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AFDELING MATHEMATISCHE BESLISKUNDE BW 136/81 FEBRUARI
(DEPARTMENT OF OPERATIONS RESEARCH)

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CONTROLLED INVARIANCE FOR NONLINEAR SYSTEMS

Preprint

kruislaan 413 1098 SJ amsterdam

Printed at the Mathematical Centre, 413 Kruislaan, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

1980 Mathematics subject classification: 93C10, 53C05

Controlled invariance for nonlinear systems*

by

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ABSTRACT

Necessary and sufficient conditions are derived for "(A,B)-invariance", called here controlled invariance, for nonlinear systems $\dot{x} = f(x,u)$. The obtained results generalize and elucidate already known results about systems $\dot{x} = A(x) + \sum_{i=1}^{m} u_i B_i(x)$. A new and direct differential geometric interpretation of the concept of controlled invariance and the derived conditions is given.

KEY WORDS & PHRASES : *Nonlinear control systems, controlled invariant quotient systems, distributions, connections*

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* This report will be submitted elsewhere.

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§1. INTRODUCTION

Basic to the solution of various problems in linear systems theory is the notion of (A,B)-invariance or, also called, *controlled invariance* (cf.[1,13]). Recently several people studied the problem of generalizing this notion to nonlinear systems of the form

$$(1.1)\qquad \dot{x} = A(x) + \sum_{i=1}^{m} u_i B_i(x)$$

(cf.[4,5,6,7,8,9])
Actually, very recently conditions have been found which seem very conclusive for this class of systems (cf.[6,9]).

The aim of this paper is to generalize the concept further to general nonlinear systems

$$(1.2)\qquad \dot{x} = f(x,u)$$

and to derive conditions similar to those derived for systems of the form (1.1). In the course of doing this it became clear that the concept of controlled invariance can be translated, in a natural and clarifying way, into classical differential geometric notions like integrability conditions and connections on fiberbundles. Actually, we will show that this point of view also elucidates the already known results about systems of the form (1.1) (we will call these systems affine systems)

Before going on we will briefly summarize some of the ideas and results about controlled invariance for linear and affine systems (for an introduction see also [4,5,8]). First we define the related notion of *invariance*. Consider a linear system

$$(1.3)\qquad \dot{x} = Ax + Bu, \quad x \in X := \mathbb{R}^n, \quad u \in U := \mathbb{R}^m$$

We call a linear subspace $V \subset X$ *invariant* if $AV \subset V$. We can interpret this condition in the following way. The collection of affine subspaces $x + V, x \in \mathbb{R}^n$, can be regarded as the leaves of a foliation of $\mathbb{R}^n$.

Then $A\mathcal{V} \subset \mathcal{V}$ is equivalent to saying that the system (1.3) leaves the foliation invariant; i.e.take two arbitrary points x_1 and x_2 on a same leaf and take an arbitrary inputfunction $\bar{u}(\cdot)$, then the integral curves starting from x_1 and x_2, generated by $\dot{x} = Ax + B\bar{u}$, intersect at every time t the same leaf.

This idea can be generalized to nonlinear systems

$$(1.4) \qquad \dot{x} = f(x,u), \quad x \in M, \quad M \text{ a manifold}$$

Take instead of a linear subspace $\mathcal{V}$ an involutive *distribution* D on M. The maximal integral manifolds of this distribution are the leaves of a foliation of M. Then we say that the distribution D is invariant if again for every input function $\bar{u}(\cdot)$, the system $\dot{x} = f(x,\bar{u})$ leaves the foliation invariant.

Actually it is a standard fact from differential geometry, that this condition is, just as in the linear case, equivalent with an *infinitesimal* condition, namely

$$[f(\cdot,\bar{u}), D] \subset D$$

(see for notation the end of this §)
Controlled invariance is defined as follows. An involutive distribution D is called controlled invariant if there exists a feedback $u \mapsto v := \alpha(x,u)$ such that after applying this feedback D is invariant with respect to the modified dynamics

$$\dot{x} = \tilde{f}(x,v)$$

Within the "category" of linear systems feedbacks should have the form

$$u \mapsto v := u - Fx$$

and for affine systems

$$u \mapsto v := M(x)u - \nu(x)$$

The defect in this definition of controlled invariance is that it requires the *existence* of a feedback. Therefore conditions should be sought on the distribution D and the system $\dot{x} = f(x,u)$ which ensure the existence of a feedback which makes D invariant. In fact for linear systems (1.3) it can be easily proven that

$$A\mathcal{V} \subset \mathcal{V} + \mathrm{Im}B$$

is necessary and sufficient for the existence of a matrix F such that $(A+BF)\mathcal{V} \subset \mathcal{V}$.

Very recently, in [6] and independently in [9] the following result has been proven for affine systems

$$\dot{x} = A(x) + \sum_{i=1}^{m} u_i B_i(x)$$

Define the affine distribution Δ by $\Delta(x) := A(x) + \mathrm{Span}\,\{B_1(x),\ldots,B_m(x)\}$ and the distribution Δ_0 by $\Delta_0(x) := \mathrm{Span}\,\{B_1(x),\ldots,B_m(x)\}$. Then a distribution D is controlled invariant iff

$$[\Delta, D] \subset D + \Delta_0$$

(see for the notation the end of this §), where we suppose, to avoid technical difficulties, that the dimension of $D \cap \Delta_0$ is constant. This last result includes an earlier result in [4].

Finally, in this paper we will give the conditions for controlled invariance for general systems $\dot{x} = f(x,u)$ (see §4).

The outline of the paper is as follows. §2 contains preliminaries about definitions of nonlinear control systems which will clear up the way to the definitions of controlled invariance in §3. It will be argued that a natural concept for controlled invariance is the idea of an *(integrable) connection*, which will be dealt with in §4. It will be shown here that for affine systems the vanishing of the *torsion* and the *curvature* tensor of an affine connection exactly gives the integrability conditions needed for the construction of a feedback. Furthermore the condition for controlled invariance for general nonlinear systems is derived. §5 contains the conclusion.

Some notation

Our basic reference to differential gemometry will be [11]. All our objects like manifolds, maps, etc. are C^∞. We call Δ an affine distribution on a manifold M if Δ in every $x \in M$ is given by an affine subspace $\Delta(x) \subset T_xM$ (in a smooth way). Given two (affine) distributions D_1, D_2, then we define the distribution

$$[D_1,D_2] := \{[X,Y] \mid X \in D_1, Y \in D_2\}.$$

where [,] is the Liebracket. Given a distribution D on M, then we define $\dot{D}$, a distribution on TM, as follows. Let X be a vectorfield on M. X generates a group of diffeomorphisms $X_t : M \longrightarrow M$ (t small), such that $t \longrightarrow X_t(x)$ is the integralcurve of X starting from x. Then $(X_t)_*$: $TM \longrightarrow TM$ is a group of diffeomorphisms which in the same way belongs to a vectorfield on TM. Denote this vectorfield by $\dot{X}$. Next, define for a vectorfield Y on M, the trivial extension $\bar{Y}$ of Y as the vectorfield on TM, which, restricted to M, is equal to Y and which, restricted to the fibers of TM, is zero. Then define

$$\dot{D} := \{\dot{X} \mid X \in D\} \cup \{\bar{Y} \mid Y \in D\}$$

If D is a k-dimensional involutive distribution we can give the following simple description of $\dot{D}$ in local coordinates. Take coordinates $(x_1, \ldots, x_n)$ for M (from now on we shall always assume M to be a n-dimensional manifold) such that

$$D = \{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}\}, \text{ with } k \leq n$$

