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DUALITY AND THE NONLINEAR ROUND-TRIP ℓ-CENTER AND COVERING PROBLEMS ON A TREE

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Duality and the nonlinear round-trip ℓ-center and covering problems on a tree*)

by

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ABSTRACT

The round-trip ℓ-center problem is to locate ℓ new facilities with respect to a finite number of pairs of existing facilities on a tree so as to minimize the maximum round-trip cost. The round-trip cost of serving a pair of existing facilities is a nonlinear function of the round-trip distance from the nearest new facility. The round-trip covering problem is to find the minimum number of new facilities such that each round-trip cost is no more than a given constant. For both problems a dual problem is defined and a strong duality result is proved. We give algorithms to solve both round-trip location problems in polynomial time.

KEY WORDS & PHRASES: *location theory, round-trip, ℓ-center problem, perfect graphs*

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*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

Consider the problem of a transportation service which has to transport goods from point p_i to point q_i; let us call this job i (i = 1,...,m). Given a network of roads, the distance a vehicle located at a depot at point x has to travel to execute job i and to return to its depot is given by the round-trip distance $d(x,p_i) + d(p_i,q_i) + d(q_i,x)$, where distances are measured on the network. We assume that the cost associated with executing job i is an increasing function f_i of the traveled distance. Due to budget constraints the transportation service can only build ℓ vehicle depots to execute the jobs. The question is where to locate these ℓ vehicle depots so as to minimize the maximum round-trip cost. The solution of problems of this type is the subject of this paper.

We shall assume that the underlying network contains no cycles, i.e., has a tree structure. Let $T = (V,E)$ be a tree with vertex set $V = \{v_1,v_2,...,v_n\}$ and edge set $E = \{e_1,e_2,...,e_{n-1}\}$. Each edge $e \in E$ has a nonnegative length $\ell(e)$. A point x on T can be a vertex or a point anywhere along an edge. The length of the shortest path from x to y on T is denoted by $d(x,y)$. Let $P_i = \{p_i,q_i\}$ (i = 1,...,m) be pairs of vertices. Let $X = \{x_1,x_2,...,x_\ell\}$ be a set of points on T. We define the distance $D(X,P_i)$ between the set X and the pair P_i by

$$D(X,P_i) = \min_{x \in X} \{d(p_i,x) + d(q_i,x)\}.$$

We say that P_i is covered within distance r whenever $D(X,P_i) \le r$. Let $f_i: \mathbb{R} \to \mathbb{R}$ be a strictly increasing continuous function (i = 1,...,m). The *round-trip ℓ-center problem* is given by

$$\min_{X:|X|=\ell} \max_{1\le i\le m} \{f_i(D(X,P_i))\}.$$

The closely related *round-trip covering problem* is given by

$$\min |X|$$

$$\text{s.t. } f_i(D(X,P_i)) \le r, \qquad i = 1,2,...,m,$$

where r is a given constant. If we define $r_i = f_i^{-1}(r)$ (assuming this is well defined (see Section 4)), this problem is equivalent to

$$\min |X|$$
$$\text{s.t. } D(X,P_i) \le r_i, \qquad i = 1,2,\ldots,m,$$

which is the form in which we will study the problem in Section 2.

The round-trip 1-center problem on a tree with linear functions f_i was solved by CHAN & FRANCIS [3]. The round-trip 1-center problem in the plane with rectilinear distances was solved by CHAN & HEARN [4]. Our work was motivated by the paper of TANSEL, FRANCIS & LOWE [8] on the nonlinear ℓ-center problem.

In Section 2 we present an algorithm to solve the round-trip covering problem in O(nm) time.

In Section 3 we show the existence of a dual problem to the round-trip covering problem and give both a constructive and theoretical proof of a strong duality result for these problems.

In Section 4 we give an algorithm to solve the round-trip ℓ-center problem, which uses the algorithm derived in Section 2 as a subroutine; again we show the existence of a dual problem and prove a strong duality result.

In Section 5 we give an interpretation of the duality results in the case that $f_i(D(X,P_i))$ is equal to the smallest distance from a point $x \in X$ to the shortest path from p_i to q_i, i.e., $f_i(D(X,P_i)) = \frac{1}{2}[D(X,P_i)-d(p_i,q_i)]$.

2. THE ROUND-TRIP COVERING PROBLEM

Given m pairs of vertices $\{p_i,q_i\}$ and positive numbers r_i satisfying $d(p_i,q_i) \le r_i$ $(i = 1,2,\ldots,m)$, the round-trip covering problem is to find the minimum number of new facilities on the tree such that each pair $\{p_i,q_i\}$ is covered within distance r_i, i.e.,

$$\min |X|$$
$$\text{s.t. } D(X,P_i) \le r_i, \qquad i = 1,2,\ldots,m.$$

During each interation of the algorithm the original tree is partitioned into two subgraphs: one green, the other brown. The gree subgraph is always a tree, denoted by GT, while the brown subgraph consists of one or more subtrees of the original tree T, each of which is "rooted" at a vertex of the green tree. By convention, a root t will be in both GT and the associated brown subtree, denoted by BT(t). Given a brown subtree BT(t) we have the following sets and values assigned to the root t:

- I_t^1 denotes the set of all indices i such that either p_i or q_i belongs to BT(t), and such that $\{p_i,q_i\}$ is not covered within distance r_i by any new facility located at BT(t).
- I_t^2 denotes the set of all indices i such that both p_i and q_i belong to BT(t), and such that $\{p_i,q_i\}$ is not covered within distance r_i by any new facility located at BT(t).
- $a_i(t) = \frac{1}{2}[r_i-d(p_i,t)-d(q_i,t)]$ for every $i \in I_t^2$.
- $a^0(t) = \min_{i\in I_t^2} \{a_t(t)\}$.
- b(t) denotes the distance from t to the nearest new facility located at BT(t).

