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CONTROLLED INVARIANCE FOR AFFINE CONTROL SYSTEMS

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Controlled invariance for affine control systems*)

by

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ABSTRACT

In this paper we will give a solution of the Disturbance Decoupling Problem for nonlinear systems. The main difference with related other works in this field is the construction of a new (state-dependent) basis for the inputs.

KEY WORDS & PHRASES: *Nonlinear systemstheory controlled-invariant distributions*

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*) The results of this report appeared at the same time in another preprint [11]

1. INTRODUCTION

The geometric approach for linear systems is a succesfull way to solve various synthesis problems in control theory, for example the Disturbance Decoupling Problem and other related decoupling problems (cf. [10]). It would be interesting to develop an analogue theory for nonlinear systems. Apparently differential geometry is the adequate apparatus. In this paper we want to discuss nonlinear A (mod B) invariance or as it is also called controlled invariance.

The systems that will be treated here have the form (locally) $\dot{x}(t) = X_0(x(t)) + \Sigma_{i=1}^{1cn} u_i(t)X_i(x(t))$.

We assume the reader is familiar with some basic concepts of differential geometry (see [1],[6]). Throughout the paper M is an n-dimensional smooth manifold. The set of all smooth vectorfields on M is denoted by V(M) and the set of all smooth functions on M is given by C(M). The Lie bracket of two vector fields X and Y is denoted by [X,Y] and by $[D_1,D_2]$, where D_1 and D_2 are (affine) distributions - we mean the set of all [X,Y] with $X \in D_1$ and $Y \in D_2$.

The organization of the paper is as follows. In section 2 we give a coordinatefree definition of a control system. In section 3 we introduce the concept of controlled invariant distributions and we prove here our main theorems. A conceptual algorithm for computing a controlled invariant distribution is given in section 4. In section 5 we discuss the concept of 'input-insensitivity' and in the last part we give a brief discussion of our results.

2. AFFINE CONTROL SYSTEMS.

DEFINITION 2.1. An *affine distribution* Δ on M will be a map Δ which assigns to each point x in M an affine subspace of T_xM. Δ is m-dimensional if the affine subspace $\Delta(m)$ is m-dimensional for all x.

DEFINITION 2.2. A C^∞ m-dimensional *affine control system* on M will be a m-dimensional affine distribution Δ on M such that for all x there exists a neighbourhood U(x) and vectorfields $X_0,\ldots,X_m \in V(M)$ such that

$$\forall x' \in U(x)\ \Delta(x') = X_0(x') + \mathrm{Span}\ \{X_1(x'),\ldots,X_m(x')\}$$

REMARKS.

(i) In fact an affine control system (or briefly control system) will be considered as a family of trajectories in state space M (See also Willems [9] for linear systems)

- Δ "=" $\{x:\mathbb{R} \to M \mid x$ absolutely continuous and $\dot{x}(t) = X_0(x(t))$ + an element of Span $\{X_1(x(t)),\ldots,X_m(x(t)\}$ almost everywhere$\}$

  $= \{x:\mathbb{R} \to M \mid x$ absolutely continuous and $\exists u_1,\ldots,u_m:\mathbb{R} \to \mathbb{R}$ such that $\dot{x}(t) = X_0(x(t)) + u_1(t)X_1(x(t)) +\ldots+u_m(t)X_m(x(t))$ almost everywhere$\}$

(Locally we identify Δ with $X_0 + \mathrm{Span}\{X_1,\ldots,X_m\}$)

It follows easily that Δ is 'feedback invariant' i.e. Δ does not depend on the choice of a basis of $\mathrm{Span}\{X_1,\ldots,X_m\}$ and

$\Delta = \{x:\mathbb{R} \to M \mid x$ absolutely continuous and $\exists v_1,\ldots,v_m:\mathbb{R} \to \mathbb{R}$ such that $x(t) = X_0(x(t)) + \Sigma_{i=1}^m \alpha_i(x(t))X_i(x(t)) + \Sigma_{i=1}^m v_i(t)X_i(x(t))$ almost everywhere$\}$ ($\alpha_i \in C(M), i=1,\ldots,m$).

