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COMPUTER AIDED COMPLEXITY CLASSIFICATION OF DETERMINISTIC SCHEDULING PROBLEMS

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ABSTRACT

We present the results obtained by a computer program that has been used to maintain a record of the known complexity results for a class of 4,536 machine scheduling problems. The input of the program consists of a listing of known "easy" problems and a listing of known "hard" problems. The program employs the structure of the problem class to determine the implications of these results. The output provides a listing of essential results in the form of maximal easy and minimal hard problems as well as listings of minimal and maximal open problems, which are helpful in indicating the direction of future research. The reader is assumed to be familiar with a companion paper by the same authors, "Computer aided complexity classification of combinatorial problems" (Report BW 137, Mathematisch Centrum, Amsterdam, 1981). In that paper, the program is described, its application to a restricted class of 120 single-machine problems is demonstrated, and possible refinements and extensions to other research areas are suggested.

KEY WORDS & PHRASES: *machine scheduling, computational complexity, polynomial algorithm, NP-hardness, computer program*

1. INTRODUCTION

In this report, we present the results obtained by a computer program that has been used to maintain a record of the known complexity results for a class of 4,536 deterministic machine scheduling problems.

The input of the program consists of a listing of known *easy* problems, which can be solved in polynomial time, and a listing of known *hard* problems, which are NP-hard. The program systematically employs a certain *partial ordering* defined on the problem class to determine the implications of these results. The output provides a count of *easy*, *open* and *hard* problems as well as complete listings of *maximal easy*, *minimal open*, *maximal open* and *minimal hard* problems.

We assume that the reader is familiar with the contents of a companion paper [Lageweg, Lawler, Lenstra & Rinnooy Kan 1981]. In that paper, we describe the program MSPCLASS in general terms, we demonstrate its application to a restricted class of 120 single-machine scheduling problems, we discuss the benefits of this approach, and we suggest possible refinements and extensions to other research areas.

Section 2 contains a schematic definition of our class of 4,536 machine scheduling problems and Section 3 presents the state of the art as exhibited by application of the program to this class. It is our intention to publish a next edition of this report whenever newly obtained results have changed the picture significantly.

2. THE CLASS OF MACHINE SCHEDULING PROBLEMS

The class of machine scheduling problems to which the program MSPCLASS is applied is defined in this section (*cf.* [Graham, Lawler, Lenstra & Rinnooy Kan 1979]).

Suppose that n *jobs* J_j (j = 1,...,n) have to be processed on m *machines* M_i (i = 1,...,m). Each machine can process at most one job at a time and each job can be processed on at most one machine at a time. Various job, machine and scheduling characteristics are reflected by a three-field classification $\alpha|\beta|\gamma$, to be introduced below. Let $\circ$ denote the empty symbol.

2.1. Machine environment

The first field $\alpha = \alpha_1\alpha_2$ specifies the machine environment.

If $\alpha_1 \in \{\circ,P,Q,R\}$, each J_j consists of a *single operation* that can be processed on *any* M_i; the processing time of J_j on M_i is p_{ij}. The four values are characterized as follows.

- $\alpha_1 = \circ$: *Single machine*; $m = 1$; $p_{1j} = p_j$.
- $\alpha_1 = P$: *Identical parallel machines*; $p_{ij} = p_j$ $(i = 1,\ldots,m)$.
- $\alpha_1 = Q$: *Uniform parallel machines*; $p_{ij} = p_j/s_i$ for a given *speed* s_i of M_i $(i = 1,\ldots,m)$.
- $\alpha_1 = R$: *Unrelated parallel machines*.

If $\alpha_1 \in \{O,F,J\}$, each J_j consists of m_j *operations* $O_{1j},\ldots,O_{m_jj}$; O_{ij} has to be processed on a *given* machine μ_{ij} during p_{ij} time units. The three values are characterized as follows.

- $\alpha_1 = O$: *Open shop*; $m_j = m$; $\mu_{ij} = M_i$ $(i = 1,\ldots,m)$; the order in which the operations are executed is immaterial.
- $\alpha_1 = F$: *Flow shop*; $m_j = m$; $\mu_{ij} = M_i$ $(i = 1,\ldots,m)$; $O_{i-1,j}$ has to be completed before O_{ij} can start $(i = 2,\ldots,m)$.
- $\alpha_1 = J$: *Job shop*; $\mu_{i-1,j} \neq \mu_{ij}$ and $O_{i-1,j}$ has to be completed before O_{ij} can start $(i = 2,\ldots,m_j)$.

If $\alpha_2 \in \{1,2,3\}$, then m is *constant* and equal to α_2. If $\alpha_2 = \circ$, then m is *variable*.

2.2. Job characteristics

The second field $\beta \subset \{\beta_1,\ldots,\beta_5\}$ indicates a number of job characteristics, which are defined as follows.

(1) $\beta_1 \in \{pmtn,\circ\}$.

- $\beta_1 = pmtn$: *Preemption* is allowed; the processing of any operation may be interrupted and resumed at a later time (or, if $\alpha_1 \in \{P,Q,R\}$, at the same time on a different machine) without penalty.
- $\beta_1 = \circ$: Preemption is not permitted.

(2) $\beta_2 \in \{prec,tree,\circ\}$.

- $\beta_2 = prec$: *Precedence constraints* between the jobs are given in the form of an acyclic digraph $G = (\{1,...,n\},A)$; if $(j,k) \in A$, then J_j has to be completed before J_k can start.

- β_2 = *tree*: G is a *rooted tree* with either indegree at most one for each vertex or outdegree at most one for each vertex.
- $\beta_2 = \circ$: G has no arcs (the jobs are *independent*).

(3) $\beta_3 \in \{r_j,\circ\}$.

- $\beta_3 = r_j$: Each J_j has a given nonnegative integer *release date* r_j, on which it becomes available for processing.
- $\beta_3 = \circ$: All $r_j = 0$.

(4) $\beta_4 \in \{d_j,\circ\}$.

- $\beta_4 = d_j$: Each J_j has a given nonnegative integer *deadline* d_j, by which it has to be completed.
- $\beta_4 = \circ$: All $d_j = \infty$.

(5) $\beta_5 \in \{p_j=1,\circ\}$.

- $\beta_5 = p_j=1$: Each operation has a *unit processing requirement*, *i.e.*, if $\alpha_1 \in \{\circ,P,Q\}$, then all $p_j = 1$, and if $\alpha_1 \in \{R,O,F,J\}$, then all $p_{ij} = 1$.
- $\beta_5 = \circ$: All p_j (p_{ij}) are given arbitrary nonnegative integers.

