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COMPUTER AIDED COMPLEXITY CLASSIFICATION
OF DETERMINISTIC SCHEDULING PROBLEMS

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COMPUTER AIDED COMPLEXITY CLASSIFICATION OF DETERMINISTIC SCHEDULING PROBLEMS

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ABSTRACT

We present the results obtained by a computer program that has been used to maintain a record of the known complexity results for a class of 4,536 machine scheduling problems. The input of the program consists of a listing of known "easy" problems and a listing of known "hard" problems. The program employs the structure of the problem class to determine the implications of these results. The output provides a listing of essential results in the form of maximal easy and minimal hard problems as well as listings of minimal and maximal open problems, which are helpful in indicating the direction of future research. The reader is assumed to be familiar with a companion paper by the same authors, "Computer aided complexity classification of combinatorial problems" (Report BW 137, Mathematisch Centrum, Amsterdam, 1981). In that paper, the program is described, its application to a restricted class of 120 single-machine problems is demonstrated, and possible refinements and extensions to other research areas are suggested.

KEY WORDS & PHRASES: machine scheduling, computational complexity, polynomial algorithm, NP-hardness, computer program
1. INTRODUCTION

In this report, we present the results obtained by a computer program that has been used to maintain a record of the known complexity results for a class of 4,536 deterministic machine scheduling problems.

The input of the program consists of a listing of known easy problems, which can be solved in polynomial time, and a listing of known hard problems, which are NP-hard. The program systematically employs a certain partial ordering defined on the problem class to determine the implications of these results. The output provides a count of easy, open and hard problems as well as complete listings of maximal easy, minimal open, maximal open and minimal hard problems.

We assume that the reader is familiar with the contents of a companion paper [Lageweg, Lawler, Lenstra & Rinnooy Kan 1981]. In that paper, we describe the program MSPCLASS in general terms, we demonstrate its application to a restricted class of 120 single-machine scheduling problems, we discuss the benefits of this approach, and we suggest possible refinements and extensions to other research areas.

Section 2 contains a schematic definition of our class of 4,536 machine scheduling problems and Section 3 presents the state of the art as exhibited by application of the program to this class. It is our intention to publish a next edition of this report whenever newly obtained results have changed the picture significantly.

2. THE CLASS OF MACHINE SCHEDULING PROBLEMS

The class of machine scheduling problems to which the program MSPCLASS is applied is defined in this section (cf. [Graham, Lawler, Lenstra & Rinnooy Kan 1979]).

Suppose that \( n \) jobs \( J_j \) \((j = 1, \ldots, n)\) have to be processed on \( m \) machines \( M_i \) \((i = 1, \ldots, m)\). Each machine can process at most one job at a time and each job can be processed on at most one machine at a time. Various job, machine and scheduling characteristics are reflected by a three-field classification \( a|b|c \), to be introduced below. Let \( \cdot \) denote the empty symbol.
2.1. Machine environment

The first field $\alpha = \alpha_1 \alpha_2$ specifies the machine environment.

If $\alpha_1 \in \{\circ, P, Q, R\}$, each $J_j$ consists of a single operation that can be processed on any $M_i$; the processing time of $J_j$ on $M_i$ is $p_{ij}$. The four values are characterized as follows.

- $\alpha_1 = \circ$: Single machine; $m = 1$; $p_{1j} = p_j$.
- $\alpha_1 = P$: Identical parallel machines; $p_{ij} = p_j$ ($i = 1, \ldots, m$).
- $\alpha_1 = Q$: Uniform parallel machines; $p_{ij} = p_j/s_i$ for a given speed $s_i$ of $M_i$ ($i = 1, \ldots, m$).
- $\alpha_1 = R$: Unrelated parallel machines.

If $\alpha_1 \in \{0, F, J\}$, each $J_j$ consists of $m_j$ operations $O_{1j}, \ldots, O_{mj}$; $O_{ij}$ has to be processed on a given machine $M_i$ during $p_{ij}$ time units. The three values are characterized as follows.

- $\alpha_1 = 0$: Open shop; $m_j = m$; $\mu_{ij} = M_i$ ($i = 1, \ldots, m$); the order in which the operations are executed is immaterial.
- $\alpha_1 = F$: Flow shop; $m_j = m$; $\mu_{ij} = M_i$ ($i = 1, \ldots, m$); $O_{i-1,j}$ has to be completed before $O_{ij}$ can start ($i = 2, \ldots, m$).
- $\alpha_1 = J$: Job shop; $\mu_{i-1,j} \neq \mu_{ij}$ and $C_{i-1,j}$ has to be completed before $O_{ij}$ can start ($i = 2, \ldots, m$).

If $\alpha_2 \in \{1, 2, 3\}$, then $m$ is constant and equal to $\alpha_2$. If $\alpha_2 = \circ$, then $m$ is variable.

2.2. Job characteristics

The second field $\beta \in \{\beta_1, \ldots, \beta_5\}$ indicates a number of job characteristics, which are defined as follows.