(ii) A control system Δ induces a distribution Δ_0, defined by $\Delta_0 := \Delta - \Delta = \{X-Y \mid X,Y \in \Delta\}$ or if locally $\Delta = X_0 + \mathrm{Span}\{X_1,\ldots,X_m\}$ then $\Delta_0 = \mathrm{Span}\{X_1,\ldots,X_m\}$, i.e. the subspace of directions in which we can steer.

3. CONTROLLED INVARIANCE

Now we are going to discuss the generalized notion of (A,B)-invariance. Recently several people studied this problem: In Isidori et al [4] (see also [3]) the control theoretic setting of the disturbance decoupling problem for nonlinear systems is given in terms of invariant distributions. Hirschorn also gives under specific conditions in [2] a solution of the nonlinear disturbance decoupling problem. The present approach unifies both works in the sense that it fits the special solution of Hirschorn to the general setting of Isidori et al.

In [9] Willems has given a collection of (equivalent) definitions of controlled-invariance for A (mod B) invariance for linear systems. The most useful ones for our purpose are given here.

Let Σ: $\dot{x} = Ax + Bu$ $x \in \mathbb{R}^n =: \mathcal{X}$, $u \in \mathbb{R}^m =: \mathcal{U}$ A and B matrices of appropriate dimensions.

DEFINITION 3.1 (a) A linear subspace $\mathcal{V} \subset \mathcal{X}$ is A (mod B)-invariant if there exists a linear feedback $F:\mathcal{X} \to \mathcal{U}$ such that $A_F\mathcal{V} \subset \mathcal{V}$ where $A_F := A + BF$

(b) A linear subspace $\mathcal{V} \subset \mathcal{X}$ is A (mod B)-invariant if $A\mathcal{V} \subset \mathcal{V} + \mathcal{B}$ ($\mathcal{B} := \mathrm{Im}B$)

(c) A linear subspace $\mathcal{V} \subset \mathcal{X}$ is A (mod B)-invariant if Σ (mod $\mathcal{V}$) is a linear system.

We first give the 'distributional version' of this definition. (See also [2]). Let $\mathcal{V}$ be a linear subspace of $\mathcal{X}$. We can associate a distribution $D_{\mathcal{V}}$ with the linear subspace $\mathcal{V}$ by defining $D_{\mathcal{V}}(x) = \mathcal{V} \subseteq T_x\mathbb{R}^n$ where we use the natural identification of $\mathbb{R}^n$ with $T_x\mathbb{R}^n$. Another way of defining $D_{\mathcal{V}}$ if given by the following: Let $\{v_1,\dots,v_k\}$ be a basis of $\mathcal{V}$ then $D_{\mathcal{V}}$ is given by Span $\{\frac{\partial}{\partial v_1},\dots,\frac{\partial}{\partial v_k}\}$.

The condition $A_F\mathcal{V} \subset \mathcal{V}$ will transform in

$$[A_F,\frac{\partial}{\partial v_i}](x) \in \mathrm{Span}\{\frac{\partial}{\partial v_1}\Big|_x,\dots,\frac{\partial}{\partial v_k}\Big|_x\} \qquad i = 1,\dots,k.$$

$\Longleftrightarrow$ A_F has the form (with respect to a basis $\{v_1,\dots,v_k,v_{k+1},\dots,v_n\}$

$$A_F = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline \underbrace{0}_{k} & A_{22} \end{array}\right)\}k \qquad (*)$$

Now we will give the generalization of definition (a). In the context of the definition of a control system A (mod B)-invariance becomes Δ (mod Δ_0)-invariance.

DEFINITION 3.2. A smooth involutive distribution D (of fixed dimension) will be called a *controlled invariant* distribution for a control system Δ or Δ *(mod* Δ_0*)-invariant* if there exists an $X \in \Delta$ such that $[X,D] \subseteq D$.

REMARKS.

(i) Throughout this paper we will use Δ (mod Δ_0)-invariance - in analogy with A (mod B)-invariance for linear systems - rather than controlled invariance.