Denote the corresponding coordinates for TM by $(x_1, --, x_n, \dot{x}_1, --, \dot{x}_n)$ ($\dot{x}_j : TM \longrightarrow \mathbb{R}$ is defined as: $\dot{x}_j(v) := dx_j(v)$, for $v \in TM$). Then

$$\dot{D} = \{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}, \frac{\partial}{\partial \dot{x}_1}, \ldots, \frac{\partial}{\partial \dot{x}_k}\}$$

ACKNOWLEDGEMENTS

The authors would like to thank Prof. J.C. Willems for many stimulating discussions. Also we like to thank Dr. J.H. van Schuppen for his encouragement.

§2. PRELIMINARIES

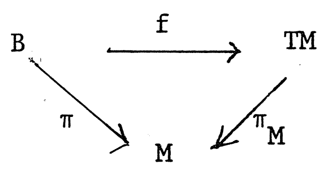
Before going to the problem of controlled invariance for general nonlinear systems, we will first review the definitions of nonlinear control systems we shall use henceforth. This new approach was proposed by WILLEMS [12], and elaborated in [10,8], and is related to recent proposals of BROCKETT [2]. In fact the problem centers around a *coordinate free* way of defining the equations

(2.1) $\dot{x} = f(x,u)$

where x is the state of the system, and u is the input. Usually this is done by looking at (2.1) as a family of globally defined vectorfields $f(\cdot,u)$ on the state space manifold, parametrized by u. However there are serious objections to this definition (cf.[2,10,12]) and moreover in many cases it happens that the input space is state dependent.

Therefore the most natural definition seems to be

<u>DEFINITION 2.1</u>. (Nonlinear control system) (cf.[12]).
A nonlinear control system Σ is a 3-tuple $\Sigma(M,B,f)$ with M a manifold, B a fiberbundle above M with projection $\pi: B \longrightarrow M$ and f a smooth map such that the diagram



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(π_M is the natural projection of TM on M)

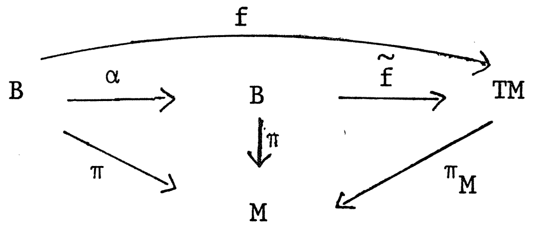
REMARK 1. M is to be considered as the state space while the fibers of B represent the (state dependent) input spaces. If we denote coordinates for M by x, and coordinates for B by (x,u), with u coordinates for the fibers, which are assumed to be m-dimensional then locally this definition comes down to (writing f as $(x,u) \longrightarrow (x,f(x,u))$, abuse of notation!)

$$\dot{x} = f(x,u)$$

REMARK 2. The usual approach is recovered by taking B a trivial bundle, i.e. $B = M \times U$, with U (most times) $\subset \mathbb{R}^m$.

REMARK 3. Note that our definition is also coordinate free with respect to the inputs, i.e. there are no a priori specified coordinates for the inputspace as in the usual approach where $U \subset \mathbb{R}^m$ and hence has already coordinates.

In this framework *feedback* can be defined in an appealing way. A system Σ (M,B,f) is *feedback equivalent* to a system $\tilde{\Sigma}(M,B,\tilde{f})$ iff there exists a bundle isomorphism $\alpha: B \longrightarrow B$ such that the diagram



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with the same abuse of notation as in remark 1 we shall write α in local coordinates as $(x,u) \longrightarrow (x,\alpha(x,u))$.

A special, but important, class of nonlinear systems is given by

DEFINITION 2.2. (Affine control system)
A nonlinear control system Σ (M,B,f) is an *affine* control system if B is a *vectorbundle* and the map f restricted to the fibers of B is an *affine* map into the fibers of TM. Also we assume, to avoid singularities, that f is an immersion.

REMARK 1. Because the fibers of B and TM are vector spaces, "affine" is well defined.

REMARK 2. If we take coordinates x for M and affine coordinates $(u_1,\dots,u_m)$ for the fibers of B (i.e. affine maps from the fibers into $\mathbb{R}$) then the system is locally described by

$$\dot{x} = A(x) + \sum_{i=1}^{m} u_i B_i(x)$$

where span $\{B_1(x),\dots,B_m(x)\}$ has constant dimension.

REMARK 3. Note that the class of feedbacks which preserve the affine structure consist of those $\alpha : B \longrightarrow B$ which restricted to the fibers are affine. Hence in coordinates as above

$$(x,u) \xrightarrow{\alpha} (x, M(x)u - \nu(x))$$

with M(x) a mxm matrix (nonsingular).

An equivalent definition is obtained by looking only at the image of the map f in TM. Because f is affine, the image of the fiber of B above a point $x \in M$ under f is an affine subspace of T_xM. Hence we obtain (cf.[8,9])

DEFINITION (2.2)'
An affine system on a manifold M is an affine distribution Δ.

REMARK. Define $\Delta_0 := \Delta - \Delta := \{X-Y \mid X,Y \in \Delta\}$. Then Δ_0 is a distribution, given in local coordinates as above by span $\{B_1(x), \dots, B(x)\}$. We denote the affine system by (Δ, Δ_0)

As already noted, our definition is also coordinate free with respect to the inputs. A local coordinatization of B is given by a trivializing chart, i.e. an open neighborhood 0 such that $\pi^{-1}(0) \simeq 0xF$, where $\simeq$ stands for isomorphic and F is the so called standard fiber. Notice that a coordinatization of 0 and F immediately gives a coordinatization (x,u) of $\pi^{-1}(0)$ such that x are coordinates for $0 \subset M$. We will call these kind of coordinates *fiber respecting*.
In general there are many trivializing charts, and hence many fiber respecting coordinatizations of B.

In this context it is easy to see that, given a local fiber respecting coordinatization of B, *feedback* $(x,u) \longmapsto (x, \alpha(x,u))$ can be interpreted as defining a new fiber respecting coordinatization (x,v) with $v = \alpha(x,u)$. This idea, translating feedback into choice of coordinates will be used in the sequel.

Finally we will define the extended system, introduced in [10], which will be important henceforth.