1. $\beta_1 \in \{\text{pmtn, } \circ\}$.
   - $\beta_1 = \text{pmtn}$: Preemption is allowed; the processing of any operation may be interrupted and resumed at a later time (or, if $\alpha_1 \in \{P, Q, R\}$, at the same time on a different machine) without penalty.
   - $\beta_1 = \circ$ : Preemption is not permitted.

2. $\beta_2 \in \{\text{prec, tree, } \circ\}$.
   - $\beta_2 = \text{prec}$: Precedence constraints between the jobs are given in the
form of an acyclic digraph $G = \langle\{1,\ldots,n\}, A\rangle$; if $(j,k) \in A$, then $J_j$ has to be completed before $J_k$ can start.

- $\beta_2 =$ tree: $G$ is a rooted tree with either indegree at most one for each vertex or outdegree at most one for each vertex.
- $\beta_2 = \circ$ : $G$ has no arcs (the jobs are independent).

(3) $\beta_3 \in \{r_j, \circ\}$.
- $\beta_3 = r_j$ : Each $J_j$ has a given nonnegative integer release date $r_j$, on which it becomes available for processing.
- $\beta_3 = \circ$ : All $r_j = 0$.

(4) $\beta_4 \in \{d_j, \circ\}$.
- $\beta_4 = d_j$ : Each $J_j$ has a given nonnegative integer deadline $d_j$, by which it has to be completed.
- $\beta_4 = \circ$ : All $d_j = \infty$.

(5) $\beta_5 \in \{p_j = 1, \circ\}$.
- $\beta_5 = p_j = 1$: Each operation has a unit processing requirement, i.e., if $a_j \in \{\circ, P, Q\}$, then all $p_j = 1$, and if $a_j \in \{R, O, F, J\}$, then all $p_{ij} = 1$.
- $\beta_5 = \circ$ : All $p_j$ ($p_{ij}$) are given arbitrary nonnegative integers.

2.3. Optimality criterion

The third field $\gamma$ refers to the optimality criterion chosen. Each $J_j$ may have a given nonnegative integer due date $\bar{d}_j$ and/or a given integer weight $w_j$. Any schedule defines for each $J_j$:
- a completion time $C_j$;
- a lateness $L_j = C_j - \bar{d}_j$;
- a tardiness $T_j = \max\{0, C_j - \bar{d}_j\}$;
- a unit penalty $U_j = 0$ if $C_j \leq \bar{d}_j$, $U_j = 1$ if $C_j > \bar{d}_j$.

The optimality criterion, which is to be minimized, is one of the following eight types.
- $\gamma = \max C_j$: the maximum completion time $\max\{C_j\}$;
- $\gamma = \max L_j$: the maximum lateness $\max\{L_j\}$;
- $\gamma = \sum C_j$: the total completion time;
- $\gamma = \sum w_j C_j$: the total weighted completion time;
- $\gamma = \sum T_j$: the total tardiness;
- $\gamma = \sum_{j=1}^{n} t_j$: the total weighted tardiness;
- $\gamma = \sum_{j=1}^{n} u_j$: the number of late jobs;
- $\gamma = \sum_{j=1}^{n} w_j u_j$: the weighted number of late jobs.

2.4. Number of problems

Each eight-tuple $a_1 a_2 | b_1, b_2, b_3, b_4, b_5 | \gamma$ corresponds to a machine scheduling problem. Several combinations of values for the components are excluded, for the following reasons.

- Clearly, $a_1 = 0$ if and only if $a_2 = 1$.
- If $a_1 = R$, then $b_5 = 0$, since the choice $b_5 = p_j = 1$ leads to a problem with $a_1 = P$.
- If $\gamma \in \{ C_{max}, L_{max} \}$, then $b_4 = 0$, since for the combinations $b_4 | \gamma$ the choices $d_j | C_{max}, L_{max}$ and $d_j | L_{max}$ yield equivalent problems (cf. [Lageweg, Lawler, Lenstra & Rinnooy Kan 1981]).

Further, we have decided to eliminate the choice $b_5 = p_j = 1$ in the case that $b_1 = pmtn$: preemptable jobs have arbitrary processing requirements.

It is not hard to see that, under these restrictions, the class of problems defined by the above classification contains 4,536 different problem types.

2.5. Partial ordering

The program MSCPCLASS employs a certain partial ordering $\triangleright$ of the problem class, which has the property that, if $P \triangleright P'$ for two problems $P$ and $P'$, then $P$ is reducible to $P'$. This relation is defined as follows.

Each member of our class of machine scheduling problems corresponds to a seven-tuple $(\pi_i)_{i=0}^6$, where $\pi_i$ is a vertex of the digraph $G_i$ shown in Figure 1 ($i = 0, \ldots, 6$); $\pi_0$ corresponds to $\alpha$, $\pi_1$ to $b_1$ ($i = 1, \ldots, 5$), and $\pi_6$ to $\gamma$. There is an arc from $\pi$ to $\pi'$ if $\pi'$ is a direct generalization of $\pi$. For two problems $F = (\pi_i)_{i=0}^6$ and $F' = (\pi'_i)_{i=0}^6$, we have $P \triangleright P'$ if either $\pi_i = \pi'_i$ or $G_i$ contains a directed path from $\pi_i$ to $\pi'_i$, for $i = 0, \ldots, 6$ (cf. [Lageweg, Lawler, Lenstra & Rinnooy Kan 1981]).
Figure 1. The graphs $G_i$ ($i = 0, \ldots, 6$).