The following observation clarifies the meaning of $a_i(t)$, which will be non-negative. Since $\{p_i,q_i\}$ $(i\in I_t^2)$ is not covered within distance r_i by any new facility located at BT(t), it must be covered by a new facility x not at BT(t), i.e., $d(p_i,x)+d(q_i,x) \le r_i$. Since $p_i,q_i \in BT(t)$, we have $d(p_i,x)+d(q_i,x) = d(p_i,t)+d(q_i,t)+2d(t,x)$. It follows that x covers $\{p_i,q_i\}$ within distance r_i if and only if $d(t,x) \le a_i(t)$.

Our algorithm is as follows.

Algorithm:

Step 0: For all vertices v set $BT(v) = \{v\}$, $I_v^2 = \emptyset$, $I_v^1 = \{i \mid 1 \le i \le m,\ v \in \{p_i,q_i\}\}$, $a_i(v) = \infty$; $b(v) = \infty$, and set $k = 0$, $I = \{1,2,\ldots,m\}$, GT = T.

Step 1: If $|I| = 0$, then stop. Set $k = k+1$. Choose a tip vertex t of GT. If GT = {t}, then define $x_k = t$ and stop. Let c(t) by the vertex of GT adjacent to t. If $a^0(t) < d(t,c(t))$, then go to Step 2, else go to Step 3.

Step 2: Define x_k to be the point at distance $a^0(t)$ from t on the edge (t,c(t)). Delete (t,c(t)) from GT. Add BT(t) and (t,c(t)) to BT(c(t)) to form a new BT(c(t)). Derive the sets and values assigned to c(t). Delete from I all indices i such that $\{p_i,q_i\}$ intersects BT(c(t)) and is covered within distance r_i by a new facility located at BT(c(t)). Go to Step 1.

Step 3: Delete (t,c(t)) from GT. Add BT(t) and (t,c(t)) to BT(c(t)) to form a new BT(c(t)). Derive the sets and values assigned to c(t). Delete from I all indices i such that $\{p_i,q_i\}$ intersects BT(c(t)) and is covered within distance r_i by a new facility located at BT(c(t)). Go to Step 1.

We define as an iteration of the algorithm any sequence of steps beginning with Step 1, through the last step before returning to Step 1. The algorithm takes at most (n-1) iterations since at each iteration an edge is deleted from GT. Let us see how we can derive the sets and values assigned to c(t) during an iteration and how we can determine an index $i \in I$ such that $\{p_i,q_i\}$ intersects BT(c(t)) and is covered within distance r_i by a new facility located at BT(c(t)).

Case A. First assume Step 2 is executed.

Case A1. For the pairs $\{p_i,q_i\}$ $(i \in I)$ in BT(c(t)) there are three possibilities.

Case A1.1. Both points are in BT(t), i.e., $i \in I_t^2$.

Since
$$\begin{aligned} d(p_i,x_k)+d(q_i,x_k) &= d(p_i,t)+d(q_i,t)+2d(t,x_k) \\ &= d(p_i,t)+d(q_i,t)+2a^0(t) \\ &\le d(p_i,t)+d(q_i,t)+2a_i(t) = r_i, \end{aligned}$$
$\{p_i,q_i\}$ is covered within distance r_i by x_k, and i can be deleted from I.

Case A1.2. Both points are in the old BT(c(t)), i.e., $i \in I_{c(t)}^2$.

$\{p_i,q_i\}$ is not covered by any new facility at the old BT(c(t)), but is covered by a new facility at BT(t) if and only if it is covered by x_k.

Since
$$\begin{aligned} d(p_i,x_k)+d(q_i,x_k) &= d(p_i,c(t))+d(q_i,c(t))+2d(c(t),x_k) \\ &= d(p_i,c(t))+d(q_i,c(t))+2(d(t,c(t))-a^0(t)) \\ &= r_i+2(d(t,c(t))-a^0(t)-a_i(c(t))), \end{aligned}$$
we have $d(p_i,x_k)+d(q_i,x_k) \le r_i$ if and only if $d(t,c(t))-a^0(t) \le a_i(c(t))$. If this condition holds, then i can be deleted from I; otherwise, i belongs to $I_{c(t)}^2$.

Case A1.3. One point is in BT(t), the other point in the old BT(c(t)), i.e., $i \in I_t^1 \cap I_{c(t)}^1$.

Since $d(p_i,x_k)+d(q_i,x_k) = d(p_i,q_i) \le r_i$, the pair $\{p_i,q_i\}$ is covered within distance r_i by x_k, and i can be deleted from I.

The values of $a_i(c(t))$ for those i which are not deleted remain the same during the iteration. It is easily seen that $b(c(t)) := \min(b(c(t)), d(t,c(t))-a^0(t))$.

<u>Case A2.</u> If exactly one point of the pair $\{p_i,q_i\}$ ($i\in I$) is in BT(c(t)) there are two possibilities.

<u>Case A2.1</u> The point is in BT(t), i.e., $i \in I_t^1 \setminus I_{c(t)}^1$.

Since $d(p_i,x_k)+d(q_i,x_k) = d(p_i,q_i) \le r_i$, i can be deleted from I.

<u>Case A2.2</u> The point is in the old BT(c(t)), i.e., $i \in I_{c(t)}^1 \setminus I_t^1$.

$\{p_i,q_i\}$ is not covered by any new facility located at the old BT(c(t)), but it is covered by a new facility located at BT(t) if and only if it is covered by x_k.

Since
$$\begin{aligned} d(p_i,x_k)+d(q_i,x_k) &= d(p_i,c(t))+d(q_i,c(t))+2d(c(t),x_k) \\ &= d(p_i,q_i)+2(d(t,c(t))-a^0(t)), \end{aligned}$$
we have $d(p_i,x_k)+d(q_i,x_k) \le r_i$ if and only if $d(t,c(t))-a^0(t) \le \le \tfrac{1}{2}[r_i-d(p_i,q_i)]$. If this condition holds, then i can be deleted from I; otherwise, i belongs to $I_{c(t)}^1$.

<u>Case B.</u> Now assume Step 3 is executed.

<u>Case B1.</u> For the pairs $\{p_i,q_i\}$ ($i\in I$) in BT(c(t)) there are three possibilities.