(ii) It will be clear what we mean by locally Δ (mod Δ_0)-invariant namely: Locally we can find $X \in \Delta$ such that $[X,D] \subseteq D$.

If we work out a coordinate version of this definition then we get the following appealing result: Locally we can find around each point in M a coordinate system such that in that chart the distribution D is spanned by the vectorfields $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}$. That such a coordinate system exists is a consequence of the local version of Frobenius' theorem (cf. [1],[6]). Writing the equation $[X,D] \subset D$ now gives

$$\frac{\partial X_i(x)}{\partial x_j} = 0 \qquad \begin{array}{l} i = k+1,\ldots,n \\ j = 1,\ldots,k \end{array} \qquad \text{if } X(x) = \sum_{i=1}^{n} X_i(x)\frac{\partial}{\partial x_i}\Big|_x$$

or equivalently, if we write $\underline{x}_1 = (x_1,\ldots,x_k)$ and $\underline{x}_2 = (x_{k+1},\ldots,x_n)$ then we get the following form for X

$$X(x) = \begin{pmatrix} X_1(\underline{x}_1,\underline{x}_2) \\ \vdots \\ X_k(\underline{x}_1,\underline{x}_2) \\ X_{k+1}(\underline{x}_2) \\ \vdots \\ X_n(\underline{x}_2) \end{pmatrix}$$

which is the nonlinear analogue of (*)!

Because we cannot check for a given distribution D if there exists an $X \in \Delta$ such that $[X,D]$ for all $X \in \Delta$!) this definition as it stands is not very useful. What we need is an analogue statement of definition 3.1(b) for nonlinear systems.

REMARK. In [4] also def. 3.1(c) is discussed. The basic problem for this (global!) definition is the requirement that the quotient M (mod D) is again a manifold. See for a study of this problem Sussmann [8].

The main result of this paper is given by the following

THEOREM 3.3. *Let* D *be an involutive distribution of fixed dimension* (f.d). *If* $D \cap \Delta_0$ *has fixed dimension then we have the following equivalence*: D *is locally* Δ (mod Δ_0)*-invariant iff* $[\Delta, D] \subseteq D + \Delta_0$ □

Before we will prove this theorem we formulate the result of Hirschorn [2]. Let as before locally Δ be given as $\Delta = X_0 + \mathrm{Span}\{X_1, \ldots, X_m\}$ then

THEOREM 3.4. [2] *An involutive distribution* D(f.d.) *which satisfies*

$$[X_0, D] \subseteq D + \Delta_0$$

and $$[X_i, D] \subseteq D \qquad i = 1, \ldots, m$$

is locally a controlled invariant distribution. □

As will be clear Hirschorn's result depends on the choice of the basis $X_1, \ldots, X_m$ for the 'inputspace' Δ_0 (In general another basis $\tilde{X}_1, \ldots, \tilde{X}_m$ of Δ_0 does not satisfy $[\tilde{X}_i, D] \subseteq D$ $i = 1, \ldots, m$ if $[X_i, D] \subseteq D$ $i = 1, \ldots, m$). This fact is rather unsatisfactory and therefore we will first prove a technical result of independent interest.

THEOREM 3.5. *Let* D *be an involutive distribution* (f.d.) *which satisfies* $[\Delta_0, D] \subseteq D + \Delta_0$ *and* $D \cap \Delta_0$ *has fixed dimension then locally there exists a basis* $X_1, \ldots, X_m$ *for* Δ_0 *such that* $[X_i, D] \subseteq D$ $i = 1, \ldots, m$. □