3. THE STATE OF THE ART

The program MSPCLASS has been implemented in PASCAL on the Control Data Cyber 170-750 of the SARA Computing Centre in Amsterdam. Its application to the
class of machine scheduling problems defined in the previous section is demonstrated below.

We note that separate runs are made for two different inputs, corresponding to different assumptions about the encoding of problem instances containing numerical data. Under a standard binary encoding, the easy problems are solvable in strictly polynomial time and the hard problems are NP-hard in the ordinary sense. Under a unary encoding, the easy problems are solvable in pseudopolynomial time and the hard problems are NP-hard in the strong sense (cf. [Garey & Johnson 1978; Lenstra & Rinnooy Kan 1979]).

Moreover, the program deals separately with nonpreemptive and preemptive problems, which reflects the fact that $G_1$ consists of two isolated vertices. Many results for preemptive scheduling, however, follow immediately from those for nonpreemptive scheduling. For example, there is no advantage to preemption for problems with a single machine and equal release dates [McNaughton 1959] and for problems of the types $\texttt{P2|\sum w_jC_j}$ [McNaughton 1959] and $\texttt{F2|C_{max}}$ [Gonzalez & Sahni 1978].

The output of the latest runs is shown on the next pages; note that the value of the component $b_1$ appears in the heading of the listings and not in each individual problem type. The listings present useful additional information in the form of references to the literature where the results in question can be found, an indication of general algorithmic techniques applicable to some easy problems, the symbol # for open problems that are both minimal and maximal, and acronyms of convenient "starting problems" for transformations in the case of hard problems.

Combining the results for nonpreemptive and preemptive scheduling, we get an overview of the total numbers of easy, open and hard problems as shown in Table 1.

<table>
<thead>
<tr>
<th></th>
<th>binary</th>
<th>unary</th>
</tr>
</thead>
<tbody>
<tr>
<td>easy</td>
<td>416 (9%)</td>
<td>471 (10%)</td>
</tr>
<tr>
<td>open</td>
<td>432 (10%)</td>
<td>547 (12%)</td>
</tr>
<tr>
<td>hard</td>
<td>3688 (81%)</td>
<td>3518 (78%)</td>
</tr>
<tr>
<td>total</td>
<td>4536 (100%)</td>
<td>4536 (100%)</td>
</tr>
</tbody>
</table>

Table 1. Summary of results.
## NONPREEMPTIVE SCHEDULING, BINARY ENCODING

**DATE:** 81:03:13

<table>
<thead>
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<th>NUMBER OF PROBLEMS</th>
<th>MINIMAL</th>
<th>TOTAL</th>
<th>MAXIMAL</th>
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<tr>
<td>EASY</td>
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<td>29</td>
<td></td>
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<tr>
<td>OPEN</td>
<td>35</td>
<td>310</td>
<td>18</td>
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<td>HARD</td>
<td>42</td>
<td>2311</td>
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</tr>
<tr>
<td><strong>TOTAL</strong></td>
<td></td>
<td>2940</td>
<td></td>
</tr>
</tbody>
</table>