<u>Case B1.1</u> Both points are in BT(t), i.e., $i \in I_t^2$.

$\{p_i,q_i\}$ is not covered by any new facility located at BT(t), but it is covered by a new facility in the old BT(c(t)) if and only if it is covered by the new facility x in BT(c(t)) located closest to c(t), i.e., for which $d(x,c(t)) = b(c(t))$.

Since
$$\begin{aligned} d(p_i,x)+d(q_i,x) &= d(p_i,t)+d(q_i,t)+2(d(t,c(t))+d(c(t),x)) \\ &= r_i+2(d(t,c(t))+b(c(t))-a_i(t)), \end{aligned}$$
we have $d(p_i,x)+d(q_i,x) \le r_i$ if and only if $d(t,c(t))+b(c(t)) \le \le a_i(t)$. If this condition holds, then i can be deleted from I; otherwise, i belongs to $I_{c(t)}^2$ and $a_i(c(t)) := a_i(t)-d(t,c(t))$.

<u>Case B1.2</u> Both points are in the old BT(c(t)), i.e., $i \in I_{c(t)}^2$.

Analogous to Case B1.1. If $d(t,c(t))+b(t) \le a_i(c(t))$, then i can be deleted from I; otherwise, i belongs to $I_{c(t)}^2$ and $a_i(c(t))$ remains the same.

<u>Case B1.3</u> One point is in BT(t), the other point in the old BT(c(t)), i.e., $i \in I_t^1 \cap I_{c(t)}^1$.

$\{p_i,q_i\}$ is not covered by any new facility located at BT(c(t)). Hence i belongs to $I_{c(t)}^2$ and $a_i(c(t)) = \tfrac{1}{2}[r_i-d(p_i,c(t))-d(q_i,c(t))] = = \tfrac{1}{2}[r_i-d(p_i,q_i)]$.

It is easily seen that $b(c(t)) := \min(b(c(t)), b(t)+d(t,c(t)))$.

Case B2. If exactly one point of $\{p_i,q_i\}$ ($i \in I$) belongs to $BT(c(t))$ we have two possibilities.

Case B2.1. The point is in $BT(t)$, i.e., $i \in I_t^1 \setminus I_{c(t)}^1$.

$\{p_i,q_i\}$ is covered by a new facility in the old $BT(c(t))$ if and only if it is covered by the new facility x located closest to $c(t)$, i.e., $d(c(t),x) = b(c(t))$.

Since
$$\begin{aligned} d(p_i,x)+d(q_i,x) &= d(p_i,c(t))+d(q_i,c(t))+2d(c(t),x) \\ &= d(p_i,q_i)+2b(c(t)), \end{aligned}$$
we have $d(p_i,x)+d(q_i,x) \le r_i$ if and only if $b(c(t)) \le$ $\le \frac{1}{2}[r_i-d(p_i,q_i)]$.

If this condition holds, then i can be deleted from I; otherwise, i belongs to $I_{c(t)}^1$.

Case B2.2. The point is in the old $BT(c(t))$, i.e., $i \in I_{c(t)}^1 \setminus I_t^1$.

Analogous to Case B2.1. If $b(t)+d(t,c(t)) \le \frac{1}{2}[r_i-d(p_i,q_i)]$, then i can be deleted from I; otherwise, i belongs to $I_{c(t)}^1$.

We leave it to the reader to verify that each iteration requires $O(m)$ calculations. Therefore the complexity of the algorithm is $O(mn)$. We now proceed to prove the correctness of the algorithm.

LEMMA 2.1. *The algorithm constructs a feasible solution* X *to the round-trip covering problem with* $|X| \le m$.

PROOF. An index i is deleted from $\{1,2,\ldots,m\}$ during an iteration whenever $\{p_i,q_i\}$ is covered by a new facility. The algorithm stops when all indices have been deleted or when the green tree consists of one single point t. In the former case, all pairs are covered. In the latter case, the set I of facilities which have not been deleted is equal to I_t^2. Since $a_i(t) =$ $= \frac{1}{2}[r_i-d(p_i,t)-d(q_i,t)]$ and $a_i(t) \ge 0$ for all $i \in I_t^2$ we have $d(p_i,t)+d(q_i,t) \le$ $\le r_i$ for all $i \in I_t^2$. Hence all pairs $\{p_i,q_i\}$ ($i \in I_t^2$) are covered by the new facility located at t.

Let the algorithm construct a feasible solution $x = \{x_1,x_2,\ldots,x_\ell\}$. We define t_i to be the tip of GT chosen by the algorithm in the iteration during

which x_i was located. Note that after this iteration all pairs $\{p_k, q_k\}$ which have at least one point in $BT(t_i)$ are covered by a new facility located at $BT(t_i)$, and k has been deleted from I. If a new facility i is located at Step 2 of the algorithm, i.e., x_i is the point at distance $a^0(t_i) = \min_{j \in I^2_{t_i}} \{a_j(t_i)\}$ from t_i on the edge $(t_i, c(t_i))$, then we may assume without loss of generality that $a_i(t_i) = a^0(t_i)$.
Since $d(t_i, x_i) = a^0(t_i) = a_i(t_i) = \frac{1}{2}[r_i - d(p_i, t_i) - d(q_i, t_i)]$, we have $d(p_i, x_i) + d(q_i, x_i) = r_i$. If new facility ℓ is located at Step 1 of the algorithm, i.e., $x_\ell = t_\ell$, then we may assume without loss of generality that $\ell \in I^2_{t_\ell}$.

LEMMA 2.2. $d(p_i, q_j) + d(q_i, p_j) > r_i + r_j$ for all $i, j \in \{1, 2, \ldots, \ell\}$, $i < j$.

PROOF. First assume that new facility ℓ is located at Step 1 of the algorithm. Since $\ell \in I^2_{t_\ell}$ we know that $\{p_\ell, g_\ell\}$ is not covered by any new facility i $(i = 1, 2, \ldots, \ell-1)$. This implies two things:

(a) both p_ℓ and q_ℓ do not belong to $BT(t_i)$;

(b) $d(p_\ell, x_i) + d(g_\ell, x_i) > r_\ell \quad (i = 1, 2, \ldots, \ell-1)$.