It is easily seen that thm. 3.4 and thm. 3.5 imply thm. 3.3. We first give an outline of the proof of thm. 3.5. In fact we will reduce it to a set of partial differential equations. Suppose

$$(1) \qquad [D, \Delta_0] \subseteq D + \Delta_0$$

Choose around point p in M a coordinate system (U(p),x) as in the local Frobenius' theorem, so $D = \mathrm{Span}\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}\}$ and choose an arbitrary basis for $\Delta_0 : X_1, \ldots, X_m$. Define the (n,m) matrix B(x) by

$$(2) \qquad B(x) = (X_1(x) \ldots X_m(x))$$

Then from (1) we have

$$\frac{\partial B\ (x)}{\partial x_i} = B(x).M_i(x) \pmod{D} \qquad i = 1,\dots,k) \tag{3}$$

where $M_i(x)$ are (n,m) matrices and mod D means modulo a (n,m) matrix of the form

$$\left(\frac{*}{0}\right)\}k.$$

From

$$\frac{\partial^2 B(x)}{\partial x_j \partial x_i} = \frac{\partial^2 B(x)}{\partial x_i \partial x_j} \qquad i,j = 1,\dots,k$$

we deduce

$$B(x)\left[\frac{\partial M_i(x)}{\partial x_j} - \frac{\partial M_j(x)}{\partial x_i} + M_j(x)M_i(x) - M_i(x)M_j(x)\right] = 0 \pmod{D} \tag{4}$$

We would like to find another basis $\tilde{X}_1,\dots,\tilde{X}_m$ for Δ_0 which satisfies

$$\frac{\partial \tilde{X}_\ell(x)}{\partial x_j} = 0 \pmod{D} \qquad \begin{array}{l} \ell = 1,\dots,m \\ j = 1,\dots,k \end{array}$$

or if $\tilde{B}(x) = (\tilde{X}_1(x)\dots\tilde{X}_m(x))$ then

$$\frac{\partial \tilde{B}(x)}{\partial x_j} = 0 \pmod{D} \tag{5}$$

The two bases $\mathcal{B}$ and $\tilde{B}$ of Δ_0 are connected via a nonsingular (m,m) matrix A such that

$$B(x) = \tilde{B}(x).A(x) \tag{6}$$

and so

$$\frac{\partial B(x)}{\partial x_j} = \frac{\partial \tilde{B}(x)}{\partial x_j}.\ A(x) + \tilde{B}(x).\frac{\partial A(x)}{\partial x_j} \qquad j = 1,\dots,m$$

and according to (5) and (6):

$$(7) \qquad \frac{\partial B(x)}{\partial x_j} = B(x).A^{-1}(x).\frac{\partial A(x)}{\partial x_j} \pmod{D}$$

combining (7) with (3) we get:

$$(8) \qquad B(x)[A^{-1}(x).\frac{\partial A(x)}{\partial x_j} - M_j(x)] = 0 \pmod{D} \qquad j = 1,\dots,m$$

In the following we will show that we can solve the set of partial differential equations:

$$A^{-1}(x)\frac{\partial A(x)}{\partial x_j} - M_j(x) = 0 \qquad j = 1,\dots,k$$

with for example $A(0) = I_m$ the -identity matrix i.e. we are looking for a (m,m)-matrix A(x) which satisfies

$$(9) \qquad \begin{aligned} &A(0) = I_m \\ &\frac{\partial A(x)}{\partial x_j} = A(x).M_j(x) \qquad j = 1,\dots,k \end{aligned}$$

where the matrices $M_j(x)$ are given and satisfy the "integrability conditions" (see (6))

$$(10) \qquad \frac{\partial M_i(x)}{\partial x_j} - \frac{\partial M_j(x)}{\partial x_i} + M_j(x)M_i(x) - M_i(x).M_j(x) = 0 \qquad i,j = 1,\dots,k$$

The required basis for Δ_0 then will be given by the columns of the matrix $B(x).A^{-1}(x)$.

Thus we can formulate the following equivalent.

<u>LEMMA 3.6</u>. *The set of partial differential equations* (9) *has a solution on a neighbourhood of* 0 *if and only if the integrability conditions* (10) *are satisfied.*

The proof of thm. 3.5 and therefore of Lemma 3.6 is rather involved and can be skipped by the reader who is only interested in the systems theory. Basically our proof will be geometric. We will first treat the 'most regular' case i.e. $D \cap \bar{\Delta}_0 = \underline{0}$ and later on indicate how the other cases can be treated. ($\bar{\Delta}_0$ is the involutive closure of Δ_0, i.e. it contains all vectorfields in Δ_0 and iterated Lie brackets of vectorfields in Δ_0.