2.3. Optimality criterion

The third field γ refers to the optimality criterion chosen. Each J_j may have a given nonnegative integer *due date* $\bar{d}_j$ and/or a given integer *weight* w_j. Any schedule defines for each J_j:

- a *completion time* C_j;
- a *lateness* $L_j = C_j-\bar{d}_j$;
- a *tardiness* $T_j = \max\{0,C_j-\bar{d}_j\}$;
- a *unit penalty* $U_j = 0$ if $C_j \le \bar{d}_j$, $U_j = 1$ if $C_j > \bar{d}_j$.

The optimality criterion, which is to be minimized, is one of the following eight types.

- $\gamma = C_{max}$: the *maximum completion time* $\max\{C_j\}$;
- $\gamma = L_{max}$: the *maximum lateness* $\max\{L_j\}$;
- $\gamma = \sum C_j$: the *total completion time*;
- $\gamma = \sum w_jC_j$: the *total weighted completion time*;
- $\gamma = \sum T_j$: the *total tardiness*;

- $\gamma = \sum w_jT_j$: the *total weighted tardiness;*
- $\gamma = \sum U_j$: the *number of late jobs;*
- $\gamma = \sum w_jU_j$: the *weighted number of late jobs.*

2.4. Number of problems

Each eight-tuple $\alpha_1\alpha_2|\beta_1,\beta_2,\beta_3,\beta_4,\beta_5|\gamma$ corresponds to a machine scheduling problem. Several combinations of values for the components are excluded, for the following reasons.

- Clearly, $\alpha_1 = \circ$ if and only if $\alpha_2 = 1$.
- If $\alpha_1 = R$, then $\beta_5 = \circ$, since the choice $\beta_5 = p_j{=}1$ leads to a problem with $\alpha_1 = P$.
- If $\gamma \in \{C_{max}, L_{max}\}$, then $\beta_4 = \circ$, since for the combinations $\beta_4|\gamma$ the choices $d_j|C_{max}$, $|L_{max}$ and $d_j|L_{max}$ yield equivalent problems (*cf.* [Lageweg, Lawler, Lenstra & Rinnooy Kan 1981]).

Further, we have decided to eliminate the choice $\beta_5 = p_j{=}1$ in the case that $\beta_1 = pmtn$: preemptable jobs have arbitrary processing requirements.

It is not hard to see that, under these restrictions, the class of problems defined by the above classification contains 4,536 different problem types.

2.5. Partial ordering

The program MSPCLASS employs a certain partial ordering $\rightarrow$ of the problem class, which has the property that, if $P \rightarrow P'$ for two problems P and P', then P is *reducible* to P'. This relation is defined as follows.

Each member of our class of machine scheduling problems corresponds to a seven-tuple $(\pi_i)_{i=0}^{6}$, where π_i is a vertex of the digraph G_i shown in Figure 1 $(i = 0,\ldots,6)$; π_0 corresponds to α, π_i to β_i $(i = 1,\ldots,5)$, and π_6 to γ. There is an arc from π to π' if π' is a direct generalization of π. For two problems $P = (\pi_i)_{i=0}^{6}$ and $P' = (\pi'_i)_{i=0}^{6}$, we have $P \rightarrow P'$ if either $\pi_i = \pi'_i$ or G_i contains a directed path from π_i to π'_i, for $i = 0,\ldots,6$ (*cf.* [Lageweg, Lawler, Lenstra & Rinnooy Kan 1981]).

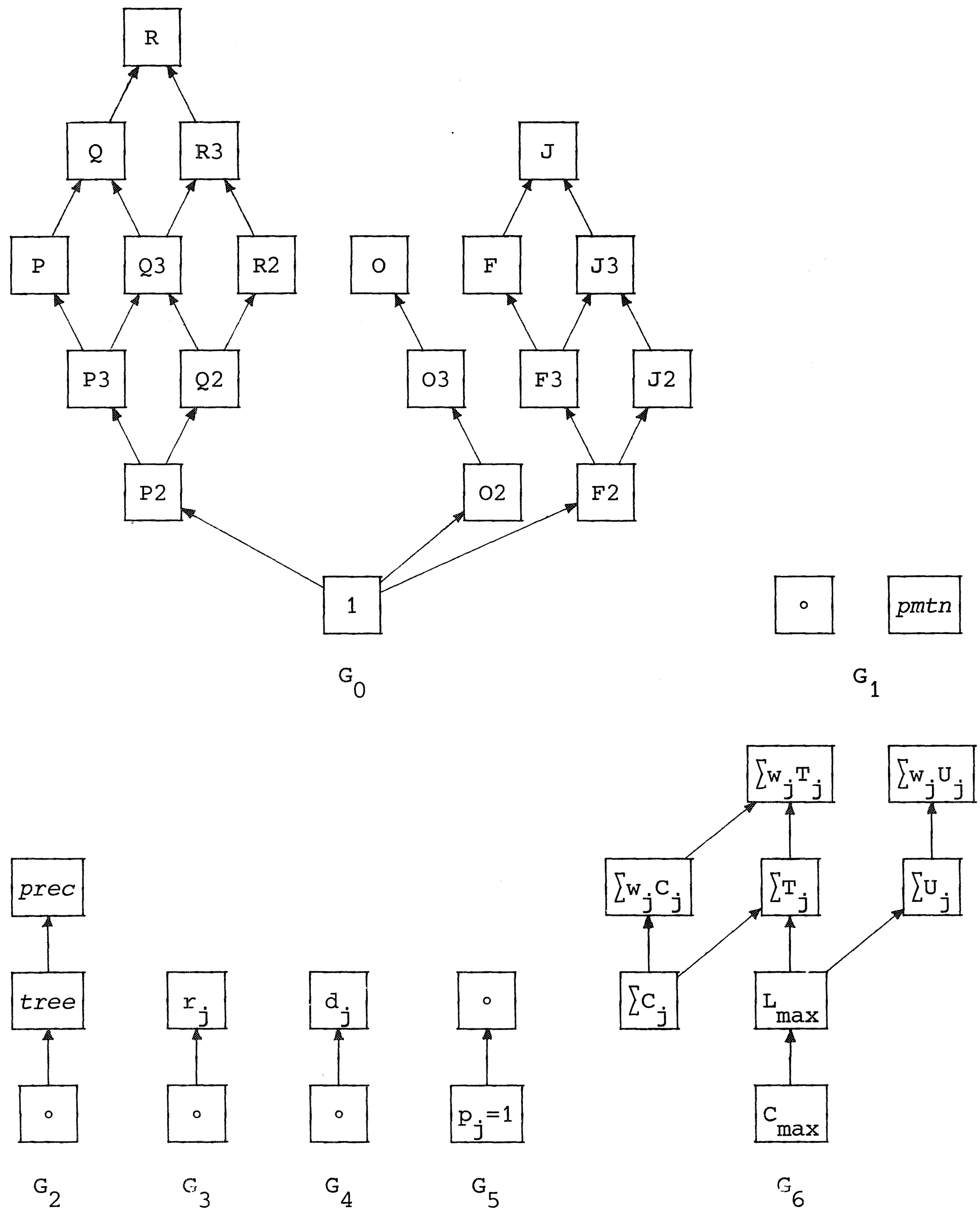

Figure 1. The graphs G_i $(i = 0,\ldots,6)$.