DEFINITION 2.3 (Extended system)
Let Σ (M,B,f) be a control system (def. 2.1). The *extended system*, denoted Σ^e (M,B,f), is an affine system (def(2.2)') constructed in the following way. Take as state spce the manifold B. Let $(\bar{x},\bar{v})$ be a point in B. We construct an affine subset $\Delta^e(\bar{x},\bar{v})$ of $T_{(\bar{x},\bar{v})}B$ as follows. The map $f:B \longrightarrow TM$ gives a vector $f(\bar{x},\bar{v}) \in T_{\bar{x}}M$. Now define

$$\Delta^e(\bar{x},\bar{v}) := \{X \in T_{(\bar{x},\bar{v})}B \mid \pi_* X = f(\bar{x},\bar{v})\}$$

Then Δ^e, in every (x,v) defined as above, is an affine distribution on B. It is easy to see that $\Delta_0^e := \Delta^e - \Delta^e = \{X \in TB \mid \pi_* X = 0\}$. Hence (Δ^e, Δ_0^e) is an affine system on B, denoted by $\Sigma^e(M,B,f)$.

§3. CONTROLLED INVARIANCE FOR NONLINEAR CONTROL SYSTEMS.

1. As we saw in the Introduction the underlying idea of (A,B)-invariance or controlled invariance is the following. Let D be a distribution, which is involutive and therefore induces a foliation. Then D is invariant with respect to the dynamics of a system $\dot{x} = f(x,u)$ if for any two points x_1 and x_2 on a same leaf of the foliation and for all input functions $u(\cdot)$ the integralcurves starting from x_1 and x_2 with a fixed $\bar{u}(\cdot)$ will be on the same leaf at the same time t. D is controlled invariant if this holds after applying feedback. The infinitesimal translation of this gives:

(PRELIMINARY) DEFINITION 3.1. Le Σ (M,B,f) be a control system. Let (x,u) be fiber respecting coordinates for B, in which the control system has the form $\dot{x} = f(x,u)$. A distribution D (involutive) on M is called *controlled invariant* if there exist a *feedback*, i.e. a bundle isomorphism

$\alpha: B \longrightarrow B$, in coordinates given by

$$(x,u) \overset{\alpha}{\longmapsto} (x, v := \alpha(x,u))$$

such that the control system in these new coordinates (x,v) given by $\dot{x} = \tilde{f}(x,v)$ satisfies

$$[\tilde{f}(\cdot,v), D] \subset D, \qquad \text{for every } v \text{ constant}$$

REMARK 1. This readily implies that for every time function $\bar{v}(\cdot)$ also $[\tilde{f}(\cdot,\bar{v}), D] \subset D$

The defect of this definition is that it already assumes a choice of input coordinates u. By doing this, it obscures the problem, because the former definition is easily seen (see end §2) to be equivalent to:

DEFINITION 3.2 (Controlled invariance)
Let Σ (M,B,f) be a control system. An involutive distribution D on M is called controlled invariant, if there are fiber respecting coordinates (x,u) for B, such that for every fixed u

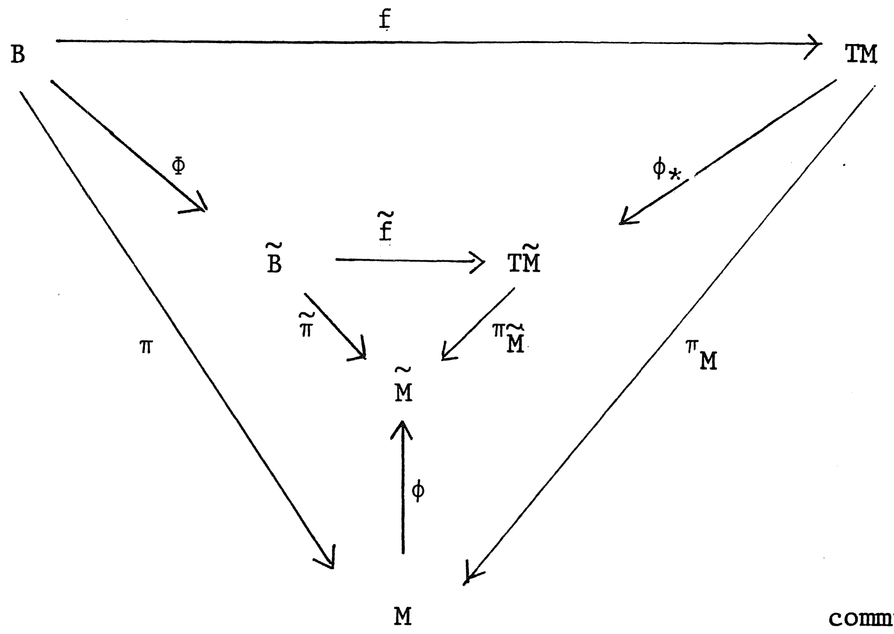
$$[f(\cdot,u), D] \subset D$$

where $\dot{x} = f(x,u)$ is the coordinate representation of Σ.

In fact, this last definition can be made totally coordinate free. For this we need the concept of an (integrable) connection, which will be treated in the next §. The final formulation will be given in th. 4.10.

In applications the concept of controlled invariance is many times used to factor out a part of the state space (cf.[4,5]). Def. 3.1 and 3.2 only ensure that locally the controlled invariant distribution can be factored out, and in fact there may be obstructions to do this globally (cf.[5]). Therefore we could also go the other way around and see what we mean by globally factoring out. Actually we will give a definition of a *quotient system* which locally implies controlled invariance.

DEFINITION 3.3. (Quotient system)
Let $\Sigma(M,B,f)$ be a control system. A control system $\tilde{\Sigma}(\tilde{M},\tilde{B},\tilde{f})$ is called a *quotient system* of Σ if there exist surjective submersions Φ and ϕ such that the diagram



(3.1)

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REMARK: compare this with the definition of minimality in [10].

In order to see that this definition locally implies controlled invariance, we have to make the following observations (cf. also [10]). Because Φ and ϕ are surjective submersions they induce the involutive distributions

$$E := \{X \in TB \mid \Phi_* X = 0\} \quad \text{resp.}$$
$$D := \{X \in TM \mid \phi_* X = 0\}$$

LEMMA 3.4. *Let* $\tilde{\Sigma}$ *be a quotient system of* Σ *as in def.* 3.3. *Let* D *be defined as above, then* D *is controlled invariant with respect to* Σ.

Proof: Diagram (3.1) has two commuting subdiagrams which respectively give

i) $\pi_* E = D$
ii) $f_* E \subset \dot{D}$

(because it is readily seen that ϕ_* induces the distribution $\dot{D}$, see §1). Now, the distribution E in fact defines fiber respecting coordinates above the leaves of the foliation generated by D, in the following way. Take a leaf F of the foliation. Restrict the bundle B to this leaf. Denote this new fiber bundle above F by B_F. Because $\pi_* E = D$ and E is involutive, E defines sections in B_F which project onto F. (the sections are the maximal integral manifolds of E). We can define coordinates u for the fibers of B_F, such that $u^{-1}(c)$, with c constant, are the sections of E in B_F.

Assume for a moment that Φ_* restricted to the fibers of B is bijective. Then one can see that, given an arbitrary fiber respecting coordinatization of $\tilde{B}$, the process above generates in a unique way fiber respecting coordinates for B. When Φ_* restricted to the fibers has a nontrivial null space, then for this part of the fibers we may arbitrarily complete the coordinates.