Since both p_i and q_i are in $BT(t_i)$ and $d(p_i, x_i) + d(q_i, x_i) = r_i$, we have

$$\begin{aligned} d(p_i, q_\ell) + d(q_i, p_\ell) &= d(p_i, x_i) + d(q_i, x_i) + d(p_\ell, x_i) + d(q_\ell, x_i) \\ &= r_i + d(p_\ell, x_i) + d(q_\ell, x_i) > r_i + r_\ell \quad (i = 1, 2, \ldots, \ell-1). \end{aligned}$$

If both new facilities i and j (i<j) are located at Step 2 of the algorithm we have to consider two cases.

Case 1. If $BT(t_i) \subseteq BT(t_j)$, then we have $p_j, q_j \notin BT(t_i)$ and $\{p_j, q_j\}$ is not covered by x_i, i.e., $d(p_j, x_i) + d(q_j, x_i) > r_j$, and hence

$$\begin{aligned} d(p_i, q_j) + d(p_j, q_i) &= d(p_i, x_i) + d(q_i, x_i) + d(p_j, x_i) + d(q_j, x_i) \\ &= r_i + d(q_j, x_i) + d(p_j, x_i) > r_i + r_j. \end{aligned}$$

Case 2. If $BT(t_i) \nsubseteq BT(t_j)$, then we have

$$\begin{aligned} d(p_i, q_j) + d(p_j, q_i) &= d(p_i, x_i) + d(x_i, x_j) + d(x_j, q_j) + d(q_i, x_i) + d(x_i, x_j) + d(x_j, p_j) = \\ &= r_i + r_j + 2d(x_i, x_j) > r_i + r_j. \quad \square \end{aligned}$$

The following lemma indicates when two pairs can be covered by the same point.

LEMMA 2.3. *There is an* $x \in T$ *such that* $d(p_i, x) + d(q_i, x) \le r_i$ *and* $d(p_j, x) + d(q_j, x) \le r_j$ *with* $r_i \ge d(p_i, q_i)$ *and* $r_j \ge d(p_j, q_j)$ *if and only if*

$d(p_i,q_j)+d(p_j,q_i) \leq r_i + r_j$.

PROOF. If $d(p_i,x)+d(q_i,x) \leq r_i$ and $d(p_j,x)+d(q_j,x) \leq r_j$, then obviously $d(p_i,q_j)+d(p_j,q_i) \leq d(p_i,x)+d(x,q_j)+d(p_j,x)+d(x,q_i) \leq r_i + r_j$. Conversely, let $d(p_i,q_j)+d(p_j,q_i) \leq r_i + r_j$. If the shortest path from p_i to q_i intersects the shortest path from p_j to q_j, then any point x at the intersection satisfies $d(p_i,x)+d(q_i,x) = d(p_i,q_i) \leq r_i$, and $d(p_j,x)+d(q_j,x) = d(p_j,q_j) \leq r_j$. Suppose now that the two shortest paths do not intersect. Let c_i be the unique point on the shortest path from p_i to q_i closest to the shortest path from p_j to q_j; the point c_j is defined analogously. Since $d(p_i,q_j)+d(p_j,q_i) \leq r_i + r_j$, we have

$$d(c_i,c_j) = \tfrac{1}{2}(d(p_i,q_j)+d(p_j,q_i)-d(p_i,q_i)-d(p_j,q_j))$$

$$\leq \tfrac{1}{2}(r_i-d(p_i,q_i))+\tfrac{1}{2}(r_j-d(p_j,q_j)).$$

It follows that there is a point x on the shortest path from c_i to c_j such that $d(c_i,x) \leq \tfrac{1}{2}(r_i-d(p_i,q_i))$ and $d(c_j,x) \leq \tfrac{1}{2}(r_j-d(p_j,q_j))$. Hence

$$d(p_i,x)+d(q_i,x) \leq d(p_i,c_i)+d(q_i,c_i)+2d(c_i,x)$$

$$= d(p_i,q_i)+2d(c_i,x) \leq r_i,$$

and similarly $d(p_j,x)+d(q_j,x) \leq r_j$. □

THEOREM 2.4. *The solution* $X = \{x_1,\ldots,x_\ell\}$ *constructed by the algorithm is an optimal solution.*

PROOF. We know from Lemma 2.2 that $d(p_i,q_j)+d(p_j,q_i) > r_i + r_j$ $(i,j = 1,2,\ldots,\ell,\ i < j)$. According to Lemma 2.3 no two pairs $\{p_i,q_i\}$ and $\{p_j,q_j\}$ $(i,j = 1,2,\ldots,\ell,\ i < j)$ can be covered by the same new facility. Therefore any feasible solution needs at least ℓ points. Since X is a feasible solution (Lemma 2.1) and $|X| = \ell$, X is an optimal solution. □

EXAMPLE. We consider the tree given by Fig.1 with $r_1 = 8$, $r_2 = 10$ and $r_3 = 7$

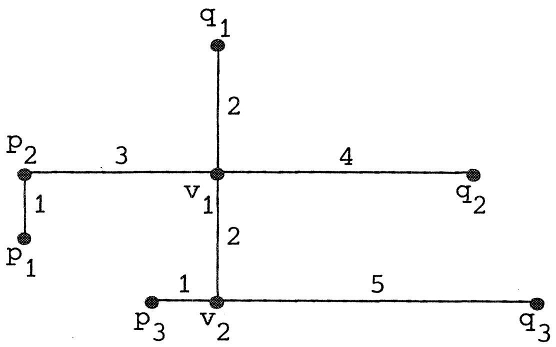

Fig. 1. Tree for the example.

Iteration 1: Choose p_1 : $I^1_{p_2} = \{1\}$.

Iteration 2: Choose p_2 : $I^1_{v_1} = \{1,2\}$.