3. THE STATE OF THE ART

The program MSPCLASS has been implemented in PASCAL on the Control Data Cyber 170-750 of the SARA Computing Centre in Amsterdam. Its application to the class of machine scheduling problems defined in the previous section is demonstrated below.

We note that separate runs are made for two different inputs, corresponding to different assumptions about the encoding of problem instances containing numerical data. Under a standard *binary* encoding, the easy problems are solvable in *strictly polynomial* time and the hard problems are NP-hard in the *ordinary* sense. Under a *unary* encoding, the easy problems are solvable in *pseudopolynomial* time and the hard problems are NP-hard in the *strong* sense (*cf.* [Garey & Johnson 1978; Lenstra & Rinnooy Kan 1979]).

Moreover, the program deals separately with *nonpreemptive* and *preemptive* problems, which reflects the fact that G_1 consists of two isolated vertices. Many results for preemptive scheduling, however, follow immediately from those for nonpreemptive scheduling. For example, there is no advantage to preemption for problems with a single machine and equal release dates [McNaughton 1959] and for problems of the types $P||\sum w_jC_j$ [McNaughton 1959] and $F2||C_{max}$ [Gonzalez & Sahni 1978].

The output of the latest runs is shown on the next pages; note that the value of the component β_1 appears in the heading of the listings and not in each individual problem type. The listings present useful additional information in the form of references to the literature where the results in question can be found, an indication of general algorithmic techniques applicable to some easy problems, the symbol # for open problems that are both minimal and maximal, and acronyms of convenient "starting problems" for transformations in the case of hard problems.

Combining the results for nonpreemptive and preemptive scheduling, we get an overview of the total numbers of easy, open and hard problems as shown in Table 1.

	binary		unary	
easy	416	(9%)	471	(10%)
open	432	(10%)	547	(12%)
hard	3688	(81%)	3518	(78%)
total	4536	(100%)	4536	(100%)

Table 1. Summary of results.

NONPREEMPTIVE SCHEDULING, BINARY ENCODING

DATE: 81:03:13

NUMBER OF PROBLEMS	MINIMAL	TOTAL	MAXIMAL
EASY		319	29
OPEN	35	310	18
HARD	42	2311	
TOTAL		2940	

MAXIMAL EASY PROBLEMS

1/PREC,RJ,DJ,PJ=1/SUMCJ	LAWLER: COFFMAN GRAHAM 1972
1//SUMUJ	MOORE 1968
1/DJ/SUMCJ	SMITH 1956
1/TREE/SUMWJCJ	HORN 1972; SIDNEY 1975
1/PREC/LMAX	LAWLER 1973
1/PREC,RJ/CMAX	LAWLER 1973
P2/PREC,PJ=1/SUMCJ	GAREY 1975: COFFMAN GRAHAM 1972
P2/PREC,RJ,PJ=1/LMAX	GAREY JOHNSON 1977
P/RJ,DJ,PJ=1/SUMWJTJ	LAWLER 1964 (TRANSPORTATION PROBLEM)
P/RJ,DJ,PJ=1/SUMWJUJ	LAWLER 1964 (TRANSPORTATION PROBLEM)
P/TREE,PJ=1/CMAX	HU 1961
Q3/RJ,PJ=1/SUMCJ	LAGEWEG (DYNAMIC PROGRAMMING)
Q/DJ,PJ=1/SUMWJTJ	GRAHAM LAWLER LENSTRA RINNOOY KAN 1979 (TRANSPORTATION PROBLEM)
Q/DJ,PJ=1/SUMWJUJ	GRAHAM LAWLER LENSTRA RINNOOY KAN 1979 (TRANSPORTATION PROBLEM)
Q/RJ,PJ=1/CMAX	LAWLER LENSTRA
R//SUMCJ	HORN 1973; BRUNO COFFMAN SETHI 1974
O2/RJ,PJ=1/LMAX	LAWLER
O2//CMAX	GONZALEZ SAHNI 1976
O3/PJ=1/SUMUJ	LAWLER
O3/RJ,PJ=1/CMAX	LAWLER
O/PJ=1/CMAX	LENSTRA
O/PJ=1/SUMWJCJ	LENSTRA
F2/TREE,PJ=1/CMAX	LAGEWEG
F2/TREE,PJ=1/SUMCJ	LAGEWEG
F2//CMAX	JOHNSON 1954
F/RJ,DJ,PJ=1/SUMWJTJ	LAGEWEG (TRANSPORTATION PROBLEM)
F/RJ,DJ,PJ=1/SUMWJUJ	LAGEWEG (TRANSPORTATION PROBLEM)
J2/PJ=1/LMAX	BRUCKER 1981
J2/RJ,PJ=1/CMAX	BRUCKER 1981