### MAXIMAL EASY PROBLEMS

- 1/PREC,RJ,DJ,PJ=1/SUMCJ
  - LAWLER: COFFMAN GRAHAM 1972
- 1//SUMUJ
  - NOORE 1968
- 1/DJ/SUMCJ
  - SMITH 1956
- 1/TREE/SUMAUCJ
  - HORI 1972; SIDNEY 1975
- 1/PREC/LMAX
  - LAWLER 1973
- 1/PREC,RJ/CMAX
  - LAWLER 1973
- P2/PREC,RJ,PJ=1/SUMCJ
  - GAREY 1975: COFFMAN GRAHAM 1972
- P2/PREC,RJ,PJ=1/LMAX
  - GAREY JOHNSON 1977
- P/RJ,DJ,PJ=1/SUMWJUJ
  - LAWLER 1964 (TRANSPORTATION PROBLEM)
- P/RJ,DJ,PJ=1/SUMWJUJ
  - LAWLER 1964 (TRANSPORTATION PROBLEM)
- P/TREE,PJ=1/CMAX
  - HU 1961
- Q2/RJ,PJ=1/SUMCJ
  - LAGEWEG (DYNAMIC PROGRAMMING)
- Q/RJ,PJ=1/SUMWJUJ
  - GRAHAM LAWLER LENSTRA RINNOY KAN 1979
  - (TRANSPORTATION PROBLEM)
- Q/RJ,PJ=1/CMAX
  - GRAHAM LAWLER LENSTRA RINNOY KAN 1979
  - (TRANSPORTATION PROBLEM)
- Q/RJ,PJ=1/SUMWJUJ
  - LAWLER LENSTRA
- R//SUMCJ
  - FORN 1973; RRuno COFFMAN SETHI 1974
- R//CMAX
  - LAWLER
- O2/RJ,PJ=1/LMAX
  - GONZALEZ SAINHI 1976
- O2/CMAX
  - LAWLER
- O3/RJ,PJ=1/SUMUJ
  - LAWLER
- O3/RJ,PJ=1/CMAX
  - LENSTRA
- O/PJ=1/SUMWJUJ
  - LENSTRA
- F2/TREE,PJ=1/CMAX
  - LAGEWEG
- F2/TREE,PJ=1/SUMCJ
  - LAGEWEG
- F2/CMAX
  - JOHNSON 1954
- F/RJ,DJ,PJ=1/SUMWJUJ
  - LAGEWEG (TRANSPORTATION PROBLEM)
- F/RJ,DJ,PJ=1/SUMWJUJ
  - LAGEWEG (TRANSPORTATION PROBLEM)
- J2/PJ=1/LMAX
  - ERUCKER 1981
- J2/RJ,PJ=1/CMAX
  - ERUCKER 1981
<table>
<thead>
<tr>
<th>NUMBER OF PROBLEMS</th>
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<th>TOTAL</th>
<th>MAXIMAL</th>
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<tbody>
<tr>
<td>EASY</td>
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<td>357</td>
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<tr>
<td>OPEN</td>
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<td>344</td>
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<td>HARD</td>
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<td><strong>TOTAL</strong></td>
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</table>

**MAXIMAL EASY PROBLEMS**

1/prec,rj,dj,pj=1/sumcj
1//sumij
1/dj/sjmcj
1/tree/sumwjcj
1/prec/lmax
1/prec,rj/cmax
p2/prec,pj=1/sumcj
p2/prec,rj,pj=1/lmax
p/rj,dj,pj=1/sumwjttj
p/rj,dj,pj=1/sumwjuj
p/tree,pj=1/cmax
q3/rj,pj=1/sumcj
q3//sumwjcj
q/dj,fj=1/sumwjttj
q/dj,fj=1/sumwjuj
q/rj,pj=1/cmax
r3//sumwjuj
r3/rj/cmax
r//sumcj
o2/rj,pj=1/lmax
o2//cmax
o3/pj=1/sunkwjuj
o3/rj,pj=1/cmax
o/pj=1/cmax
o/pj=1/sumwjcj
p2/tree,pj=1/cmax
p2/tree,pj=1/sumcj
p2//cmax
f/rj,dj,pj=1/sumwjttj
f/rj,dj,pj=1/sumwjuj
j2/pj=1/lmax
j2/rj,pj=1/cmax

LAWLER: COFFMAN GRAHAM 1972
LAWLER 1977
SMITH 1956
HCRN 1972; SIDNEY 1975
LAWLER 1973
LAWLER 1973
GARREY 1975; COFFMAN GRAHAM 1972
GARREY JOHNSON 1977
LAWLER 1964 (TRANSPORTATION PROBLEM)
LAWLER 1964 (TRANSPORTATION PROBLEM)
HU 1961
LAGEWEG (DYNAMIC PROGRAMMING)
LAWLER (DYNAMIC PROGRAMMING)
GRAHAM LAWLER LENSTRA RINNOY KAN 1979
(TRANSPORTATION PROBLEM)
GRAHAM LAWLER LENSTRA RINNOY KAN 1979
(TRANSPORTATION PROBLEM)
LAWLER LENSTRA
LAWLER (DYNAMIC PROGRAMMING)
LAWLER (DYNAMIC PROGRAMMING)
HORN 1973; BRUNO COFFMAN SETHI 1974
LAWLER
GONZALEZ SAHNI 1976
LAWLER
LAWLER
LENSTRA
LENSTRA
LAGEWEG
LAGEWEG
JOHNSON 1954
LAGEWEG (TRANSPORTATION PROBLEM)
LAGEWEG (TRANSPORTATION PROBLEM)
BRUCKER 1981
BRUCKER 1981
### Minimal Open Problems

<table>
<thead>
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MINIMAL OPEN PROBLEMS