Iteration 3: Choose q_2 : $I^1_{v_1} = \{1\}$, $I^2_{v_1} = \{2\}$, $a_2(v_1) = 1\frac{1}{2}$.

Iteration 4: Choose q_1 : $I^1_{v_1} = \emptyset$, $I^2_{v_1} = \{1,2\}$, $a_1(v_1) = 1$, $a_2(v_1) = 1\frac{1}{2}$, $a^0(v_1) = 1$.

Iteration 5: Choose v_1 : Locate x_1 at distance 1 from v_1 on the edge (v_1,v_2), $b(v_2) = 1$, $I = \{3\}$.

Iteration 6: Choose p_3 : Since $b(v_2) > \frac{1}{2}[r_3-d(p_3,q_3)] = \frac{1}{2}$, $\{p_3,q_3\}$ is not covered by x_1, $I^1_{v_2} = \{3\}$.

Iteration 7: Choose v_2 : $I^2_{q_3} = \{3\}$.

Iteration 8: Choose q_3 : Locate x_2 at q_3 and stop.

Note that $16 = d(p_1,q_3)+d(p_3,q_1) > r_1 + r_3 = 15$.

3. A STRONG DUALITY RESULT

In this section we formulate a dual problem to the round-trip covering problem and give both a constructive and theoretical proof of a strong duality result for these problems. The dual problem is given by

$$\max |I|$$

$$\text{s.t. } d(p_i,q_j)+d(p_j,q_i) > r_i + r_j \quad \forall i,j \in I,\ i \neq j,$$

$$I \subseteq \{1,2,\ldots,m\}.$$

THEOREM 3.1. (*Weak Duality*). *Let* X *be a feasible solution to the covering problem and let* I *be a feasible solution to the dual problem. Then* $|I| \leq |X|$.

PROOF. According to Lemma 2.3 no two pairs $\{p_i,q_i\}$ and $\{p_j,q_j\}$ $(i,j \in I, i \neq j)$ can be served by the same point $x \in X$. Hence $|I| \leq |X|$. □

THEOREM 3.2. (*Strong Duality*) $\min\{|X| \mid X \subseteq T,\ D(X,P_i) \leq r_i,\ i = 1,2,\ldots,m\}$ $= \max\{|I| \mid I \subseteq \{1,2,\ldots,m\},\ d(p_i,q_j)+d(p_j,q_i) > r_i + r_j,\ i,j \in I\}$.

PROOF. It follows from Theorem 3.1 that $|I| \leq |X|$ for any feasible solutions I and X. However the algorithm of Section 2 constructs a feasible dual solution I (Lemma 2.2) and a feasible solution X (Lemma 2.1) such that $|I| = |X|$. This shows that a strong duality result holds. □

The proof of Theorem 3.2 is constructive. In order to give a theoretical proof we need some definitions from graph theory. A clique in a graph is a subset of vertices such that every two vertices in the subset are adjacent. A covering by cliques of a graph is a family of cliques such that each vertex of the graph is in at least one clique belonging to this family. An independent set is a subset of vertices no two of which are adjacent. A graph is chordal if every simple cycle of order greater than three contains a chord, i.e., an edge between two vertices which are not adjacent in the cycle. It is well known that a chordal graph is a perfect graph [1]. A perfect graph has among others the following property.

LEMMA 3.3. [1]. *In a perfect graph the minimum number of cliques in a covering by cliques is equal to the maximum cardinality of an independent set.*

GAVRIL [6] gives polynomial time algorithms to find a minimum covering by cliques and a maximum independent set of a chordal graph. It has recently been proved by GRÖTSCHEL, LOVÁSZ & SCHRIJVER [7] that one can find a minimum covering by cliques and a maximum independent set in an arbitrary perfect graph in polynomial time.

A subtree is a connected subset of T not necessarily a subgraph. Given a tree T and a family $F = \{T_i \mid i = 1,\dots,m\}$ of subtrees of T the intersection graph of F is a graph with vertices $1,2,\dots,m$ and edges (i,j) if and only if $T_i \cap T_j \neq \emptyset$.

The following lemmas are concerned with a family of subtrees of T.

LEMMA 3.4. (BUNEMAN [2]). *Let F be a family of subtrees of a tree* T. *Then the intersection graph of F is a chordal graph.*

LEMMA 3.5. (Helly property [1]). *Let $F = \{T_i \mid i = 1,\dots,m\}$ be a family of subtrees of a tree* T *such that $T_i \cap T_j \neq \emptyset$ for all* i,j. *Then* $\bigcap(T_i \mid T_i \in F) \neq \emptyset$.

Returning to the round-trip covering problem on a tree T, we define the subtrees T_i of T by $T_i = \{x \in T \mid d(p_i,x)+d(q_i,x) \leq r_i\}$ $(i = 1,2,\dots,m)$. Note that since $r_i \geq d(p_i,q_i)$, we have $T_i \neq \emptyset$ $(i = 1,2,\dots,m)$. According to Lemma 2.3, $T_i \cap T_j \neq \emptyset$ if and only if $d(p_i,q_j)+d(p_j,q_i) \leq r_i+r_j$. Construct the intersection graph G corresponding to this family of subtrees. We now have the following relations.

By Lemma 3.5:

$$\min\{|X| \mid D(X,P_i) \leq r_i,\ i = 1,\dots,m\} = \text{minimum number of cliques in a covering by cliques of G.}$$

By lemma 2.3:

$$\max\{|I| \mid I \subseteq \{1,2,\dots,m\},\ d(p_i,q_j)+d(p_j,q_i) > r_i+r_j, i,j \in I\} = \text{maximum cardinality of an independent set of G.}$$

Using the fact that the intersection graph G is chordal and therefore perfect the strong duality result follows from Lemma 3.3.