We need the following two lemmas.

<u>LEMMA 3.7.</u> *If* $[D,\Delta_0] \subseteq D + \Delta_0$ *then* $D + \bar{\Delta}_0$ *is involutive.*

<u>PROOF</u>. Use the Jacobi-identity. □

<u>LEMMA 3.8.</u> (*Frobenius' theorem extended*). *Let* D *and* $D + \bar{\Delta}_0$ *be involutive distributions of fixed dimension, then around each point* p *in* M *there exists a coordinate system* $(U(p),x)$ *such that in that chart*

$$D = \mathrm{Span}\{\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_q}\}$$

and

$$D + \bar{\Delta}_0 = \mathrm{Span}\{\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_p},\dots,\frac{\partial}{\partial x_q}\}.$$

<u>PROOF</u>. See Jakubczyk [6]. □

Now if $D \cap \bar{\Delta}_0 = \underline{0}$ we have that in the local chart of lemma 3.8 $\bar{\Delta}_o$ is spanned by $\ell := q-p$ vector fields

$$Y_1 = \frac{\partial}{\partial x_{p+1}} + \sum_{i=1} \alpha_i^1(x)\frac{\partial}{\partial x_i},\dots,Y_\ell = \frac{\partial}{\partial x_q} + \sum \alpha_i^\ell(x)\frac{\partial}{\partial x_i}.$$

Moreover $[Y_s,Y_t] = 0 \quad s,t = 1,\dots,\ell$. Furthermore it will be clear that $[D,Y_s] \subseteq D$ $s = 1,\dots,\ell$.

<u>PROOF OF THEOREM 3.5</u> (and Lemma 3.6). Let $D \cap \Delta_0 = 0$. Choose locally an arbitrary basis $X_1,\dots,X_m$ for Δ_0 and select some other vectorfields $X_{m+1},\dots,X_\ell$ such that $\mathrm{Span}\{X_1,\dots,X_\ell\} = \bar{\Delta}_0 = \mathrm{Span}\{Y_1,\dots,Y_\ell\}$ (See lemma 3.8). (Here $X_1,\dots,X_\ell$ are linearly independent in each point).

Define (n,ℓ) matrices $B(x)$ and $\tilde{B}(x)$ by

$$B(x) = (X_1(x)\dots X_\ell(x))$$
$$\tilde{B}(x) = (Y_1(x)\dots Y_\ell(x)).$$

Now $\{X_1,\dots,X_\ell\}$ as well as $\{Y_1,\dots,Y_\ell\}$ form a basis for $\bar{\Delta}_0$ so there exist a non-singular (ℓ,ℓ)-matrix A such that

$$B(x) = \tilde{B}(x).A(x).$$

Therefore $\dfrac{\partial B(x)}{\partial x_i} = \dfrac{\partial \tilde{B}(x)}{\partial x_i}.A(x) + \tilde{B}(x).\dfrac{\partial A(x)}{\partial x_i}$

$$(11) \qquad \Rightarrow \frac{\partial B(x)}{\partial x_i} = B(x).A^{-1}(x)\frac{\partial A(x)}{\partial x_i} \quad (\mathrm{mod}\ D) \qquad i = 1,\dots,k$$

Furthermore we may assume that $A(0) = I_\ell$.

On the other hand we deduce from $[D,\Delta_0] \subseteq D + \Delta_0$ that $[D,\bar{\Delta}_0] \subset D + \bar{\Delta}_0$ (lemma 3.7) and

$$(12) \qquad \frac{\partial B(x)}{\partial x_i} = B(x).\bar{M}_i(x) \quad (\mathrm{mod}\ D) \qquad i = 1,\dots,k$$

where the (ℓ,ℓ)-matrix $\bar{M}_i(x)$ has the form

$$(13) \qquad \left(\begin{array}{c|c} M_i(x) & * \\ \hline 0 & * \end{array}\right)\begin{array}{l}\}\,m \\ \\ \end{array}$$

$$\underbrace{\qquad}_{m}$$

Combining (11) and (12) leads to:

$$(14) \qquad B(x).[A^{-1}(x).\frac{\partial A(x)}{\partial x_i} - \bar{M}_i(x)] = \underline{0} \qquad (\mathrm{mod}\ D), \qquad i = 1,\dots,k$$

and by the fact that $D \cap \bar{\Delta}_0 = 0$ it follows

$$A^{-1}(x)\frac{\partial A(x)}{\partial x_i} - \bar{M}_i(x) = 0 \qquad i = 1,\dots,k$$