NONPREEMPTIVE SCHEDULING, UNARY ENCODING

DATE: 81:03:13

NUMBER OF PROBLEMS	MINIMAL	TOTAL	MAXIMAL
EASY		357	32
OPEN	40	344	22
HARD	44	2239	
TOTAL		2940	

MAXIMAL EASY PROBLEMS

1/PREC,RJ,DJ,PJ=1/SUMCJ	LAWLER: COFFMAN GRAHAM 1972
1//SUMTJ	LAWLER 1977
1/DJ/SUMCJ	SMITH 1956
1/TREE/SUMWJCJ	HORN 1972; SIDNEY 1975
1/PREC/LMAX	LAWLER 1973
1/PREC,RJ/CMAX	LAWLER 1973
P2/PREC,PJ=1/SUMCJ	GAREY 1975: COFFMAN GRAHAM 1972
P2/PREC,RJ,PJ=1/LMAX	GAREY JOHNSON 1977
P/RJ,DJ,PJ=1/SUMWJTJ	LAWLER 1964 (TRANSPORTATION PROBLEM)
P/RJ,DJ,PJ=1/SUMWJUJ	LAWLER 1964 (TRANSPORTATION PROBLEM)
P/TREE,PJ=1/CMAX	HU 1961
Q3/RJ,PJ=1/SUMCJ	LAGEWEG (DYNAMIC PROGRAMMING)
Q3//SUMWJCJ	LAWLER (DYNAMIC PROGRAMMING)
Q/DJ,PJ=1/SUMWJTJ	GRAHAM LAWLER LENSTRA RINNOOY KAN 1979 (TRANSPORTATION PROBLEM)
Q/DJ,PJ=1/SUMWJUJ	GRAHAM LAWLER LENSTRA RINNOOY KAN 1979 (TRANSPORTATION PROBLEM)
Q/RJ,PJ=1/CMAX	LAWLER LENSTRA
R3//SUMWJUJ	LAWLER (DYNAMIC PROGRAMMING)
R3/RJ/CMAX	LAWLER (DYNAMIC PROGRAMMING)
R//SUMCJ	HORN 1973; BRUNO COFFMAN SETHI 1974
O2/RJ,PJ=1/LMAX	LAWLER
O2//CMAX	GONZALEZ SAHNI 1976
O3/PJ=1/SUMWJUJ	LAWLER
O3/RJ,PJ=1/CMAX	LAWLER
O/PJ=1/CMAX	LENSTRA
O/PJ=1/SUMWJCJ	LENSTRA
F2/TREE,PJ=1/CMAX	LAGEWEG
F2/TREE,PJ=1/SUMCJ	LAGEWEG
F2//CMAX	JOHNSON 1954
F/RJ,DJ,PJ=1/SUMWJTJ	LAGEWEG (TRANSPORTATION PROBLEM)
F/RJ,DJ,PJ=1/SUMWJUJ	LAGEWEG (TRANSPORTATION PROBLEM)
J2/PJ=1/LMAX	BRUCKER 1981
J2/RJ,PJ=1/CMAX	BRUCKER 1981

NONPREEMPTIVE SCHEDULING, BINARY ENCODING (CONTINUED)

MINIMAL OPEN PROBLEMS	MAXIMAL OPEN PROBLEMS
1/TREE,PJ=1/SUMTJ	1/DJ/SUMTJ
1//SUMTJ	1/TREE/SUMTJ
P2/TREE,DJ,PJ=1/SUMCJ	Q3/TREE,RJ,PJ=1/SUMTJ
P2/TREE,RJ,PJ=1/SUMCJ	Q3/PREC,RJ,DJ,PJ=1/SUMCJ
P3/TREE,PJ=1/LMAX	Q/RJ,DJ,PJ=1/SUMWJTJ
P3/TREE,PJ=1/SUMCJ	Q/RJ,DJ,PJ=1/SUMWJUJ
P3/TREE,RJ,PJ=1/CMAX	Q/TREE,PJ=1/CMAX
P3/PREC,PJ=1/CMAX	Q/TREE,PJ=1/SUMCJ
Q2/RJ,PJ=1/LMAX	O3//SUMCJ
Q2/RJ,PJ=1/SUMWJCJ	O/RJ,DJ,PJ=1/SUMWJTJ
Q2/TREE,PJ=1/CMAX	O/RJ,DJ,PJ=1/SUMWJUJ
Q2/TREE,PJ=1/SUMCJ	O/TREE,PJ=1/SUMWJCJ
Q/RJ,PJ=1/SUMCJ	O/TREE,RJ,PJ=1/SUMTJ
O2/PJ=1/SUMTJ	O/PREC,RJ,DJ,PJ=1/SUMCJ
O2/PJ=1/SUMWJUJ	F/TREE,RJ,PJ=1/SUMTJ
O2/DJ,PJ=1/SUMCJ	F/PREC,RJ,DJ,PJ=1/SUMCJ
O2/DJ,PJ=1/SUMUJ	J2/RJ,DJ,PJ=1/SUMWJTJ
O2/RJ,PJ=1/SUMCJ	J2/RJ,DJ,PJ=1/SUMWJUJ
O2/RJ,PJ=1/SUMUJ	
O2/TREE,PJ=1/CMAX	
O2/TREE,PJ=1/SUMCJ	
O2//SUMCJ	
O3/RJ,PJ=1/LMAX	
O/PJ=1/LMAX	
O/RJ,PJ=1/CMAX	
F2/TREE,PJ=1/LMAX	
F2/TREE,RJ,PJ=1/CMAX	
F2/TREE,RJ,PJ=1/SUMCJ	
F2/PREC,PJ=1/CMAX	
F2/PREC,PJ=1/SUMCJ	
F3/TREE,PJ=1/CMAX	
F3/TREE,PJ=1/SUMCJ	
J2/PJ=1/SUMCJ	
J2/PJ=1/SUMUJ	
J2/RJ,PJ=1/LMAX	

NONPREEMPTIVE SCHEDULING, UNARY ENCODING (CONTINUED)

MINIMAL OPEN PROBLEMS

1/TREE,PJ=1/SUMTJ
1/DJ/SUMTJ
1/DJ/SUMUJ
P2/TREE,DJ,PJ=1/SUMCJ
P2/TREE,RJ,PJ=1/SUMCJ
P2//SUMTJ
P2/DJ/SUMCJ
P2/TREE/CMAX
P3/TREE,PJ=1/LMAX
P3/TREE,PJ=1/SUMCJ
P3/TREE,RJ,PJ=1/CMAX
P3/PREC,PJ=1/CMAX
Q2/RJ,PJ=1/LMAX
Q2/RJ,PJ=1/SUMWJCJ
Q2/TREE,PJ=1/CMAX
Q2/TREE,PJ=1/SUMCJ
Q/RJ,PJ=1/SUMCJ
R2//SUMWJCJ
O2/PJ=1/SUMTJ
O2/DJ,PJ=1/SUMCJ
O2/DJ,PJ=1/SUMUJ
O2/RJ,PJ=1/SUMCJ
O2/RJ,PJ=1/SUMUJ
O2/TREE,PJ=1/CMAX
O2/TREE,PJ=1/SUMCJ
O2//SUMCJ
O3/RJ,PJ=1/LMAX
O3//CMAX
O/PJ=1/LMAX
O/RJ,PJ=1/CMAX
F2/TREE,PJ=1/LMAX
F2/TREE,RJ,PJ=1/CMAX
F2/TREE,RJ,PJ=1/SUMCJ
F2/PREC,PJ=1/CMAX
F2/PREC,PJ=1/SUMCJ
F3/TREE,PJ=1/CMAX
F3/TREE,PJ=1/SUMCJ
J2/PJ=1/SUMCJ
J2/PJ=1/SUMUJ
J2/RJ,PJ=1/LMAX