Finally, take coordinates $x_1, \ldots, x_n$

$$D = \{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}\}, \text{ with } k \le n.$$

Then construct fiber respecting coordinates (x,u) as above. In these coordinates

$$E = \{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}\}, \text{ and } f_* E \subset \dot{D} \text{ implies}$$

$$j^{th} \text{ component } \left(\frac{\partial f}{\partial x_i}(x,u)\right) = 0, \quad \text{with } \begin{array}{l} i = 1, \ldots, k \\ j = k+1, \ldots, n \end{array}$$

(where $\dot{x} = f(x,u)$ is the representation of Σ in (x,u)). This is equivalent with :

$$[f(\cdot,u), D] \subset D$$

<u>REMARK 1</u>. This proof also exactly shows which freedom one has in choosing coordinates (or in constructing feedback) such that in these coordinates D is invariant. In fact, loosely speaking, outside of the distribution

D the coordinates for the fibers are arbitrary. Above the distribution D the coordinates for the fibers are uniquely determined except for the part which Φ_* send to zero. This last part consists exactly of the inputs which are factored out in diagram (3.1) and which do not appear anymore in the quotientsystem.

REMARK 2.

An interesting special case of controlled invariance is when $f_*(\pi_*^{-1}(D)) \subset \dot{D}$. The proof above shows that in this situation D is invariant for all fiber respecting coordinatizations (x,u) of B. Also, it is easily seen that the system factored out by D, is autonomous (the inputspace consists of only one point).

Finally we can also relate controlled invariance in a system $\Sigma(M,B,f)$ with controlled invariance in the extended system $\Sigma^e(M,B,f)$ denoted by (Δ^e, Δ_0^e) (see def 2.3). In local coordinates it is easily proven (see also [10]):

LEMMA 3.5

$$f_* E \subset \dot{D},\ \textit{with}\ \pi_* E = D,\ \textit{is equivalent to}$$
$$[\Delta^e, E] \subset E + \Delta_0^e$$

As is known from recent work ([6,9], see the introduction) the last expression $[\Delta^e, E] \subset E + \Delta_0^e$ is equivalent to the controlled invariance of E with respect to the affine system (Δ^e, Δ_0^e). Therefore, combining conditions i) and ii) (in the proof of lemma 3.4) and lemma 3.5, gives

PROPOSITION 3.6

An involutive distribution D is controlled invariant with respect to Σ (M,B,f) *iff there exists an involutive distribution E, with* $\pi_* E = D$, *such that* E *is controlled invariant with respect to* Σ^e (M,B,f).

2. We have defined controlled invariance by requiring that, after

applying feedback, the modified dynamics leave the foliation invariant for *all* input functions. Of course, this demand might be too strong and we could be content if the foliation is invariant for only a *part* of the inputs. We will call this *degenerate controlled invariance*. Definitions 3.1 - 3.3 can be readily adapted to cover this situation. For instant we require in def. 3.1 no longer that α is an isomorphism, and in def. 3.3 we allow Φ to be a *partial* map.
However finding necessary and sufficient conditions for degenerate controlled invariance seems to be harder than for the (full) controlled invariant case, and we will leave it for the moment.
(Note that in the linear case degenerate controlled invariance implies full controlled invariance).

§4. CONTROLLED INVARIANCE AND CONNECTIONS

In this section we introduce the concept of a connection on a fiber bundle and we will relate this to the controlled invariance as introduced in section 3. For a more detailed treatment of a connection the reader is referred to the litterature on differential geometry. (See e.g. [3])

DEFINITION 4.1 Let $\pi : B \longrightarrow M$ be a smooth (fiber) bundle. A tangent vector $v \in T_pB$, $p \in B$, is said to be *vertical* if $\pi_{*p}(v) = 0$. $V(p)$ denotes the set of all vertical tangent vectors in p. A distribution H on B is said to be *horizontal* if $T_pB = H(p) \oplus V(p)$ for all $p \in B$.

REMARK: We see that $H \subset V(B)$ is horizontal implies that for all $p \in M$, $H(p)$ is a linear subspace of T_pB with the following properties:

$$\dim H(p) = \dim M$$
$$H(p) \cap V(p) = 0$$

π_* maps $H(p)$ isomorphically onto $T_{\pi(p)}M$.

Now the next definition will be clear:

DEFINITION 4.2

A curve $\sigma : \mathbb{R} \longrightarrow B$ is *horizontal* with respect to a horizontal distribution B if $\sigma'(t) \in H(\sigma(t))$ for all $t \in \mathbb{R}$., i.e. σ is an integral curve of a vector field which belongs to the horizontal distribution H on B.

We are now able to define a connection:

DEFINITION 4.3

Let $\pi : B \longrightarrow M$ be a smooth bundle, and let H be a horizontal distribution on B. H determines a *nonlinear connection* for $\pi : B \longrightarrow M$ which is defined by the following *lifting* procedure:

For every curve σ_1 $\mathbb{R} \longrightarrow M$ and each point $p \in \pi^{-1}(\sigma_1(0))$ there is a horizontal curve $\sigma : \mathbb{R} \longrightarrow B$ such that

$$\pi(\sigma(t)) = \sigma_1(t) \quad , \quad \sigma(0) = p$$

REMARKS

(i) We claim that every curve in M can globally be lifted to an integral curve of H, that is a *complete* nonlinear connection. In general a nonlinear connection is not complete; a curve in M can only locally be lifted to an integral curve of H. For the results of this paper we do not need the completeness, but it makes it somewhat easier to handle.

(ii) In the litterature there exists a couple of different definitions of a connection (introduced by different people). The above definition in fact defines the Ehresmann connection.

The next proposition gives a uniqueness property of the lift σ of σ_1 in definition 4.3.

PROPOSITION 4.5

Let H *be a horizontal distribution on B which defines a nonlinear connection for* $\pi : B \longrightarrow M$ *then the lift* $\sigma : \mathbb{R} \longrightarrow B$ *of a curve* $\sigma_1 : \mathbb{R} \longrightarrow M$ *defined by definition* 4.3 *is unique.*

And so we have as a direct consequence

PROPOSITION 4.6

Let H *be a horizontal distribution on B which defines a nonlinear connection for* $\pi : B \longrightarrow M$. *The connection determines a diffeomorphism between every two fibers of* π, i.e. *for all* $m_1, m_2 \in M$ *we have a diffeomorphism* $h : \pi^{-1}(m_1) \rightarrow \pi^{-1}(m_2)$

Next we will define an important class of nonlinear connections.

DEFINITION 4.7

Let $\pi : B \longrightarrow M$ be a *vector bundle*, i.e. for all $m \in M, \pi^{-1}(m)$ is a real vector space. A nonlinear connection defined by a horizontal distribution is called an *affine* connection if the fiber diffeomorphisms defined by the connection are affine isomorphisms between the vector space fibers.

Another useful property isgiven by:

DEFINITION 4.8

Let $\pi : B \longrightarrow M$ be a smooth bundle. Let H be a horizontal distribution on B. which defines a nonlinear connection. The connection is *integrable* if $[H,H] \subset H$, i.e. H is integrable as a vector field system.