(This could be seen by skipping the first k-rows of equation (14) and by observing that the reduced matrix B (i.e. a $(n-k,\ell)$-matrix has rank ℓ). So we have given an unique solution of the set of partial differential equations

$$\frac{\partial A(x)}{\partial x_i} = A(x)\bar{M}_i(x) \qquad i = 1,\dots,k$$

$$A(0) = I_\ell .$$

Now if we partition the (ℓ,ℓ)-matrix $A(x)$ in the following way

$$A(x) = \left(\begin{array}{c|c} A_1(x) & A_2(x) \\ \hline \underbrace{A_3(x)}_{m} & A_4(x) \end{array}\right) \begin{array}{l} \}m \\ \end{array}$$

we see that $A_1(x)$ satisfies

$$\frac{\partial A_1}{\partial x_i}(x) = A_1(x)M_i(x) \qquad i = 1,\dots,k$$

$$A_1(0) = I_m$$

Therefore the columns of the matrix $(X_1(x)\dots.X_k(x)) - A_1^{-1}$ forms a nice basis for Δ_0.

Now we will shortly indicate the proof for the more general situation where we assume that the distribution $D \cap \Delta_0$ has fixed dimension. In that case we do not bother about the part of Δ_0 which is contained in $D \cap \Delta_0$ (every basis of $D \cap \Delta_0$ already has the required properties); the essential part of the proof remains the same for an arbitrary basis of Δ_0 (mod $\Delta_0 \cap D$).

Finally we note that the assumption that $D + \bar{\Delta}_0$ has fixed dimension (see Lemma 3.8) is not essential in the proof. By using Sussmann's method of generalizing Chow's theorem [7] we can embed $D + \bar{\Delta}_0$ in an involutive distribution of fixed dimension. □

4. A NAIVE ALGORITHM

For the disturbance decoupling problem (see Section 6) it is relevant to solve the following problem:

Find the largest Δ (mod Δ_0)-invariant distribution S contained in a given distribution K of fixed dimension. The existence of S is shown by Hirschorn [2] and Isidon et.al. [4].

Let Δ and Δ_0 be given as before and let D be an arbitrary distribution (not necessarily involutive). Then define: $\Delta^{-1}(\Delta_0 + D) :=$ $\{X \in V(M) | [\Delta,X] \subseteq \Delta_0 + D\}$. It is straightforward to show that $\Delta^{-1}(\Delta_0 + D) \cap D$ is a distribution.

The following theorem is a slight modification of the linear algorithm [10].

<u>THEOREM 4.1</u>. *Let* K *be a given involutive distribution of dimension* k. *Consider the algorithm*

$$\begin{aligned} D^0 &= K \\ D^{m+1} &= D^m \cap \Delta^{-1}(\Delta_0 + D^m) \qquad m = 0,1,2,\ldots \end{aligned}$$

then

$$\lim_{m\to\infty} D^m = D^k$$

and assume that the involutive closure $\bar{D}^k$ *has fixed dimension then* $\bar{D}^k$ *equals the largest* Δ (mod Δ_0)*-invariant distribution in* K.

<u>PROOF</u>. By induction we show $D^s \subset D^{s-1}$, $s = 1,2,\ldots$. Clearly $D^1 \subset D^0$ and if $D^s \subset D^{s-1}$ then $D^{s+1} = D^s \cap \Delta^{-1}(\Delta_0 + D^s) \subseteq D^{s-1} \cap \Delta^{-1}(\Delta_0 + D^{s-1}) = D^s$. Furthermore it is easy to see that the algorithm ends after at most k steps. The resulting distribution D^k is Δ (mod Δ_0) invariant but not necessarily involutive. By using the Jacobi identity we can show that the involutive closure of D^k is also Δ (mod Δ_0) invariant.