MAXIMAL OPEN PROBLEMS

1/TREE/SUMTJ
P3/TREE/CMAX
Q3/TREE,RJ,PJ=1/SUMTJ
Q3/PREC,RJ,DJ,PJ=1/SUMCJ
Q/RJ,DJ,PJ=1/SUMWJTJ
Q/RJ,DJ,PJ=1/SUMWJUJ
Q/TREE,PJ=1/CMAX
Q/TREE,PJ=1/SUMCJ
R3//SUMWJCJ
R3/DJ/SUMTJ
R3/DJ/SUMWJUJ
O3//CMAX
O3//SUMCJ
O/RJ,DJ,PJ=1/SUMWJTJ
O/RJ,DJ,PJ=1/SUMWJUJ
O/TREE,PJ=1/SUMWJCJ
O/TREE,RJ,PJ=1/SUMTJ
O/PREC,RJ,DJ,PJ=1/SUMCJ
F/TREE,RJ,PJ=1/SUMTJ
F/PREC,RJ,DJ,PJ=1/SUMCJ
J2/RJ,DJ,PJ=1/SUMWJTJ
J2/RJ,DJ,PJ=1/SUMWJUJ

NONPREEMPTIVE SCHEDULING, BINARY ENCODING (CONTINUED)

MINIMAL HARD PROBLEMS

1/TREE,PJ=1/SUMWJTJ	3PT	LENSTRA RINNOOY KAN 1980
1/TREE,PJ=1/SUMUJ	S3P	LENSTRA RINNOOY KAN 1980
1/TREE,DJ,PJ=1/SUMWJCJ	3PT	LENSTRA RINNOOY KAN 1980
1/TREE,DJ,PJ=1/SUMTJ	VC	LENSTRA
1/TREE,RJ,PJ=1/SUMWJCJ	3PT	LENSTRA RINNOOY KAN 1980
1/PREC,PJ=1/SUMWJCJ	LA	LAWLER 1978; LENSTRA RINNOOY KAN 1978
1/PREC,PJ=1/SUMTJ	CL	LENSTRA RINNOOY KAN 1978
1//SUMWJTJ	3PT	LAWLER 1977; LENSTRA RINNOOY KAN BRUCKER 1977
1//SUMWJUJ	KS	KARP 1972
1/DJ/SUMWJCJ	3PT	LENSTRA RINNOOY KAN BRUCKER 1977
1/DJ/SUMUJ	KS	LAWLER
1/RJ/LMAX	3PT	GAREY JOHNSON 1977; LENSTRA RINNOOY KAN BRUCKER 1977
1/RJ/SUMCJ	3PT	LENSTRA RINNOOY KAN BRUCKER 1977
1/TREE,DJ/SUMCJ	3PT	LENSTRA
1/PREC/SUMCJ	LA	LAWLER 1978; LENSTRA RINNOOY KAN 1978
P2/TREE,PJ=1/SUMWJCJ	3PT	LENSTRA RINNOOY KAN 1980
P2//CMAX	PT	KARP 1972
P2//SUMWJCJ	PT	BRUNO COFFMAN SETHI 1974; LENSTRA RINNOOY KAN BRUCKER 1977
P2/TREE/SUMCJ	3PT	SETHI 1977
P/TREE,PJ=1/LMAX	VC	BRUCKER GAREY JOHNSON 1977
P/TREE,RJ,PJ=1/CMAX	VC	BRUCKER GAREY JOHNSON 1977
P/TREE,RJ,PJ=1/SUMCJ	VC	LENSTRA: BRUCKER GAREY JOHNSON 1977
P/PREC,PJ=1/CMAX	CL	ULLMAN 1975; LENSTRA RINNOOY KAN 1978
P/PREC,PJ=1/SUMCJ	CL	LENSTRA RINNOOY KAN 1978
O2//LMAX	3PT	LAWLER LENSTRA RINNOOY KAN 1981
O2//SUMWJCJ	3PT	LENSTRA
O2/RJ/CMAX	3PT	LAWLER LENSTRA RINNOOY KAN 1981
O2/TREE/CMAX	3PT	LAGEWEG LENSTRA
O2/TREE/SUMCJ	3PT	LAGEWEG LENSTRA
O3//CMAX	PT	GONZALEZ SAHNI 1976
O//SUMCJ	G3C	GONZALEZ 1979
F2/TREE,PJ=1/SUMWJCJ	3PT	LENSTRA RINNOOY KAN 1980
F2//LMAX	3PT	LENSTRA RINNOOY KAN BRUCKER 1977
F2//SUMCJ	3PT	GAREY JOHNSON SETHI 1976
F2/RJ/CMAX	3PT	LENSTRA RINNOOY KAN BRUCKER 1977
F2/TREE/CMAX	3PT	LENSTRA RINNOOY KAN BRUCKER 1977
F3//CMAX	3PT	GAREY JOHNSON SETHI 1976; LENSTRA RINNOOY KAN BRUCKER 1977
J2/TREE,PJ=1/CMAX	3PT	LENSTRA
J2/TREE,PJ=1/SUMCJ	3PT	LENSTRA
J2//CMAX	3PT	GAREY JOHNSON SETHI 1976; LENSTRA RINNOOY KAN BRUCKER 1977
J3/PJ=1/CMAX	3PT	LENSTRA RINNOOY KAN 1979
J3/PJ=1/SUMCJ	3PT	LENSTRA

NONPREEMPTIVE SCHEDULING, UNARY ENCODING (CONTINUED)