1/TREE,PJ=1/SUMTJ
1/DJ/SUMTJ
1/DJ/SUMUJ
P2/TREE,DJ,PJ=1/SUMCJ
P2/TREE,RJ,PJ=1/SUMCJ
P2//SUMTJ
P2/DJ/SUMCJ
P2/TREE/CMAX
P2/TREE,PJ=1/LMAX
P3/TREE,PJ=1/SUMCJ
P3/TREE,RJ,PJ=1/CMAX
P3/PREC,PJ=1/CMAX
Q2/RJ,PJ=1/LMAX
Q2/RJ,PJ=1/SUMWJCJ
Q2/TREE,PJ=1/CMAX
Q3/TREE,PJ=1/SUMCJ
Q/RJ,PJ=1/SUMCJ
R2//SUMWJCJ
O2/PJ=1/SUMCJ
O2/DJ,PJ=1/SUMCJ
O2/DJ,PJ=1/SUMUJ
O2/RJ,PJ=1/SUMCJ
O2/RJ,PJ=1/SUMUJ
O2/TREE,PJ=1/CMAX
O2/TREE,PJ=1/SUMCJ
O2/SUMCJ
O3/RJ,PJ=1/\Sigma_{MAX}
O3/CMAX
O/PJ=1/LMAX
O/RJ,PJ=1/CMAX
F2/TREE,PJ=1/LMAX
F2/TREE,RJ,PJ=1/CMAX
F2/TREE,RJ,PJ=1/SUMCJ
F2/PREC,PJ=1/CMAX
F2/PREC,PJ=1/SUMCJ
F3/TREE,PJ=1/CMAX
F3/TREE,PJ=1/SUMCJ
J2/PJ=1/SUMCJ
J2/PJ=1/SUMUJ
J2/RJ,PJ=1/LMAX

MAXIMAL OPEN PROBLEMS

1/TREE/SUMTJ
P3/TREE/CMAX
Q3/TREE,RJ,PJ=1/SUMTJ
Q3/PREC,RJ,DJ,PJ=1/SUMCJ
Q/RJ,DJ,PJ=1/SUMWJTJ
Q/RJ,DJ,PJ=1/SUMWJUJ
Q/TREE,PJ=1/CMAX
Q/TREE,PJ=1/SUMCJ
R3//SUMWJCJ
R3/DJ/SUMTJ
R3/DJ/SUMWJUJ
# O3//CMAX
03//SUMCJ
0/RJ,DJ,PJ=1/SUMWJTJ
0/RJ,DJ,PJ=1/SUMWJUJ
0/TREE,PJ=1/SUMWJCJ
0/TREE,RJ,PJ=1/SUMTJ
0/PREC,RJ,DJ,PJ=1/SUMCJ
F/TREE,RJ,PJ=1/SUMTJ
F/PREC,RJ,DJ,PJ=1/SUMCJ
J2/RJ,DJ,PJ=1/SUMWJTJ
J2/RJ,DJ,PJ=1/SUMWJUJ
NONPREEMPTIVE SCHEDULING, BINARY ENCODING (CONTINUED)

MINIMAL HARD PROBLEMS

1/TREE, PJ=1/SUMWJ TJ
1/TREE, PJ=1/SUMWJ
1/TREE, DJ, PJ=1/SUMWJCJ
1/TREE, DJ, PJ=1/SUMTJ
1/TREE, RJ, PJ=1/SUMWJCJ
1/PREC, PJ=1/SUMWJCJ
1/PREC, PJ=1/SUMTJ
1/SUMWJTJ
1/SUMWJUJ
1/DJ/SUMWJCJ
1/DJ/SUMUJ
1/RJ/LMAX
1/RJ/SUMCJ
1/TREE, DJ/SUMCJ
1/PREC/SUMCJ
P2/TREE, PJ=1/SUMWJCJ
P2/CMAX
P2/SUMWJCJ
P2/TREE/SUMCJ
P/TREE, PJ=1/LMAX
P/TREE, RJ, PJ=1/CMAX
P/TREE, RJ, PJ=1/SUMCJ
P/PREC, PJ=1/CMAX
P/PREC, PJ=1/SUMCJ
O2/LMAX
O2/SUMWJCJ
O2/RJ/CMAX
O2/TREE/CMAX
O2/TREE/SUMCJ
O3/CMAX
O/SUMCJ
F2/TREE, PJ=1/SUMWJCJ
F2/LMAX
F2/SUMCJ
F2/RJ/CMAX
F2/TREE/CMAX
F3/CMAX
J2/TREE, PJ=1/CMAX
J2/TREE, PJ=1/SUMCJ
J2/CMAX
J3/PJ=1/CMAX
J3/PJ=1/SUMCJ