Finally let D be a Δ (mod Δ_0) invariant distribution in K then $D \subset D^0$ and if $D \subset D^{s-1}$ then $D \subset D^{s-1} \cap \Delta^{-1}(D + \Delta_0) \subset D^{s-1} \cap \Delta^{-1}(D^{s-1} + \Delta_0) =$ D^s. Therefore $\bar{D}_k$ is the greatest Δ (mod Δ_0) invariant distribution. □

5. INPUT-INSENSITIVITY

Theorem 3.3, as it stands, can be considered as a closed loop formulation; given $X_0 \in \Delta$ with $[X_0,D] \subseteq D + \Delta_0$ we can find a vectorfield $B \in \Delta_0$ such that $[X_0 + B,D] \subseteq D$. Before we will treat a mixed (i.e. closed *and* open loop) situation we want to investigate the linear dituation somewhat deeper.

Let $\dot{x} = Ax + Bu$ and $V \subset X$ be an (A,B)-invariant subspace. Let F be a feedback such that $(A + BF)V \subset V$. We can associate - as in the beginning of section 3 - an involutive distribution D_V with the linear subspace V: $D_V(x) = \mathrm{Span}\{\frac{\partial}{\partial v_1}\Big|_x, \ldots, \frac{\partial}{\partial v_k}\Big|_x\}$.

Now let $b_1,\ldots,b_m$ be the columns of B, then for each vectorfield $D \in D_V$ we have $[b_i,D] \subseteq D_V$ $i = 1,\ldots,m$ (Compare with theorem 3.5). If we apply a feedback of the form $u = Fx + v$ then we have the system $\dot{x} = A_F x + Bv$ and $[A_F x + Bv, D(x)] \in D_V(x)$ $\forall D \in D_V$. Geometrically this implies that when we start at $t = t_0$ in two different points of the same leaf of D_V then these points will be following the integral curves of $A_F x + Bv$ - at any time $t = t_1 > t_0$ at the same leaf of D. This fact is called 'input insensitive' (see [9]).

For nonlinear systems the situation is a bit different. From theorem 3.5 it will be clear that we can give Δ_0 a basis such that the modified system (i.e. first selecting $X \in \Delta$ with $[X,D] \subseteq D$ and then selecting the basis $X_1,\ldots,X_k$ of theorem 3.5 for Δ_0) is input insensitive. So for the system $\dot{x}(t) = X(x(t)) + u_1(t)X_1(x(t)) + \ldots + u_k(t)X_k(x)$ we have that starting in two points on the same leaf of D will remain on the same leaf of D for every $u_1,\ldots,u_k: \mathbb{R} \to \mathbb{R}$. But as noted before this is in general not true for an arbitrary basis of Δ_0. So the mixed situation of input insensitivity depends on the choice of a basis of Δ_0.

6. DISCUSSION

It is very natural to apply the results of sections 3 and 4 to the Disturbance Decoupling Problem (D.D.P. cf. [10]). For an extensive treatment the reader is referred to Isidorietal [4]. We will briefly indicate how to solve D.D.P. Let us assume that there is also given an output map $h: M \to N$, where N is another manifold and h is a smooth function. With h we can associate a distribution K (K of kernel compare [10]) consisting of all $X \in V(M)$ which satisfy $h_*(X) = 0$ ($h_*: TM \to TN$, given in local coordinates by $(\frac{\partial h}{\partial x})$). The algorithm of section 4 will provide you with the greatest Δ (mod Δ_0) invariant distribution in K. Disturbances contained in this invariant distribution - i.e. vectorfields which cause disturbances - do not influence the output. Theorem 3.3 guarantees that we can find an appropriate $X \in \Delta$ which leaves this distribution invariant. The main difference with the paper of Hirschorn, except for the fact that we solved the D.D.P. in a far more general way, lies in the input-insensitivity as sketched in section 5.

In a forthcoming paper we will also introduce the concept of controlled invariance for arbitrary nonlinear systems.

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