The integrability of a connection of a horizontal distribution H implies that through each point $p \in B$ there passes an unique maximal connected integral submanifold M' of H (according to Frobenius'theorem) and this submanifold M' is transversal to the fibers of π, i.e. for all $q \in M'$ we have $T_q B = T_q M' \oplus V(q)$.

For later use we will investigate the integrability of an affine connection in detail.

According to definition 4.7 we can choose a (affine) coordinate system for $B : (x,v) = (x_1, \ldots, x_n, v_1, \ldots v_m)$ where small $(x_1, \ldots, x_n)$ is a coordinatization of M such that the linear subspace $H(x,v) \subset T_{(x,v)}B$ (def. 4.1) has a basis $X_1, \ldots, X_n$ of the following form (See [3]).

$$(4.1) \qquad X_i(x,v) = \frac{\partial}{\partial x_i} + [h_i(x) + K_i(x)v]\frac{\partial}{\partial v} \qquad i = 1, \ldots, n$$

where:

$h_i(x)$ is a m-vector

$K_i(x)$ is a m×m-matrix

$\frac{\partial}{\partial v} = (\frac{\partial}{\partial v_1}, \ldots, \frac{\partial}{\partial v_m})^t$ $((\ldots)^t$ denotes transposed)

Now $[H,H] \subset H$ implies

$$(4.2) \quad [X_i, X_j](x,v) = [\frac{\partial}{\partial x_i} + (h_i(x) + K_i(x)v)\frac{\partial}{\partial v}, \frac{\partial}{\partial x_j} + (h_j(x)+K_j(x)v)\frac{\partial}{\partial v}]$$

$$= [\frac{\partial h_j(x)}{\partial x_i} - \frac{\partial h_i(x)}{\partial x_j} + K_j(x)h_i(x) - K_i(x)h_j(x)]\frac{\partial}{\partial v} +$$

$$[\frac{\partial K_j(x)}{\partial x_i} - \frac{\partial K_i(x)}{\partial x_j} + K_j(x)K_i(x) - K_i(x)K_j(x)]v\frac{\partial}{\partial v}$$

$$= 0 \text{ (by (4.2)) for all } (x,v)$$

Therefore:

$$(4.3) \quad \frac{\partial h_j(x)}{\partial x_i} - \frac{\partial h_i(x)}{\partial x_j} + K_j(x)h_i(x) - K_i(x)h_h(x) = 0$$

and

$$(4.4) \quad \frac{\partial K_j(x)}{\partial x_i} - \frac{\partial K_i(x)}{\partial x_j} + K_j(x)K_i(x) - K_i(x)K_j(x) = 0$$

for $i,j = 1,\ldots,n$.

We can also work out the integrability condition (4.2) in a dual fashion, dual in the sense that we translate eq. (4.2) to the cotangent space of B. The integrability of H then guarantees that two 2-forms, called the *torsion tensor* and the *curvature tensor*, vanish (See e.g [3]). This requirement is exactly equivalent to the equations (4.3) and (4.4), and thus we will call this the *torsion equation* resp. the *curvature* *equation*. Conversely an integrable affine connection will be defined by the vector fields given by (4.1) where $h_i(x)$ and $K_i(x)$ satisfy the torsion and curvature equation.

Let D be an involutive distribution of fixed dimension k on M. Let H be an horizontal distribution on B which induces an integrable affine connection on $\pi : B \longrightarrow M$. Then this connection defines an unique lifting procedure for the distribution D (See def 4.3). In fact, choose a coordinate system $(x_1,\ldots,x_n)$ for M as in the Frobenius' theorem then D is spanned by the vectorfields $\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_k}$;

Let $H(x,v) \subset T_{(x,v)}B$ be spanned by (as in (4.1))

$$X_i = \frac{\partial}{\partial x_i} + [h_i(x) + K_i(x)v]\frac{\partial}{\partial v} \qquad i = 1,\ldots,n$$

Then the lifting of the distribution D gives a new involutive distribution D_ℓ which is spanned by

$$X_i = \frac{\partial}{\partial x_i} + [h_i(x) + K_i(x)v]\frac{\partial}{\partial v} \qquad i = 1,\ldots,k \tag{4.5}$$

REMARK:

The basis $X_1,\ldots,X_k$ for D_ℓ defined by (4.5) satisfies

$$\begin{aligned} &\frac{\partial h_j(x)}{\partial x_i} - \frac{\partial h_i(x)}{\partial x_j} + K_j(x)h_i(x) - K_i(x)h_j(x) = 0 \\ &\frac{\partial K_j(x)}{\partial x_i} - \frac{\partial K_i(x)}{\partial x_j} + K_j(x)K_i(x) - K_i(x)K_j(x) = 0 \\ &\qquad\qquad\qquad\qquad \text{for } i,j = 1,\ldots,k \end{aligned} \tag{4.6}$$

Now assume we have given an affine control system (Δ,Δ_0) as in definition 2.2'. We will denote the extended system (See def. 2.3) by Δ^e with 'input space' Δ_0^e.

After these preparations we state

THEOREM 4.9

D is a controlled invariant distribution for an affine system (Δ,Δ_0) *iff* *there exists an integrable affine connection for* $\pi: B \longrightarrow M$ *such that* $[\Delta^e, D_\ell] \subset D_\ell + \Delta_0^e$

PROOF

($\Leftarrow$) Suppose there exists an integrable affine connection for $\pi: B \rightarrow M$ with $[\Delta^e, D_\ell] \subset D_\ell + \Delta_0^e$. The horizontal system on B which defines the affine connection is according to (4.1) given by:

$$(4.1) \qquad X_i(x,v) = \frac{\partial}{\partial x_i} + [h_i(x) + K_i(x)v]\frac{\partial}{\partial v} \qquad i = 1,\ldots,n$$

where (x,v) is an affine coordinate system for B. By the integrability it follows that h_i and K_i satisfy the curvature and torsion equation (4.6). Let the control system om M be given by

$$(4.7) \qquad \dot{x}(t) = A(x(t)) + \Sigma_{i=1}^{m} v_i(t)B_i(x(t)) =: A(x(t)) + B(x(t))v(t)$$

where $B(x)$ a (n,m) - matrix with columns $B_i(x)$ and $v(t) =$ $= (v_1(t),\ldots,v_m(t))^t$. So the extended system has the form

$$(4.8) \qquad \begin{cases} \dot{x}(t) = A(x(t)) + B(x(t))v(t) \\ \dot{v}(t) = u(t) \end{cases}$$

From (4.5) we know that D_ℓ is spanned by

$$X_i(x,v) = \frac{\partial}{\partial x_i} + [h_i(x) + K_i(x)v]\frac{\partial}{\partial v} \qquad i = 1,\ldots,k$$