3PT LENSTRA RINNOOY KAN 1980
3PT LENSTRA RINNOOY KAN 1980
3PT LENSTRA RINNOOY KAN 1980
VC LENSTRA
3PT LENSTRA RINNOOY KAN 1980
LA LAWLER 1978; LENSTRA RINNOOY KAN 1978
CL LENSTRA RINNOOY KAN 1978
3PT LAWLER 1977; LENSTRA RINNOOY KAN BRUCKER 1977
XS KARP '72
3PT LENSTRA RINNOOY KAN BRUCKER 1977
XS LAWLER
3PT GAREY JOHNSON 1977; LENSTRA RINNOOY KAN BRUCKER 1977
3PT LENSTRA RINNOOY KAN BRUCKER 1977
3PT LENSTRA
LA LAWLER 1978; LENSTRA RINNOOY KAN 1978
3PT LENSTRA RINNOOY KAN 1980
PT KARP 1972
PT BRUNO COFF; KAN SETHI 1974; LENSTRA RINNOOY KAN BRUCKER 1977
3PT SETHI 1977
VC BRUCKER GAREY JOHNSON 1977
VC BRUCKER GAREY JOHNSON 1977
VC LENSTRA; BRUCKER GAREY JOHNSON 1977
CL ULLMAN 1975; LENSTRA RINNOOY KAN 1978
CL LENSTRA RINNOOY KAN 1978
3PT LAWLER LENSTRA RINNOOY KAN 1981
3PT LENSTRA
3PT LAWLER LENSTRA RINNOOY KAN 1981
3PT LAGEWEG LENSTRA
3PT LAGEWEG LENSTRA
PT GONZALEZ SAIHI 1976
G3C GONZALEZ 1979
3PT LENSTRA RINNOOY KAN 1980
3PT LENSTRA RINNOOY KAN BRUCKER 1977
3PT GAREY JOHNSON SETHI 1976
3PT LENSTRA RINNOOY KAN BRUCKER 1977
3PT LENSTRA RINNOOY KAN BRUCKER 1977
3PT GAREY JOHNSON SETHI 1976; LENSTRA RINNOOY KAN BRUCKER 1977
3PT LENSTRA RINNOOY KAN 1979
3PT LENSTRA

3PT LENSTRA
3PT LENSTRA
3PT GAREY JOHNSON SETHI 1976; LENSTRA RINNOOY KAN BRUCKER 1977
3PT LENSTRA RINNOOY KAN 1979
3PT LENSTRA
NONPREEMPTIVE SCHEDULING, UNARY ENCODING (CONTINUED)

MINIMAL HARD PROBLEMS

1/TREE, P_J=1/SUMWJ TJ
1/TREE, P_J=1/SUMUJ
1/TREE, D_J, P_J=1/SUMWJCJ
1/TREE, D_J, P_J=1/SUMTJ
1/TREE, R_J, P_J=1/SUMWJCJ
1/PREC, P_J=1/SUMWJCJ
1/PREC, P_J=1/SUMTJ
1///SUMWJ TJ
1/DJ/SUMWJCJ
1/RJ/LMAX

1/RJ/SMJC
1/TREE, D_J/SMJC
1/PREC/SMJC
P2/TREE, P_J=1/SMWJCJ
P2/TREE/LMAX
P2/TREE/SMJC
P2/TREE, R_J/CMAX
P2/PREC/CMAX
P/TREE, P_J=1/LMAX
P/TREE, R_J, P_J=1/CMAX
P/TREE, R_J, P_J=1/SMJC
F/PREC, P_J=1/CMAX
F/PREC, P_J=1/SMJC
F///CMAX
F///SUMWJCJ
O2/TREE/CMAX
O2///LMAX
O2///SUMWJCJ
O2/RJ/CMAX
O2/TREE/CMAX
O2/TREE/SMJC
O///CMAX
O///SUMJC
P2/TREE, P_J=1/SMWJCJ
P2///LMAX
P2///SUMJC
P2/RJ/CMAX
P2/TREE/CMAX
P3///CMAX

J2/TRJE, P_J=1/CMAX
J2/TRJE, P_J=1/SMJC
J2///CMAX

J3/P_J=1/CMAX
J3/P_J=1/SMJC

3PT LENSTRA RINNOOY KAN 1980
3PT LENSTRA RINNOOY KAN 1980
3PT LENSTRA RINNOOY KAN 1980
VC LENSTRA
3PT LENSTRA RINNOOY KAN 1980
LA LAWLER 1978; LENSTRA RINNOOY KAN 1978
CL LENSTRA RINNOOY KAN 1978
3PT LAWLER 1977; LENSTRA RINNOOY KAN BRUCKER 1977
3PT LENSTRA RINNOOY KAN BRUCKER 1977
3PT GAREY JOHNSON 1977; LENSTRA RINNOOY KAN BRUCKER 1977
3PT LENSTRA RINNOOY KAN BRUCKER 1977
3PT LENSTRA
3PT LAUER 1978; LENSTRA RINNOOY KAN 1978
3PT LENSTRA RINNOOY KAN 1980
3PT LENSTRA
3PT SETHI 1977
3PT LENSTRA
CL ULLMAN 1975; LENSTRA RINNOOY KAN 1978
VC BRUCKER GAREY JOHNSON 1977
VC BRUCKER GAREY JOHNSON 1977
VC LENSTRA; BRUCKER GAREY JOHNSON 1977
CL ULLMAN 1975; LENSTRA RINNOOY KAN 1978
CL LENSTRA RINNOOY KAN 1978
3PT GAREY JOHNSON 1975
3PT LAGEWEY LENSTRA
3PT LAGEWEY
3PT LAWLER LENSTRA RINNOOY KAN 1981
3PT LENSTRA
3PT LAWLER LENSTRA RINNOOY KAN 1981
3PT LAGEWEY LENSTRA
3PT LAGEWEY LENSTRA
3PT LENSTRA
G3C GONZALEZ 1979
3PT LENSTRA RINNOOY KAN 1980
3PT LENSTRA RINNOOY KAN BRUCKER 1977
3PT GAREY JOHNSON SETHI 1976
3PT LENSTRA RINNOOY KAN BRUCKER 1977
3PT LENSTRA RINNOOY KAN BRUCKER 1977
3PT GAREY JOHNSON SETHI 1976; LENSTRA RINNOOY KAN BRUCKER 1977
3PT LENSTRA
3PT LENSTRA
3PT GAREY JOHNSON SETHI 1976; LENSTRA RINNOOY KAN BRUCKER 1977
3PT LENSTRA RINNOOY KAN 1979
3PT LENSTRA
PREEMPTIVE SCHEDULING, BINARY ENCODING