MINIMAL HARD PROBLEMS

1/TREE,PJ=1/SUMWJTJ	3PT	LENSTRA RINNOOY KAN 1980
1/TREE,PJ=1/SUMUJ	S3P	LENSTRA RINNOOY KAN 1980
1/TREE,DJ,PJ=1/SUMWJCJ	3PT	LENSTRA RINNOOY KAN 1980
1/TREE,DJ,PJ=1/SUMTJ	VC	LENSTRA
1/TREE,RJ,PJ=1/SUMWJCJ	3PT	LENSTRA RINNOOY KAN 1980
1/PREC,PJ=1/SUMWJCJ	LA	LAWLER 1978; LENSTRA RINNOOY KAN 1978
1/PREC,PJ=1/SUMTJ	CL	LENSTRA RINNOOY KAN 1978
1//SUMWJTJ	3PT	LAWLER 1977; LENSTRA RINNOOY KAN BRUCKER 1977
1/DJ/SUMWJCJ	3PT	LENSTRA RINNOOY KAN BRUCKER 1977
1/RJ/LMAX	3PT	GAREY JOHNSON 1977; LENSTRA RINNOOY KAN BRUCKER 1977
1/RJ/SUMCJ	3PT	LENSTRA RINNOOY KAN BRUCKER 1977
1/TREE,DJ/SUMCJ	3PT	LENSTRA
1/PREC/SUMCJ	LA	LAWLER 1978; LENSTRA RINNOOY KAN 1978
P2/TREE,PJ=1/SUMWJCJ	3PT	LENSTRA RINNOOY KAN 1980
P2/TREE/LMAX	3PT	LENSTRA
P2/TREE/SUMCJ	3PT	SETHI 1977
P2/TREE,RJ/CMAX	3PT	LENSTRA
P2/PREC/CMAX	CL	ULLMAN 1975; LENSTRA RINNOOY KAN 1978
P/TREE,PJ=1/LMAX	VC	BRUCKER GAREY JOHNSON 1977
P/TREE,RJ,PJ=1/CMAX	VC	BRUCKER GAREY JOHNSON 1977
P/TREE,RJ,PJ=1/SUMCJ	VC	LENSTRA: BRUCKER GAREY JOHNSON 1977
P/PREC,PJ=1/CMAX	CL	ULLMAN 1975; LENSTRA RINNOOY KAN 1978
P/PREC,PJ=1/SUMCJ	CL	LENSTRA RINNOOY KAN 1978
P//CMAX	3PT	GAREY JOHNSON 1975
P//SUMWJCJ	3PT	LAGEWEG LENSTRA
Q2/TREE/CMAX	3PT	LAGEWEG
O2//LMAX	3PT	LAWLER LENSTRA RINNOOY KAN 1981
O2//SUMWJCJ	3PT	LENSTRA
O2/RJ/CMAX	3PT	LAWLER LENSTRA RINNOOY KAN 1981
O2/TREE/CMAX	3PT	LAGEWEG LENSTRA
O2/TREE/SUMCJ	3PT	LAGEWEG LENSTRA
O//CMAX	3PT	LENSTRA
O//SUMCJ	G3C	GONZALEZ 1979
F2/TREE,PJ=1/SUMWJCJ	3PT	LENSTRA RINNOOY KAN 1980
F2//LMAX	3PT	LENSTRA RINNOOY KAN BRUCKER 1977
F2//SUMCJ	3PT	GAREY JOHNSON SETHI 1976
F2/RJ/CMAX	3PT	LENSTRA RINNOOY KAN BRUCKER 1977
F2/TREE/CMAX	3PT	LENSTRA RINNOOY KAN BRUCKER 1977
F3//CMAX	3PT	GAREY JOHNSON SETHI 1976; LENSTRA RINNOOY KAN BRUCKER 1977
J2/TREE,PJ=1/CMAX	3PT	LENSTRA
J2/TREE,PJ=1/SUMCJ	3PT	LENSTRA
J2//CMAX	3PT	GAREY JOHNSON SETHI 1976; LENSTRA RINNOOY KAN BRUCKER 1977
J3/PJ=1/CMAX	3PT	LENSTRA RINNOOY KAN 1979
J3/PJ=1/SUMCJ	3PT	LENSTRA

PREEMPTIVE SCHEDULING, BINARY ENCODING

DATE: 81:03:13

NUMBER OF PROBLEMS	MINIMAL	TOTAL	MAXIMAL
EASY		97	11
OPEN	14	122	9
HARD	31	1377	
TOTAL		1596	

MAXIMAL EASY PROBLEMS

1/DJ/SUMCJ	SMITH 1956 (NO PMTN)
1/RJ/SUMCJ	BAKER 1974
1/RJ/SUMUJ	LAWLER (DYNAMIC PROGRAMMING)
1/TREE/SUMWJCJ	HORN 1972; SIDNEY 1975 (NO PMTN)
P/TREE/CMAX	MUNTZ COFFMAN 1970; GONZALEZ JOHNSON 1980
Q2/PREC,RJ/LMAX	LAWLER
Q3//SUMUJ	LAWLER 1979
Q//SUMCJ	GONZALEZ 1977
R/RJ/LMAX	LAWLER LABETOULLE 1978 (LINEAR PROGRAMMING)
O/RJ/LMAX	CHO SAHNI 1978 (LINEAR PROGRAMMING)
F2//CMAX	GONZALEZ SAHNI 1976; JOHNSON 1954 (NO PMTN)

MINIMAL OPEN PROBLEMS	MAXIMAL OPEN PROBLEMS
1//SUMTJ	Q3/RJ/SUMUJ
1/RJ,DJ/SUMCJ	Q3/TREE/SUMTJ
P2/DJ/SUMCJ	Q3/PREC,RJ/LMAX
P2/RJ/SUMCJ	Q/TREE/CMAX
P2/RJ/SUMUJ	Q/TREE/SUMCJ
P2/TREE/SUMCJ	R3//SUMUJ
P3/TREE/LMAX	R/RJ,DJ/SUMTJ
P3/TREE,RJ/CMAX	O3/RJ,DJ/SUMTJ
P3/PREC/CMAX	F2/RJ/SUMCJ
Q3/TREE/CMAX	
R2//SUMCJ	
R2//SUMUJ	
O2//SUMCJ	
F2//SUMCJ	

PREEMPTIVE SCHEDULING, UNARY ENCODING

DATE: 81:03:13

NUMBER OF PROBLEMS	MINIMAL	TOTAL	MAXIMAL
EASY		114	14
OPEN	21	203	10
HARD	27	1279	
TOTAL		1596	

MAXIMAL EASY PROBLEMS

1//SUMTJ	LAWLER 1977 (NO PMTN)
1/DJ/SUMCJ	SMITH 1956 (NO PMTN)
1/RJ/SUMCJ	BAKER 1974
1/RJ/SUMWJUJ	LAWLER (DYNAMIC PROGRAMMING)
1/TREE/SUMWJCJ	HORN 1972; SIDNEY 1975 (NO PMTN)
P3//SUMWJCJ	MCNAUGHTON 1959 (NO PMTN; DYNAMIC PROGRAMMING)
P/TREE/CMAX	MUNTZ COFFMAN 1970; GONZALEZ JOHNSON 1980
Q2/PREC,RJ/LMAX	LAWLER
Q//SUMCJ	GONZALEZ 1977
R3//SUMWJUJ	LAWLER
R/RJ/LMAX	LAWLER LABETOULLE 1978 (LINEAR PROGRAMMING)
O3//SUMWJUJ	LAWLER
O/RJ/LMAX	CHO SAHNI 1978 (LINEAR PROGRAMMING)
F2//CMAX	GONZALEZ SAHNI 1976; JOHNSON 1954 (NO PMTN)