So from $[\Delta^e, D_\ell] \subset D_\ell + \Delta_0^e$ we deduce for all $i = 1,\ldots,k$

$$(4.9) \qquad \begin{aligned} &[(A(x)+B(x)v)\frac{\partial}{\partial v} + u\frac{\partial}{\partial v}, \frac{\partial}{\partial x_i} + (h_i(x)+ K_i(x)v)\frac{\partial}{\partial v}] \in \\ &\text{Span}\,\{\frac{\partial}{\partial x_i} + (h_i(x)+K_i(x)v)\frac{\partial}{\partial v}, \frac{\partial}{\partial v}, \; i = 1,\ldots,k\} \end{aligned}$$

Computing the Lie bracket of (4.9) leads to

$$[\frac{\partial A(x)}{\partial x_i} + \frac{\partial B(x)}{\partial x_i}v + B(x)h_i(x) + B(x)K_i(x)v]\frac{\partial}{\partial x} \in$$

$$\text{Span }\{\frac{\partial}{\partial x_i} + (h_i(x)+K_i(x)v)\frac{\partial}{\partial v}, \frac{\partial}{\partial v} \quad i = 1,\dots,k\}$$

$$\text{for all } i=1,\dots,k$$

Therefore

$$\text{(4.10)} \qquad j^{th} \text{ component of } \left(\frac{\partial A(x)}{\partial x_i} + \frac{\partial B(x)}{\partial x_i}v + B(x)h_i(x) + B(x)K_i(x)v\right) = 0$$

$$\text{for all } (x,v) \qquad j = k+1,\dots,n \quad i = 1,\dots,k$$

Thus we have

$$\text{(4.11)} \qquad j^{th} \text{ component of } \left(\frac{\partial A(x)}{\partial x_i} + B(x)h_i(x)\right) = 0 \qquad j = k+1,\dots,n \quad i = 1,\dots,k$$

$$\text{(4.12)} \qquad j^{th} \text{ component of } \left(\frac{\partial B(x)}{\partial x_i} + B(x)K_i(x)\right) = 0 \qquad j = k+1,\dots,n \quad i = 1,\dots,k$$

where $h_i(x)$ and $K_i(x)$ satisfy (4.3) and (4.4).

Now equation (4.11) together with the curvature equation (4.4) is an old friend (cf. Nijmeijer [9], Isidori et al [6]). We deduce from [6] and [9] that there exists a nonsingular (m,m) - matrix M(x) such that

$$\text{(4.13)} \qquad j^{th} \text{ component of } \left(\frac{\partial}{\partial x_i}[B(x).M(x)]\right) = 0 \qquad j = k+1,\dots,n \quad i = 1,\dots,n$$

Let

$$\text{(4.14)} \qquad \tilde{B}(x) := B(x)M(x)$$

Furthermore we see

$$j^{th} \text{ component of } \left(\frac{\partial}{\partial x_s}[B(x)h_i(x)]\right) =$$

$$j^{th} \text{ component of } \left(\frac{\partial B(x)}{\partial x_s} h_i(x) + B(x)\frac{\partial h_i(x)}{\partial x_s}\right) \qquad \underline{\underline{(4.12)}}$$

$$j^{th} \text{ component of } \left(-B(x)K_s(x)h_i(x) + B(x)\frac{\partial h_i(x)}{\partial x_s}\right)$$

and by the torsion equation (4.3), the last expression equals

$$j^{th} \text{ component of } \left(\frac{\partial}{\partial x_i}[B(x)h_s(x)]\right) \qquad \begin{array}{l} i,s = 1,\ldots,k \\ j = k+1,\ldots,n \end{array}$$

It follows, combining (4.11) and HIRSCHORN [4] - in fact Frobenius' theorem - that there exists an m vector $\nu(x)$ such that:

$$(4.15) \qquad j^{th} \text{ component of } \left(\frac{\partial}{\partial x_i}[A(x) + B(x)\nu(x)]\right) = 0 \qquad \begin{array}{l} j = k+1,\ldots,n \\ i = i,\ldots,k \end{array}$$

Thus if we use a feedback $v(t) = M(x)v(t) + \tilde{\nu}(x)$ for the system (4.7) we get

$$(4.16) \qquad \dot{x}(t) = A(x) + B(x)\nu(x) + \tilde{B}(x)\tilde{v}(t)$$

and so by using (4.13) and (4.15) we see that the distribution D is controlled invariant for the system (4.16)

($\Rightarrow$) Let D be a controlled invariant distribution for the system given by (4.7), where D is spanned by the vector fields $\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_k}$. For the construction of an integrable affine connection we need matrices $h_i(x)$ and $K_i(x)$ which satisfy the torsion and the curvature equation (4.3) and (4.4). From Isidori et al [6] and Nijmeijer [9] we know there exist (m,m)-matrices $K_i(x)$ such that

$$\frac{\partial B(x)}{\partial x_i} = B(x)\, K_i(x) \bmod D \qquad i = 1,\ldots,k$$

and these matrices $K_i(x)$ satisfy

$$\frac{\partial K_j(x)}{\partial x_i} - \frac{\partial K_i(x)}{\partial x_j} + K_j(x)K_i(x) - K_i(x)K_j(x) = 0 \qquad i,j = 1,\dots,k$$

According to [6] (see also Remark 1 after lemma 3.4) we can also define matrices $K_{k+1}(x),\dots,K_n(x)$ such that

$$(4.4) \qquad \frac{\partial K_j(x)}{\partial x_i} - \frac{\partial K_i(x)}{\partial x_j} + K_j(x)K_i(x) - K_i(x)K_j(x) = 0 \qquad i,j = 1,\dots,n$$

i.e. the curvature equation (4.4)!

Furthermore it follows from the fact that D is controlled invariant that

$$\frac{\partial A(x)}{\partial x_i} = B(x)h_i(x) \pmod{D} \qquad i = 1,\dots,k$$

where the vectors $h_i(x)$ satisfy

$$\frac{\partial h_i(x)}{\partial x_j} - \frac{\partial h_j(x)}{\partial x_i} + K_i(x)h_j(x) - K_j(x)h_i(x) = 0 \qquad i,j = 1,\dots,k$$

In the same way as in [6] we can define vectors $h_{k+1}(x),\dots,h_n(x)$ such that

$$(4.3) \qquad \frac{\partial h_i(x)}{\partial x_j} - \frac{\partial h_j(x)}{\partial x_i} + K_i(x)h_j(x) - K_j(x)h_i(x) = 0 \qquad i,j = 1,\dots,n$$

Thus the matrices $h_i(x)$ and $K_i(x)$ define an integrable affine connection □

Next we want to investigate the situation for a general control system (M,B,f) as defined in definition 2.1.