DATE: 81:03:13

NUMBER OF PROBLEMS

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<th></th>
<th>MINIMAL</th>
<th>TOTAL</th>
<th>MAXIMAL</th>
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<td>TOTAL</td>
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MAXIMAL EASY PROBLEMS

1/DJ/SUMCJ  SMITH 1956 (NO PMTN)
1/RJ/SUMCJ  BAKER 1974
1/RJ/SUMUJ  LAWLER (DYNAMIC PROGRAMMING)
1/TREE/SUMWJJC J HORN 1972; SIDNEY 1975 (NO PMTN)
P/TREE/CMAK MUNTEM COFFMAN 1970; GONZALEZ JOHNSON 1980
Q2/PREC,RJ/LMA LAWLER
Q3/ /SUMUJ LAWLER 1979
Q3/ /SUMCJ GONZALEZ 1977
R/RJ/LMAK LAWLER LEBETTOLLE 1978 (LINEAR PROGRAMMING)
O/RJ/LMAK CHO SAHN 1978 (LINEAR PROGRAMMING)
F2//CMAK GONZALEZ SAHN 1976; JOHNSON 1954 (NO PMTN)

MINIMAL OPEN PROBLEMS

1//SUMTJ Q3/RJ/SUMUJ
1/RJ,DJ/SUMCJ Q3/TREE/SUMTJ
F2/DJ/SUMCJ Q3/PREC,RJ/LMAK
P2/RJ/SUMCJ Q/TREE/CMAK
P2/RJ/SUMUJ Q/TREE/SUMCJ
P2/TREE/SUMCJ R3//SUMUJ
P3/TREE/LMAK R/RJ,DJ/SUMTJ
P3/TREE,RJ/CMAK Q3/RJ,DJ/SUMTJ
P3/PREC/CMAK F2/RJ/SUMCJ
Q3/TREE/CMAK
R2//SUMCJ
R2//SUMUJ
Q2//SUMCJ
F2//SUMCJ
PREEMPTIVE SCHEDULING, UNARY ENCODING

DATE: 81:03:13

<table>
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<th>NUMBER OF PROBLEMS</th>
<th>MINIMAL</th>
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<th>MAXIMAL</th>
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<td>EASY</td>
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<tr>
<td>OPEN</td>
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<td>HARD</td>
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<td><strong>TOTAL</strong></td>
<td><strong>1536</strong></td>
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MAXIMAL EASY PROBLEMS

1//SUMDJ
1/DJ/SUMCJ
1/RJ/SUMCJ
1/RJ/SUMWWUJ
1/TREE/SUMWWCJ
P3//SUMWWCJ
P/TREE/CMAAX
Q2/PREC,RJ/LMAX
Q//SUMCJ
R3//SUMWWUJ
R/RJ/LMAX
C3//SUMWWUJ
C/RJ/LMAX
F2//CMAAX

LAWLER 1977 (NO PMTN)
SMITH 1956 (NO PMTN)
BAKER 1974
LAWLER (DYNAMIC PROGRAMMING)
HORN 1972; SIDNEY 1975 (NO PMTN)
MCNAUGHTON 1959 (NO PMTN; DYNAMIC PROGRAMMING)
MINTZ COFFMAN 1970; GONZALES JOHNSON '900
LAWLER
GONZALES 1977
LAWLER
LAWLER LABETTOULLE 1978 (LINEAR PROGRAMMING)
LAWLER
CHO SAHNI 1978 (LINEAR PROGRAMMING)
GONZALES SAHNI 1976; JOHNSON 1954 (NO PMTN)

MINIMAL OPEN PROBLEMS

1/DJ/SUMTJ
1/DJ/SUMUJ
1/RJ/SUMTJ
1/RJ,DJ/SUMCJ
1/TREE/SUMTJ
P2//SUMTJ
P2/DJ/SUMCJ
P2/RJ/SUMCJ
P2/RJ/SUMUJ
P2/TREE/SUMCJ
P3/TREE/CMAAX
P3/PREC/CMAAX
P/SUMUJ
Q2//SUMWWCJ
Q3/TREE/CMAAX
R2//SUMCJ
Q2//SUMCJ
Q2/RJ/SUMUJ
Q/SUMUJ
F2//SUMCJ

MAXIMAL OPEN PROBLEMS
PREEMPTIVE SCHEDULING, BINARY ENCODING (CONTINUED)

