} E_\alpha g = E_\alpha \pi_H g = E_\alpha g' \in E_\alpha K_H.$$

This implies $V = E_\alpha K_{U_\alpha}$ and the theorem is proved.

The following result has previously been obtained by EMRE & HAUTUS in [1].

COROLLARY 4.8. *If* $K \in \underline{F}(V^*(\text{Ker } C))$ *then* $K \in \underline{F}(V)$ *for every* V *that is* (A,B)*-invariant, is contained in* Ker C *and contains* $\mathcal{R}^*(\text{Ker } C)$.

PROOF. Follows from Corollaries 4.4 and 4.7.

We denote by $\mathcal{V}_*(\mathcal{B})$ the minimal (C,A)-invariant subspace that contains $\mathcal{B}$.

COROLLARY 4.9. *The following inclusion holds*

$$\mathcal{R}^*(\text{Ker } C) \subset \mathcal{V}_*(\mathcal{B}). \tag{4.17}$$

PROOF. Relation (3.8) obtained in the proof of Theorem 3.6 is equivalent to (4.17) where we use the identifications of $\mathcal{R}^*(\text{Ker } C)$ and $\mathcal{V}_*(\mathcal{B})$given by Theorem 4.2 and Corollary 3.7 respectively.

The inclusion also follows from a result of MORSE [10].

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