First we will formulate the integrability of a nonlinear connection in the same way as we have done for an affine connection. Following the notation as used after definition 4.8 we have that the nonlinear connection is spanned by vector fields $X_1,\dots,X_n$ of the following form

$$(4.17) \qquad X_i(x,v) = \frac{\partial}{\partial x_i} + h_i(x,v)\frac{\partial}{\partial v} \qquad i = 1,\dots,n$$

where $h_i(x,v)$ is a m-vector

$$\frac{\partial}{\partial v} = \left(\frac{\partial}{\partial v_1}, \dots, \frac{\partial}{\partial v_m}\right)^t$$

From the integrability we derive that

$$[\frac{\partial}{\partial x_i} + h_i(x,v)\frac{\partial}{\partial v}, \frac{\partial}{\partial x_j} + h_j(x,v)\frac{\partial}{\partial v}] =$$

$$[\frac{\partial h_j}{\partial x_i}(x,v) - \frac{\partial h_i}{\partial x_j}(x,v) + \frac{\partial h_j}{\partial v}(x,v).h_i(x,v) - \frac{\partial h_i}{\partial v}(x,v).h_j(x,v)]\frac{\partial}{\partial v} = 0$$

Remark: $\frac{\partial h_i}{\partial v}(x,v)$ is an (m,m)-matrix consisting of the columns $\frac{\partial h_i}{\partial v_j}(x,v)$.
Therefore:

$$\frac{\partial h_j}{\partial x_i}(x,v) - \frac{\partial h_i}{\partial x_j}(x,v) + \frac{\partial h_j}{\partial v}(x,v).h_i(x,v) - \frac{\partial h_i}{\partial v}(x,v).h_j(x,v) = 0 \tag{4.18}$$

Now the following theorem will be the direct generalization of theorem 4.9:

THEOREM 4.10.

D is a controlled invariant distribution for a control system $\Sigma(M,B,f)$ *iff there exists an integrable nonlinear connection for* $\pi:B \to M$ *such that* $[\Delta^e,D] \subset D_\ell + \Delta^e_0$.

PROOF. (⇐) Suppose there exists an integrable nonlinear connection for $\pi:B \to M$ with $[\Delta^e,D_\ell] \subset D_\ell + \Delta^e_0$. The horizontal system on B which defines the connection is according to (4.17) given by

$$X_i(x,v) = \frac{\partial}{\partial x_i} + h_i(x,v)\frac{\partial}{\partial v} \qquad i = 1,\dots,n$$

where the $h_i(x,v)$ satisfy (4.18).

Let the control system on M be given by

$$x(t) = f(x(t),v(t)) \tag{4.19}$$

So the extended system has the form

$$\begin{cases} \dot{x}(t) = f(x(t), v(t)) \\ \dot{v}(t) = u(t) \end{cases} \tag{4.20}$$

As in (4.5) the distribution D_ℓ is spanned by

$$X_i(x,v) = \frac{\partial}{\partial x_i} + h_i(x,v)\frac{\partial}{\partial v} \qquad i = 1,\dots,k$$

So from $[\Delta^e, D_\ell] \subset D_\ell + \Delta_0^e$ we deduce that:

$$\text{(4.21)} \qquad j^{\text{th}} \text{ component of } \left(\frac{\partial f(x,v)}{\partial x_i} + \frac{\partial f(x,v)}{\partial v} \cdot h_i(x,v)\right) = 0 \qquad \begin{matrix} j = 1,\dots,k \\ j = k+1,\dots,n \end{matrix}$$

where the $h_i(x,v)$ satisfy (4.18).

Now consider the set of partial differential equations

$$\text{(4.22)} \qquad \begin{cases} \dfrac{\partial \alpha}{\partial x_i}(x,\tilde{v}) = h_i(x,\alpha(x,\tilde{v})) & i = 1,\dots,n \\ \alpha(0,0) = I_{m.m} & \text{(See [11])} \end{cases}$$

From Frobenius' theorem (See [11]) we know that there exists a unique solution $\alpha(x,\tilde{v})$ of (4.22) iff the integrability condition (4.18) is satisfied. Hence if we apply a feedback $v(t) = \alpha(x,\tilde{v}(t))$ to the system (4.19) we get

$$\text{(4.23)} \qquad \dot{x}(t) = f(x,\alpha(x,\tilde{v}(t)))$$

and by using (4.21) we see that the distribution D is controlled invariant for (4.23).

($\Rightarrow$) Let D be a controlled invariant distribution for the system given by (4.19) where D is spanned by $\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_k}$. For the construction of an integrable nonlinear connection we need matrices $h_i(x,v)$ which satisfy (4.18). By the fact that D is controlled invariant we know that there exists a(m,m)-matrix $\alpha(x,\tilde{v})$ with $\frac{\partial \alpha(x,\tilde{v})}{\partial \tilde{v}}$ nonsingular - i.e. the map $\tilde{v} \mapsto v := \alpha(x,\tilde{v})$ is invertible. We will denote - abuse of notation! - the inverse of this map by $\alpha^{-1}(x,v)$.

Define

$$\text{(4.24)} \qquad h_i(x,v) := \left(\frac{\partial \alpha}{\partial x_i}(x,\tilde{v})\right)\bigg|_{\tilde{v} = \alpha^{-1}(x,v)} \qquad i = 1,\dots,n$$

Now from (4.24) we see that

$$\frac{\partial\alpha}{\partial x_i}(x,\tilde{v}) = h_i(x,\alpha(x,\tilde{v})) \tag{4.22}$$

and therefore:

$$\frac{\partial h_j}{\partial x_i}(x,v) - \frac{\partial h_i}{\partial x_j}(x,v) + \frac{\partial h_j}{\partial v}(x,v).h_i(x,v) - \frac{\partial h_i}{\partial v}(x,v),h_j(x,v) = 0 \tag{4.18}$$

$$i,j = 1,\dots,n$$

i.e. the integrability condition for a nonlinear connection defined by

$$X_i(x,v) = \frac{\partial}{\partial x_i} + h_i(x,v)\frac{\partial}{\partial v} \qquad i = 1,\dots,n \tag{4.17}$$ □

To conclude this section we want to give conditions under which a distribution D is controlled invariant for a control system $\Sigma(M,B,f)$. First we will solve this problem in a local fashion (coordinate dependent) and afterwards we give the main theorem 4.13. Let, as before, the control system be given (locally) by $\dot{x} = f(x,v)$ and let

$$\mathrm{Span}\{\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_k}\} = D$$

Suppose that there exist m-vectors $m_i(x,v)$ $(i = 1,\dots,k)$ such that

$$s^{th} \text{ component of } \left(\frac{\partial f}{\partial x_i}(x,v) + \frac{\partial f}{\partial v}(x,v).m_i(x,v)\right) = 0 \tag{4.25}$$

$$i = 1,\dots,k$$
$$s = k+1,\dots,n$$

Then it follows

$$s^{th} \text{ component of } \left(\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}(x,v) + \frac{\partial f}{\partial v}(x,v).m_i(x,v)\right)\right) =$$

$$s^{th} \text{ component of } \left(\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}(x,v) + \frac{\partial f}{\partial v}(x,v).m_j(x,v)\right)\right)$$

$$i,j = 1,\dots,k$$
$$s = k+1,\dots,n.$$

Hence

$$s^{th} \text{ component of } \left(\frac{\partial^2 f}{\partial x_j \partial x_i}(x,v) + \frac{\partial^2 f}{\partial x_j \partial v}(x,v) m_i(x,v) + \right.$$

$$\left. + \frac{\partial f}{\partial v}(x,v) \cdot \frac{\partial m_i}{\partial x_j}(x,v)\right) =$$

$$s^{th} \text{ component of } \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x,v) + \frac{\partial^2 f}{\partial x_i \partial v}(x,v) m_j(x,v) + \right.$$

$$\left. \frac{\partial f}{\partial v}(x,v) \cdot \frac{\partial m_j}{\partial x_i}(x,v)\right)$$