4. THE ROUND-TRIP ℓ-CENTER PROBLEM

The round-trip ℓ-center problem is given by

$$\min_{X:|X|=\ell} \max_{1\le i\le m} \{f_i(D(X,P_i))\},$$

or equivalently

$$\begin{aligned} &\min \quad r \\ &\text{s.t.} \quad f_i(D(X,P_i)) \le r, \qquad i = 1,\ldots,m, \\ &\qquad\;\; X \subseteq T,\ |X| = \ell, \end{aligned}$$

where f_i is a strictly increasing continuous function ($i = 1,\ldots,m$). Define $\delta_i = \max\{d(p_i,x)+d(q_i,x) \mid x \in T\}$. The function f_i^{-1} is a strictly increasing continuous function with domain $[f_i(d(p_i,q_i)), f_i(\delta_i)]$. Define $\alpha = \max_i\{f_i(d(p_i,q_i))\}$ and $\eta = \min_i\{f_i(\delta_i)\}$. We assume that $\alpha < \eta$, since if $\alpha = f_s(d(p_s,q_s)) \ge f_t(\delta_t) = \eta$, then $f_s(D(X,P_s)) \ge f_t(D(X,P_t))$ for all X, and the constraint corresponding to f_t can be deleted from the constraint set without changing the optimum value. The function $f_i^{-1} + f_j^{-1}$ is a strictly increasing continuous function with domain $[\max\{f_i(d(p_i,q_i)), f_j(d(p_j,q_j))\}, \min\{f_i(\delta_i), f_j(\delta_j)\}]$ and range $[L_{ij}, U_{ij}]$, where $L_{ij} = (f_i^{-1}+f_j^{-1})(\max\{f_i(d(p_i,q_i)), f_j(d(p_j,q_j))\})$ and $U_{ij} = (f_i^{-1}+f_j^{-1})(\min\{f_i(\delta_i), f_j(\delta_j)\})$. Since $\alpha < \eta$ the domain is non empty. The following lemmas plays an important role in the analysis of the round-trip ℓ-center problem.

LEMMA 4.1. *There is a* $x \in T$ *such that* $f_i(d(p_i,x)+d(q_i,x)) \le r$ *and* $f_j(d(p_j,x)+d(q_j,x)) \le r$ *if and only if* $r \ge (f_i^{-1}+f_j^{-1})^{-1}(\max\{d(p_i,q_j)+d(p_j,q_i), L_{ij}\})$.

PROOF. We will first show that the condition $r \ge \beta_{ij}$, where $\beta_{ij} = (f_i^{-1}+f_j^{-1})^{-1}(\max\{d(p_i,q_j)+d(p_j,q_i), L_{ij}\})$, is well defined, i.e., that $d(p_i,q_j)+d(p_j,q_i) \le U_{ij}$. Without loss of generality assume that $\min\{f_i(\delta_i), f_j(\delta_j)\} = f_i(\delta_i)$. Choose x to be a point on the shortest path from p_j to q_j. Then

$$\begin{aligned} d(p_i,q_j)+d(p_j,q_i) &\le d(p_i,x)+d(x,q_j)+d(p_j,x)+d(x,q_i) \\ &= d(p_j,q_j)+d(p_i,x)+d(q_i,x) \\ &\le \delta_i+d(p_j,q_j) \\ &\le (f_i^{-1}+f_j^{-1})(f_i(\delta_i)) = U_{ij}, \end{aligned}$$

since $f_j(d(p_j,q_j)) < f_i(\delta_i)$ by our assumption that $\alpha < \eta$. In order to proof the lemma we consider three cases.

(a) If $r < \max\{f_i(d(p_i,q_i)),f_j(d(p_j,q_j))\}$, then there is no $x \in T$ such that $f_i(d(p_i,x)+d(q_i,x)) \le r$ and $f_j(d(p_j,x)+d(q_j,x)) \le r$, since if such an $x \in T$ exists $f_i(d(p_i,q_i)) \le f_i(d(p_i,x)+d(q_i,x)) \le r$ and similarly $f_j(d(p_j,q_j)) \le r$.

(b) If $r \ge \min\{f_i(\delta_i),f_j(\delta_j)\} = f_i(\delta_i)$, then choose x to be a point on the shortest path from p_j to q_j. We have $f_i(d(p_i,x)+d(q_i,x)) \le f_i(\delta_i) \le r$ and $f_j(d(p_j,x)+d(q_j,x)) = f_j(d(p_j,q_j)) \le f_i(\delta_i) \le r$, where the first inequality holds since $\alpha < \eta$.

(c) If $\max\{f_i(d(p_i,q_i)),f_j(d(p_j,q_j))\} \le r < \min\{f_i(\delta_i),f_j(\delta_j)\}$, then $f_i(D(x,P_i)) \le r$ if and only if $D(x,P_i) \le f_i^{-1}(r)$ and $f_j(D(x,P_j) \le r$ if and only if $D(x,P_j) \le f_j^{-1}(r)$. According to Lemma 2.3 there is a $x \in T$ such that $D(x,P_i) \le f_i^{-1}(r)$ and $D(x,P_j) \le f_j^{-1}(r)$ if and only if $d(p_i,q_j)+d(p_j,q_i) \le f_i^{-1}(r)+f_j^{-1}(r) = (f_i^{-1}+f_j^{-1})(r)$, or equivalently $r \ge \beta_{ij}$.

Combining (a), (b) and (c) we obtain the desired result. □

We now formulate the dual problem to the round-trip ℓ-center problem. The dual problem is given by

$$\max_{I\subseteq\{1,2,\dots,m\}:|I|=\ell+1} \max\{\alpha,\min_{i,j\in I,i\neq j}\{\beta_{ij}\}\}.$$

<u>THEOREM 4.2.</u> (Weak Duality) *Let* $X(|X|=\ell)$ *be a set of points on* T *and let* I $(|I|=\ell+1)$ *be a subset of* $\{1,2,\dots,m\}$.
Then $\max\{\alpha,\min_{i,j\in I,i\neq j}\{\beta_{ij}\} \le \max_{1\le i\le m}\{f_i(D(X,P_i))\}$.