MINIMAL OPEN PROBLEMS	MAXIMAL OPEN PROBLEMS
1/DJ/SUMTJ	Q3/TREE/SUMTJ
1/DJ/SUMUJ	Q3/PREC,RJ/LMAX
1/RJ/SUMTJ	Q/TREE/CMAX
1/RJ,DJ/SUMCJ	Q/TREE/SUMCJ
1/TREE/SUMTJ	R3//SUMWJCJ
P2//SUMTJ	R/RJ,DJ/SUMTJ
P2/DJ/SUMCJ	R/RJ,DJ/SUMWJUJ
P2/RJ/SUMCJ	O3/RJ,DJ/SUMTJ
P2/RJ/SUMUJ	O/RJ,DJ/SUMWJUJ
P2/TREE/SUMCJ	F2/RJ/SUMCJ
P3/TREE/LMAX	
P3/TREE,RJ/CMAX	
P3/PREC/CMAX	
P//SUMUJ	
Q2//SUMWJCJ	
Q3/TREE/CMAX	
R2//SUMCJ	
O2//SUMCJ	
O2/RJ/SUMUJ	
O//SUMUJ	
F2//SUMCJ	

PREEMPTIVE SCHEDULING, BINARY ENCODING (CONTINUED)

MINIMAL HARD PROBLEMS

1//SUMWJTJ	3PT	LAWLER 1977; LENSTRA RINNOOY KAN BRUCKER 1977 (NO PMTN)
1//SUMWJUJ	KS	KARP 1972 (NO PMTN)
1/DJ/SUMWJCJ	3PT	LENSTRA RINNOOY KAN BRUCKER 1977 (NO PMTN)
1/DJ/SUMUJ	KS	LAWLER
1/RJ/SUMWJCJ	3PT	LABETOULLE LAWLER LENSTRA RINNOOY KAN 1979
1/TREE/SUMUJ	S3P	LENSTRA RINNOOY KAN 1980 (NO PMTN)
1/TREE,DJ/SUMCJ	3PT	LENSTRA
1/TREE,RJ/SUMCJ	3PT	LENSTRA
1/PREC/SUMCJ	LA	LAWLER 1978; LENSTRA RINNOOY KAN 1978 (NO PMTN)
P2//SUMWJCJ	PT	MCNAUGHTON 1959: BRUNO COFFMAN SETHI 1974; LENSTRA RINNOOY KAN BRUCKER 1977 (NO PMTN)
P//SUMUJ	KS	LAWLER
P/TREE/LMAX	VC	LENSTRA: BRUCKER GAREY JOHNSON 1977
P/TREE,RJ/CMAX	VC	LENSTRA: BRUCKER GAREY JOHNSON 1977
P/PREC/CMAX	CL	ULLMAN 1976
R2/RJ/SUMUJ	KS	LAWLER
R2/TREE/CMAX	3PT	LENSTRA
R2/TREE/SUMCJ	3PT	LENSTRA
O2//SUMWJCJ	3PT	LENSTRA
O2//SUMUJ	KS	LAWLER LENSTRA RINNOOY KAN 1981
O2/TREE/CMAX	3PT	LAGEWEG LENSTRA
O2/TREE/SUMCJ	3PT	LAGEWEG LENSTRA
O//SUMCJ	G3C	GONZALEZ 1979
F2//LMAX	3PT	CHO SAHNI 1978
F2//SUMWJCJ	3PT	LENSTRA
F2/RJ/CMAX	3PT	CHO SAHNI 1978
F2/TREE/CMAX	3PT	LAGEWEG LENSTRA
F2/TREE/SUMCJ	3PT	LAGEWEG LENSTRA
F3//CMAX	3PT	GONZALEZ SAHNI 1978
F3//SUMCJ	3PT	LENSTRA
J2//CMAX	3PT	GONZALEZ SAHNI 1978
J2//SUMCJ	3PT	LENSTRA

PREEMPTIVE SCHEDULING, UNARY ENCODING (CONTINUED)

MINIMAL HARD PROBLEMS

1//SUMWJTJ	3PT	LAWLER 1977; LENSTRA RINNOOY KAN BRUCKER 1977 (NO PMTN)
1/DJ/SUMWJCJ	3PT	LENSTRA RINNOOY KAN BRUCKER 1977 (NO PMTN)
1/RJ/SUMWJCJ	3PT	LABETOULLE LAWLER LENSTRA RINNOOY KAN 1979
1/TREE/SUMUJ	S3P	LENSTRA RINNOOY KAN 1980 (NO PMTN)
1/TREE,DJ/SUMCJ	3PT	LENSTRA
1/TREE,RJ/SUMCJ	3PT	LENSTRA
1/PREC/SUMCJ	LA	LAWLER 1978; LENSTRA RINNOOY KAN 1978 (NO PMTN)
P2/TREE/SUMWJCJ	3PT	LENSTRA
P//SUMWJCJ	3PT	MCNAUGHTON 1959: LAGEWEG LENSTRA (NO PMTN)
P/TREE/LMAX	VC	LENSTRA: BRUCKER GAREY JOHNSON 1977
P/TREE,RJ/CMAX	VC	LENSTRA: BRUCKER GAREY JOHNSON 1977
P/PREC/CMAX	CL	ULLMAN 1976
R2/TREE/CMAX	3PT	LENSTRA
R2/TREE/SUMCJ	3PT	LENSTRA
O2//SUMWJCJ	3PT	LENSTRA
O2/TREE/CMAX	3PT	LAGEWEG LENSTRA
O2/TREE/SUMCJ	3PT	LAGEWEG LENSTRA
O//SUMCJ	G3C	GONZALEZ 1979
F2//LMAX	3PT	CHO SAHNI 1978
F2//SUMWJCJ	3PT	LENSTRA
F2/RJ/CMAX	3PT	CHO SAHNI 1978
F2/TREE/CMAX	3PT	LAGEWEG LENSTRA
F2/TREE/SUMCJ	3PT	LAGEWEG LENSTRA
F3//CMAX	3PT	GONZALEZ SAHNI 1978
F3//SUMCJ	3PT	LENSTRA
J2//CMAX	3PT	GONZALEZ SAHNI 1978
J2//SUMCJ	3PT	LENSTRA

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REFERENCES

K.R. BAKER (1974) *Introduction to Sequencing and Scheduling*, Wiley, New York.

P. BRUCKER (1981) An efficient optimal algorithm to minimize maximum lateness for the two-machines, unit-time, job-shop scheduling problem. Unpublished manuscript.

P. BRUCKER, M.R. GAREY, D.S. JOHNSON (1977) Scheduling equal-length tasks under treelike precedence constraints to minimize maximum lateness. *Math. Oper. Res.* 2,275-284.