Therefore

$$(4.26) \quad \begin{aligned} &s^{th} \text{ component of } \left(\frac{\partial^2 f}{\partial x_j \partial v}(x,v) \cdot m_i(x,v) + \frac{\partial f}{\partial v}(x,v) \cdot \frac{\partial m_i}{\partial x_j}(x,v)\right) = \\ &s^{th} \text{ component of } \left(\frac{\partial^2 f}{\partial x_i \partial v}(x,v) \cdot m_j(x,v) + \frac{\partial f}{\partial v}(x,v) \cdot \frac{\partial m_j}{\partial x_i}(x,v)\right) \end{aligned}$$

$$i,j = 1,\dots,k$$
$$s = k+1,\dots,n$$

Now

$$s^{th} \text{ component of } \left(\frac{\partial^2 f}{\partial x_j \partial v}(x,v) \cdot m_i(x,v)\right) = \frac{\partial}{\partial v}\left(\frac{\partial f^s}{\partial x_j}(x,v)\right) \cdot m_i(x,v)$$

$$\underset{(4.25)}{=} \frac{\partial}{\partial v}\left(- \frac{\partial f^s}{\partial v}(x,v) \cdot m_j(x,v)\right) \cdot m_i(x,v)$$

$$(4.27) \quad \begin{aligned} &= - m_i^t(x,v) \frac{\partial^2 f^s}{\partial v^2}(x,v) m_j(x,v) - \\ &- \frac{\partial f^s}{\partial v}(x,v) \cdot \frac{\partial m_j}{\partial v}(x,v) \cdot m_i(x,v) \end{aligned}$$

Substituting (4.27), and a similar expression for the left hand side of (4.26), in (4.26) leads to

$$\frac{\partial f^s}{\partial v}(x,v).\frac{\partial m_i}{\partial x_j}(x,v) - m_i^t(x,v)\frac{\partial^2 f^s}{\partial v^2}(x,v)m_j(x,v) -$$

$$\frac{\partial f^s}{\partial v}(x,v).\frac{\partial m_j}{\partial v}(x,v).m_i(x,v)$$

$$= \frac{\partial f^s}{\partial v}(x,v),\frac{\partial m_j}{\partial x_i}(x,v) - m_j^t(x,v)\frac{\partial^2 f^s}{\partial v^2}(x,v)m_i(x,v) -$$

$$\frac{\partial f^s}{\partial v}(x,v)\frac{\partial m_i}{\partial v}(x,v).m_j(x,v) \qquad = 0$$

$$i,j = 1,\ldots,k$$

$$s = k+1,\ldots,n$$

So

(4.28)
$$\frac{\partial f^s}{\partial v}(x,v)[\frac{\partial m_i}{\partial x_j}(x,v) - \frac{\partial m_j}{\partial v}(x,v)m_i(x,v) -$$

$$\frac{\partial m_j}{\partial x_i}(x,v) + \frac{\partial m_i}{\partial v}(x,v).m_j(x,v)] \qquad = 0$$

$$i,j = 1,\ldots,k$$

$$s,k+1,\ldots,n$$

Suppose that the matrix

(4.29) $\left(\frac{\partial f^s}{\partial v}(x,v)\right)_{s=k+1,\ldots,n}$ has full rank

then (4.28) leads to

$$\frac{\partial m_i}{\partial x_j}(x,v) - \frac{\partial m_j}{\partial x_i}(x,v) + \frac{\partial m_i}{\partial v}(x,v)m_j(x,v) -$$

(4.30)
$$\frac{\partial m_j}{\partial v}(x,v)m_i(x,v) = 0 \qquad i,j = 1,\ldots,k$$

i.e. a *partial* integrability condition as in (4.18)!

We need the following simple but crucial lemma

LEMMA 4.11. *The set of partial differential equations*

$$(4.31)\qquad \begin{cases} \dfrac{\partial\alpha}{\partial x_i}(x,\tilde{v}) = m_i(x,\alpha(x,\tilde{v})) & i = 1,\ldots,k \\ \alpha(0,0) = I_{m,m} \end{cases}$$

has a solution.

REMARK. This set of partial differential equations (4.31) is nearly the same as in (4.22). We cannot immediately apply Frobenius' theorem, while not all partial derivatives of α are specified (Compare [9]).

PROOF. There exist $m_{k+1}(x,v),\ldots,m_n(x,v)$ such that

$$(4.18)\qquad \begin{aligned} &\frac{\partial m_i}{\partial x_j}(x,v) - \frac{\partial m_j}{\partial x_i}(x,v) + \frac{\partial m_i}{\partial v}(x,v).m_j(x,v) - \\ &\frac{\partial m_j}{\partial v}(x,v).\ m_i(x,v) = 0 \qquad i,j = 1,\ldots,n \end{aligned}$$

(See [9], See also equation (4.4); this follows from the fact that the distribution D = TM is controlled invariant).

Finally apply Frobenius' theorem. □

COROLLARY 4.12. *If there exist* $m_i(x,v)$ (i=1,...,k) *which satisfy* (4.2) *and condition* (4.29) *is fulfilled then the distribution* D *is controlled invariant for the system* $\dot{x}(t) = f(x(t),\alpha(x(t),\tilde{v}(t)))$, *where* $\alpha(x,\tilde{v})$ *is defined by lemma* 4.11.

Finally we will give in a coordinate-free way the analogue of [6] and [9]. for a nonlinear control system $\Sigma(M,B,f)$. Recall the definition of $\dot{D}$ for a given distribution D (See Notation §1).

THEOREM 4.13. *Let* $\Sigma(M,B,f)$ *be a nonlinear control system and let* D *be an involutive distribution of fixed dimension on* M. *If* $f_*(\Delta_0^e) \cap \dot{D}$ *has fixed dimension then we have the following equivalence:*
D *is locally controlled-invariant iff*

$$(4.31)\qquad f_*(\pi_*^{-1}(D)) \subset \dot{D} + f_*(\Delta_0^e)$$

PROOF. (⇒) Direct
(⇐) Work out in local coordinates, and suppose $f_*(\Delta_0^e) \cap \dot{D} = 0$. The the result is given by Corollary 4.12. In a similar way as in ISIDORI et al [6] and NIJMEIJER [9] we derive the same result in the case that $f_*(\Delta_0^e) \cap \dot{D}$ has fixed dimension. □

§5. CONCLUSION

The main result of this paper is theorem 4.13 which gives necessary and sufficient conditions for controlled invariance in general nonlinear systems. With the aid of this theorem the Disturbance Decoupling Problem (see [13]) for instance can be readily solved, analogous to [4,5]. Very surprising results are theorems 4.9 and 4.10 where the concept of controlled invariance is directly related to the well known differential geometric notion of an integrable connection.

It would be interesting to look for similar results in the case of degenerate controlled invariance, as sketched in §3, section 2.

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