PROOF. Let $r = \max_{1\le i\le m}\{f_i(D(X,P_i))\}$.
Since $d(p_i,q_i) \le d(p_i,x)+d(q_i,x)$ for all $x \in T$, we have $f_i(d(p_i,q_i) \le r$ for all $i = 1,2,\ldots,m$, and hence $\alpha \le r$. Since I contains $\ell+1$ elements there is a pair $i,j \in I$ $(i\neq j)$ and an $x \in X$ such that $f_i(d(p_i,x)+d(q_i,x)) \le r$ and $f_j(d(p_j,x)+d(q_j,x)) \le r$. By Lemma 4.1 we have $r \ge \beta_{ij} \ge \min_{i,j\in I,i\neq j}\{\beta_{ij}\}$. □

Using the duality result of Section 3 we are also able to prove a strong duality result.

THEOREM 4.3 (Strong Duality). $\max\{\alpha,\max_{I\subseteq\{1,\ldots,m\}:|I|=\ell+1}\min_{i,j\in I,i\neq j}\{\beta_{ij}\}\} = \min_{X\subseteq T:|X|=\ell}\max_{1\le i\le m}\{f_i(D(X,P_i))\}$.

PROOF. Let r_ℓ be the minimum value of the round-trip ℓ-center problem and assume $r_\ell > \max\{\alpha,\max_{I\subseteq\{1,2,\ldots,m\}:|I|=\ell+1}\min_{i,j\in I,i\neq j}\{\beta_{ij}\}\}$.

Case 1. $\alpha \ge \max_{I\subseteq\{1,\ldots,m\}:|I|=\ell+1}\min_{i,j\in I,i\neq j}\{\beta_{ij}\}$. Since $r_\ell > \alpha$ we know that the optimum value of the problem given by

$$\begin{aligned}&\min|X|\\ &\text{s.t. } f_i(D(X,P_i)) \le \alpha,\end{aligned}$$

is greater than or equal to $\ell+1$. Since $f_i(D(X,P_i)) \le \alpha$ is equivalent to $D(X,P_i) \le f_i^{-1}(\alpha)$ we find using the duality result of Section 3 that the optimum value of the problem

$$\begin{aligned}&\max|I|\\ &\text{s.t. } d(p_i,q_j)+d(p_j,q_i) > f_i^{-1}(\alpha)+f_j^{-1}(\alpha) \quad i,j \in I,\ i \neq j\end{aligned}$$

is greater than or equal to $\ell+1$. Hence there is a set I ($|I|=\ell+1$) such that for all $i,j \in I$ $(i\neq j)$ $d(p_i,q_j)+d(p_j,q_i) > (f_i^{-1}+f_j^{-1})(\alpha)$ (note that α is in the right domain) or equivalently $\alpha < (f_i^{-1}+f_j^{-1})^{-1}(d(p_i,q_j)+d(p_j,q_i))$ (note that $d(p_i,q_j)+d(p_j,q_i) \le U_{ij}$). Since $d(p_i,q_j)+d(p_j,q_i) > (f_i^{-1}+f_j^{-1})(\alpha) \ge (f_i^{-1}+f_j^{-1})(L_{ij})$ we find that

$$(f_i^{-1}+f_j^{-1})^{-1}(d(p_i,q_j) + d(p_j,q_i)) = \beta_{ij}.$$

We conclude that $\alpha < \min_{i,j\in I,i\neq j}\{\beta_{ij}\}$ which contradicts our starting assumption.

<u>Case 2</u>. $\alpha < \max_{I\subseteq\{1,\ldots,m\}:|I|=\ell+1} \min_{i,j\in I,i\neq j}\{\beta_{ij}\}$. Let $J \subseteq \{1,2,\ldots,m\}$ be the set of all indices i such that $r_\ell > f_i(\delta_i)$. Then

$$\min_{X\subseteq T:|X|=\ell} \max_{1\leq i\leq m}\{f_i(D(X,P_i))\} =$$

$$\max_{X\subseteq T:|X|=\ell} \max_{i\in J}\{f_i(D(X,P_i))\}.$$

Define $r = \max_{I\subseteq\{1,\ldots,m\}:|I|=\ell+1} \min_{i,j\in I,i\neq j}\{\beta_{ij}\}$. Since $r_\ell > r$ we know that the optimum value of the problem

$$\min|X|$$
$$\text{s.t. } f_i(D(X,P_i) \leq r, \quad i \in J,$$

is greater than or equal to $\ell+1$. Since $f_i(d(p_i,q_i)) \leq r \leq f_i(\delta_i)$ the constraint $f_i(D(X,P_i)) \leq r$ is equivalent to $D(X,P_i) \leq f_i^{-1}(r)$. Using the duality result of Section 3 we known that there is a subset $I \subseteq J$ ($|I|=\ell+1$) such that

$$d(p_i,q_j)+d(p_j,q_i) > f_i^{-1}(r)+f_j^{-1}(r) = (f_i^{-1}+f_j^{-1})(r) \quad \text{for all } i,j \in I,\ i\neq j,$$

or equivalently,

$$r < (f_i^{-1}+f_j^{-1})^{-1}(d(p_i,q_j) + d(p_j,q_i)) \qquad \text{for all } i,j \in I,\ i\neq j.$$

Since $d(p_i,q_j)+d(p_j,q_i) > (f_i^{-1}+f_j^{-1})(r) \geq (f_i^{-1}+f_j^{-1})(L_{ij})$ we find that

$$(f_i^{-1}+f_j^{-1})^{-1}(d(p_i,q_j) + d(p_j,q_i)) = \beta_{ij}.$$

We conclude that $r < \min_{i,j\in I,i\neq j}\{\beta_{ij}\}$, which contradicts our definition of r. □

The strong duality result for the round-trip ℓ-center problem shows that there are only $O(m^2)$ possible values of the optimum value for the round-trip ℓ-center problem, namely α and β_{ij} $(i,j = 1,\ldots,m,\ i \neq j)$. If we assume that inverses can be calculated in, say, unit time, then the following algorithm solves the round-trip ℓ-center problem in $O(m \cdot \max\{m,n\} \cdot \log m)$ time.