J. BRUNO, E.G. COFFMAN, JR., R. SETHI (1974) Scheduling independent tasks to reduce mean finishing time. *Comm. ACM* 17,382-387.

Y. CHO, S. SAHNI (1978) Preemptive scheduling of independent jobs with release and due times on open, flow and job shops. Technical Report 78-5, Computer Science Department, University of Minnesota, Minneapolis.

E.G. COFFMAN, JR., R.L. GRAHAM (1972) Optimal scheduling for two-processor systems. *Acta Inform.* 1,200-213.

M.R. GAREY (1975) Personal communication.

M.R. GAREY, D.S. JOHNSON (1975) Complexity results for multiprocessor scheduling under resource constraints. *SIAM J. Comput.* 4,397-411.

M.R. GAREY, D.S. JOHNSON (1977) Two-processor scheduling with start-times and deadlines. *SIAM J. Comput.* 6,416-426.

M.R. GAREY, D.S. JOHNSON (1978) "Strong" NP-completeness results: motivation, examples and implications. *J. Assoc. Comput. Mach.* 25,499-508.

M.R. GAREY, D.S. JOHNSON, R. SETHI (1976) The complexity of flowshop and job-shop scheduling. *Math. Oper. Res.* 1,117-129.

T. GONZALEZ (1977) Optimal mean finish time preemptive schedules. Technical Report 220, Computer Science Department, Pennsylvania State University, University Park, PA.

T. GONZALEZ (1979) NP-hard shop problems. Report CS-79-35, Department of

Computer Science, Pennsylvania State University, University Park, PA.

T. GONZALEZ, D.B. JOHNSON (1980) A new algorithm for preemptive scheduling of trees. *J. Assoc. Comput. Mach.* 27,287-312.

T. GONZALEZ, S. SAHNI (1976) Open shop scheduling to minimize finish time. *J. Assoc. Comput. Mach.* 23,665-679.

T. GONZALEZ, S. SAHNI (1978) Flowshop and jobshop schedules: complexity and approximation. *Oper. Res.* 26,36-52.

R.L. GRAHAM, E.L. LAWLER, J.K. LENSTRA, A.H.G. RINNOOY KAN (1979) Optimization and approximation in deterministic sequencing and scheduling: a survey. *Ann. Discrete Math.* 5,287-326.

W.A. HORN (1972) Single-machine job sequencing with treelike precedence ordering and linear delay penalties. *SIAM J. Appl. Math.* 23,189-202.

W.A. HORN (1973) Minimizing average flow time with parallel machines. *Oper. Res.* 21,846-847.

T.C. HU (1961) Parallel sequencing and assembly line problems. *Oper. Res.* 9, 841-848.

S.M. JOHNSON (1954) Optimal two- and three-stage production schedules with setup times included. *Naval Res. Logist. Quart.* 1,61-68.

R.M. KARP (1972) Reducibility among combinatorial problems. In: R.E. MILLER, J.W. THATCHER (eds.) (1972) *Complexity of Computer Computations*, Plenum Press, New York, 85-103.

J. LABETOULLE, E.L. LAWLER, J.K. LENSTRA, A.H.G. RINNOOY KAN (1979) Preemptive scheduling of uniform machines subject to release dates. Report BW 99, Mathematisch Centrum, Amsterdam.

B.J. LAGEWEG (-) Unpublished results.

B.J. LAGEWEG, E.L. LAWLER, J.K. LENSTRA, A.H.G. RINNOOY KAN (1981) Computer aided complexity classification of combinatorial problems. Report BW 137, Mathematisch Centrum, Amsterdam.

E.L. LAWLER (1964) On scheduling problems with deferral costs. *Management Sci.* 11,280-288.

E.L. LAWLER (1973) Optimal sequencing of a single machine subject to precedence constraints. *Management Sci.* 19,544-546.

E.L. LAWLER (1977) A "pseudopolynomial" algorithm for sequencing jobs to minimize total tardiness. *Ann. Discrete Math.* 1,331-342.

E.L. LAWLER (1978) Sequencing jobs to minimize total weighted completion time subject to precedence constraints. *Ann. Discrete Math.* 2,75-90.

E.L. LAWLER (1979) Preemptive scheduling of uniform parallel machines to minimize the weighted number of late jobs. Report BW 105, Mathematisch Centrum, Amsterdam.

E.L. LAWLER (-) Unpublished results.

E.L. LAWLER, J. LABETOULLE (1978) On preemptive scheduling of unrelated parallel processors by linear programming. *J. Assoc. Comput. Mach.* 25, 612-619.

E.L. LAWLER, J.K. LENSTRA, A.H.G. RINNOOY KAN (1981) Minimizing maximum lateness in a two-machine open shop. *Math. Oper. Res.*, to appear.

J.K. LENSTRA (-) Unpublished results.

J.K. LENSTRA, A.H.G. RINNOOY KAN (1978) Complexity of scheduling under precedence constraints. *Oper. Res.* 26,22-35.

J.K. LENSTRA, A.H.G. RINNOOY KAN (1979) Computational complexity of discrete optimization problems. *Ann. Discrete Math.* 4,121-140.

J.K. LENSTRA, A.H.G. RINNOOY KAN (1980) Complexity results for scheduling chains on a single machine. *European J. Oper. Res.* 4,270-275.

J.K. LENSTRA, A.H.G. RINNOOY KAN, P. BRUCKER (1977) Complexity of machine scheduling problems. *Ann. Discrete Math.* 1,343-362.

R. McNAUGHTON (1959) Scheduling with deadlines and loss functions. *Management Sci.* 6,1-12.

J.M. MOORE (1968) An *n* job, one machine sequencing algorithm for minimizing the number of late jobs. *Management Sci.* 15,102-109.

R.R. MUNTZ, E.G. COFFMAN, JR. (1970) Preemptive scheduling of real-time tasks on multiprocessor systems. *J. Assoc. Comput. Mach.* 17,324-338.

R. SETHI (1977) On the complexity of mean flow time scheduling. *Math. Oper. Res.* 2,320-330.

J.B. SIDNEY (1975) Decomposition algorithms for single-machine sequencing with precedence relations and deferral costs. *Oper. Res.* 23,283-298.

W.E. SMITH (1956) Various optimizers for single-stage production. *Naval Res. Logist. Quart.* 3,59-66.

J.D. ULLMAN (1975) *NP*-Complete scheduling problems. *J. Comput. System Sci.* 10,384-393.

J.D. ULLMAN (1976) Complexity of sequencing problems. In: E.G. COFFMAN, JR. (ed.) (1976) *Computer & Job/Shop Scheduling Theory*, Wiley, New York, 139-164.