MINIMAL HARD PROBLEMS

1//SUMWJTJ 3PT LAWLER 1977; LENSTRA RINNOY KAN BRUCKER 1977 (NO PMTN)
1//SUMWJUJ KS KARP 1972 (NO PMTN)
1/DJ/SUMWJCJ 3PT LENSTRA RINNOY KAN BRUCKER 1977 (NO PMTN)
1/DJ/SUMUJ KS LAWLER
1/RJ/SUMWJCJ 3PT LABEOFJELLE LAWLER LENSTRA RINNOY KAN 1979
1/TREE/SUMUJ S3P LENSTRA RINNOY KAN 1980 (NO PMTN)
1/TREE,DJ/SUMCJ 3PT LENSTRA
1/TREE,RJ/SUMCJ 3PT LENSTRA
1/PREC/SUMCJ LA LAWLER 1978; LENSTRA RINNOY KAN 1978 (NO PMTN)
22//SUMWJCJ PT MCNAUGHTON 1959; BRUNO COPFMAN SETHI 1974;
LENSTRA RINNOY KAN BRUCKER 1977 (NO PMTN)
2//SUMUJ KS LAWLER
2/TREE,LMAI VC LENSTRA: BRUCKER GAREY JOHNSON 1977
2/TREE,RJ/CMAI VC LENSTRA: BRUCKER GAREY JOHNSON 1977
2/PREC,CMAI CL ULLMAN 1976
R2/RJ/SUMUJ KS LAWLER
R2/TREE/CMAI 3PT LENSTRA
R2/TREE/SUMCJ 3PT LENSTRA
O2//SUMWJCJ 3PT LENSTRA
O2//SUMUJ KS LAWLER LENSTRA RINNOY KAN 1981
O2/TREE/CMAI 3PT LAGEWEG LENSTRA
O2/TREE,SUMCJ 3PT LAGEWEG LENSTRA
O/SUMCJ G3C GONZALEZ 1979
F2//LMAI 3PT CHO SANNI 1978
F2//SUMWJCJ 3PT LENSTRA
F2/RJ/CMAI 3PT CHO SANNI 1978
F2/TREE/CMAI 3PT LAGEWEG LENSTRA
F2/TREE,SUMCJ 3PT LAGEWEG LENSTRA
F3//CMAI 3PT GONZALEZ SANNI 1978
F3//SUMCJ 3PT LENSTRA
J2//CMAI 3PT GONZALEZ SANNI 1978
J2//SUMCJ 3PT LENSTRA
MINIMAL HARD PROBLEMS

1//SUMWJTJ
1/DJ/SUMWJCJ
1/RJ/SUMWJCJ
1/TREE/SUMUJ
1/TREE,DJ/SUMCJ
1/TREE,RJ/SUMCJ
1/REC/SUMCJ

P2/TREE/SUMWKJCJ
P//SUMWKJCJ
P/TREE/LMAX
P/TREE,RJ/CMAX
P/PREC/CMax
R2/TREE/CMAX
R2/TREE/SUMCJ
O2//SUMWKJCJ
O2/TREE,CMAX
O2/TREE,SUMCJ
O//SUMCJ
F2//LMAX
F2//SUMWKJCJ
F2/RJ/CMAX
F2/TREE/CMAX
F2/TREE,SUMCJ
F3//CMax
F3//SUMCJ
J2//CMax
J2//SUMCJ

3PT LAKLER 1977; LENSTRA RINNOOY KAN BRUCKER 1977 (NO PMTN)
3PT LENSTRA RINNOOY KAN BRUCKER 1977 (NO PMTN)
3PT LAFETTOULLE LAWLER LENSTRA RINNOOY KAN 1979
3PT LENSTRA RINNOOY KAN 1980 (NO PMTN)
3PT LENSTRA
3PT LENSTRA
LA LAWLER 1978; LENSTRA RINNOOY KAN 1978 (NO PMTN)
3PT LENSTRA
3PT MCNAUGHCION 1959; LAGEWEG LENSTRA (NO PMTN)
VC LENSTRA: BRUCKER GAREY JOHNSON 1977
VC LENSTRA: BRUCKER GAREY JOHNSON 1977
CL ULLMAN 1976
3PT LENSTRA
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3PT LAGEWEG LENSTRA
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G3C GONZALEZ 1979
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3PT GONZALEZ SAHNI 1978
3PT LENSTRA
3PT GONZALEZ SAHNI 1978
3PT LENSTRA
ACKNOWLEDGMENTS

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REFERENCES


M.R. GAREY (1975) Personal communication.


T. GONZALEZ (1979) NP-hard shop problems. Report CS-79-35, Department of
Computer Science, Pennsylvania State University, University Park, PA.


E.J. LAGEWES (-) Unpublished results.


E.L. LAWLER (1978) Sequencing jobs to minimize total weighted completion time


E.L. LAWLER (-) Unpublished results.


J.K. LENSTRA (-) Unpublished results.