Algorithm:

Step 1 ($O(m^2 \log m)$ time): Calculate the values α and β_{ij} $(i,j = 1,\dots,m,$ $i \neq j)$ and arrange these values in a list L in non-decreasing order.

Step 2 ($O(mn \log m)$ time): For any value r_{ij} in L the algorithm of Section 2 can be applied to find the optimum value ℓ_{ij} to the covering problem

$$\min|X|$$
$$\text{s.t. } D(X,P_k) \le f_k^{-1}(r_{ij}), \quad k \in J(r_{ij}),$$

where

$$J(r_{ij}) = \{k \mid f_k(\delta_k) \ge r_{ij}\}.$$

Perform a binary search on L to find

$$r_\ell = \min\{r_{ij} \mid \ell_{ij} \le \ell\},$$

which is the optimum value of the round-trip ℓ-center problem.

EXAMPLE. Consider the round-trip 1-center problem, with

$$f_i(D(X,P_i)) = w_i[D(X,P_i) + k_i] \quad (i = 1,\dots,m).$$

This problem wast firstly solved by CHAN & FRANCIS [3].

$$f_i^{-1}(x) = \frac{1}{w_i}x - k_i, \quad (f_i^{-1}+f_j^{-1})(x) = \frac{w_i+w_j}{w_i w_j}x - k_i - k_j,$$

$$(f_i^{-1}+f_j^{-1})^{-1}(x) = \frac{w_i w_j}{w_i+w_j}(x+k_i+k_j),$$

$$L_{ij} = \frac{w_i+w_j}{w_i w_j}(\max\{w_i(d(p_i,q_i)+k_i), w_j(d(p_j,q_j)+k_j)\}) - k_i - k_j,$$

$$\beta_{ij} = \max\left\{\frac{w_i w_j}{w_i+w_j}(d(p_i,q_j)+d(p_j,q_i)+k_i+k_j), w_i(d(p_i,q_i)+k_i), w_j(d(p_j,q_j)+k_j)\right\}.$$

By the duality result we find that the optimum value is equal to

$$\max\left\{\max_i\{w_i(d(p_i,q_i)+k_i)\},\max_{i,j}\left\{\frac{w_iw_j}{w_i+w_j}(d(p_i,q_j)+d(p_j,q_i)+k_i+k_j)\right\}\right\}. \qquad \square$$

5. INTERPRETATION OF THE DUALITY RESULTS

In this section we give an interpretation of the duality result in case the function f_i is given by $f_i(d(X,P_i)) = \frac{1}{2}[D(X,P_i) - d(p_i,q_i)]$ $(i = 1,\dots,m)$, i.e. $f_i(D(X,P_i))$ is equal to the shortest distance from a point $x \in X$ to the shortest path $SP(p_i,q_i)$ from p_i to q_i. If $SP(p_i,q_i) \cap SP(p_j,q_j) = \emptyset$, then we define c_i to be the unique point on $SP(p_i,q_i)$ closest to $SP(p_j,q_j)$; c_j is defined analogously. The shortest path from c_i to c_j is the unique shortest path connecting $SP(p_i,q_i)$ and $SP(p_j,q_j)$. For every pair $\{p_i,q_i\}$ and $\{p_j,q_j\}$ $(i,j = 1,\dots,m)$ we define the distance $d(P_i,P_j)$ between $SP(p_i,q_i)$ and $SP(p_j,q_j)$ by

$$d(P_i,P_j) = \tfrac{1}{2}[d(p_i,q_j)+d(p_j,q_i)-d(p_i,q_i)-d(p_j,q_j)].$$

In case $SP(p_i,q_i) \cap SP(p_j,q_j) \neq \emptyset$, we have $d(P_i,P_j) = 0$, and in case $SP(p_i,q_i) \cap SP(p_j,q_j) = \emptyset$ we have $d(P_i,P_j) = d(c_i,c_j)$.
By the duality result of Section 3 we have

$$\min\{|X| \mid X \subseteq T,\ \tfrac{1}{2}[D(X,P_i)-d(p_i,q_i)] \le r\} =$$

$$\max\{|I| \mid I \subseteq \{1,\dots,m\},\ d(p_i,q_j)+d(p_j,q_i) > (2r+d(p_i,q_i))+(2r+d(p_j,q_j)),\ i,j \in I\}.$$

Since $d(p_i,q_j)+d(p_j,q_i) > 4r+d(p_i,q_i)+d(p_j,q_j)$ is equivalent to $d(P_i,P_j) > 2r$, we have the following interpretation of the duality result.

(1) The minimum number of points needed to make sure that each shortest path is within distance r of such a point is equal to the maximum number of paths such that the distance between any two of them is greater than 2r.

The value β_{ij} defined in Section 4 is equal to $\frac{1}{2}d(P_i,P_j)$.
Since $\alpha = 0$ we have the following duality result.

(2) $\min_{X:|X|=\ell} \max_{1\le i\le m}\{\frac{1}{2}[D(X,P_i)-d(p_i,q_i)]\} =$
$\max_{I:|I|=\ell+1} \min_{i,j\in I,i\ne j}\{\frac{1}{2}d(P_i,P_j)\}$.

The reader familiar with the duality results concerning the ℓ-center problem on a tree (see [5]) will note the resemblance in the interpretation of the duality result. The ℓ-center problem is given by

$$\min_{X:|X|=\ell} \max_{1\le i\le n}\{D(v_i,x)\},$$

where

$$D(v_i,X) = \min_{x\in X}\{d(v_i,x)\}.$$

The corresponding covering problem is given by

$$\min|X|$$
$$\text{s.t. } D(v_i,X) \le r, \quad i = 1,\dots,n.$$

The duality results for these problems are (see [5]):

(1) The minimum number of points needed to make sure that each vertex is within distance r of such a point is equal to the maximum number of vertices such that the distance between any two of them is greater than 2r.

(2) $\min_{X:|X|=\ell} \max_{1\le i\le n}\{D(v_i,X)\} =$
$\max_{I:|I|=\ell+1} \min_{i,j\in I,i\ne j}\{\frac{1}{2}d(v_i,v_j)\}$.

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