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CONTROLLED INVARIANCE BY STATIC OUTPUT FEEDBACK FOR NONLINEAR SYSTEMS

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Controlled invariance by static output feedback for nonlinear systems *)

by

Henk Nijmeijer & Arjan van der Schaft **)

ABSTRACT

The paper deals with the notion of static output feedback for nonlinear systems. Necessary and sufficient conditions are derived for "(C,A,B)-invariance", here called measured controlled invariance, for nonlinear control systems.

KEY WORDS & PHRASES: *Invariant distributions, nonlinear controlsystems, static output feedback, connections*

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*) This report will be submitted for publication elsewhere.

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1. INTRODUCTION

In linear systems theory an important concept in the study of synthesis problems is the notion of invariant subspaces (cf[7]). Recall that for the linear system

$$(1.1)\qquad \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$, A, B and C matrices of appropriate dimensions, a subspace $V \subset \mathbb{R}^n$ is *controlled invariant*, or (A,B)-invariant, if there exists an (m,n)-matrix F defining a linear state feedback law $u = Fx + v$ such that the modified dynamics

$$(1.2)\qquad \dot{x} = (A+BF)x + Bv$$

leaves V invariant, i.e.

$$(1.3)\qquad (A+BF)V \subset V.$$

In a dual fashion we have that a subspace $V \subset \mathbb{R}^n$ is *conditionally invariant*, or (C,A)-invariant, if there exists an (n,p)-matrix K-output injection - such that the modified system

$$(1.4)\qquad \dot{x} = (A+KC)x + Bv$$

leaves V invariant, thus

$$(1.5)\qquad (A+KC)V \subset V.$$

It is well known and easy to see that (1.3) and (1.5) are equivalent to

$$(1.6)\qquad AV \subset V + \mathcal{B} \qquad (\mathcal{B}=\mathrm{Im}B)$$

respectively

(1.7) $\quad A(V \cap \mathrm{Ker} C) \subset V.$

A combination of these two notions leads to the following concept. A subspace $V \subset \mathbb{R}^n$ is *measured controlled invariant* - usually called (C,A,B)-invariant - if there exists an (m,p) matrix K, defining a static output feedback law u = Ky + v such that the modified dynamics

(1.8) $\quad \dot{x} = (A+BKC)x + Bv$

leaves V invariant. This is the same as the requirement that the state feedback u = Fx + v in (1.2) only depends on the measurements y. Again it is straightforward to show that (1.8) is equivalent to the following conditions

(1.9a) $\quad AV \subset V + B$

(1.9b) $\quad A(V \cap \mathrm{Ker}\ C) \subset V.$

Or, a subspace V is measured controlled invariant if and only if it is controlled invariant as well as conditionally invariant.

The notions controlled invariance and conditioned invariance also arise in various synthesis problems for nonlinear systems theory (cf[2], see also [3] for further references on nonlinear controlled invariance). We will briefly sketch some ideas concerning nonlinear controlled invariance. These ideas have been elaborated in our basic reference [3]; some of the necessary backgrounds also may be found in the next section. We furthermore assume that the reader is familiar with basic notions of differential geometry. Suppose there is given a smooth nonlinear control system (locally) described by

(1.10) $\quad \dot{x} = f(x,u)$

where $x \in M$, the state manifold and $u \in U$, the input manifold . The notion of an (invariant) subspace is generalized to that of an (invariant) involutive distribution. An involutive distribution D is *invariant* for the system (1.10) if

$$(1.11)\qquad [f(\cdot,\bar{u}),D] \subset D$$

for every (constant) input function $\bar{u}$. For the direct analogue of (1.2) we obtain: there exists a state feedback law

$\alpha : M \times U \to U$ such that the modified dynamics

$$(1.12)\qquad \dot{x} = f(x,\alpha(x,u)) := \hat{f}(x,u)$$

satisfies the invariantness condition

$$(1.13)\qquad [\hat{f}(\cdot,\bar{u}),D] \subset D$$

for every (constant) input function $\bar{u}$.

The distribution D is then called *controlled invariant*. Of course it is desirable to maintain as much open loop control as possible; therefore one seeks an $\alpha(\cdot,\cdot)$ such that for all $x \in M$ $\alpha(x,\cdot): U \to U$ is a diffeomorphism. Under a certain condition, which is analogous to (1.6), one can really construct such a feedback function α in a local fashion (i.e. locally around each point x_0 α can be found), see [3].
Suppose we also have a smooth output function $C : M \to Y$, where Y is the output-manifold. An involutive distribution D is *measured controlled invariant* if there exists a *static output* feedback $\beta : Y \times U \to U$ such that the modified dynamics

$$(1.14)\qquad \dot{x} = f(x,\beta(C(x)),u)) := \hat{f}(x,u)$$

satisfies

$$(1.15)\qquad [\hat{f}(\cdot,\bar{u}),D] \subset D$$

for every (constant) input function $\bar{u}$. Again we want to maintain as much open loop control as possible; therefore we seek a $\beta(\cdot,\cdot)$ such that for all $y \in Y$ $\beta(y,\cdot) : U \to U$ is a diffeomorphism. As will be clear, a distribution D is measured controlled invariant implies that D is controlled invariant; in the linear case condition (1.9a) is satisfied. In this paper we will show that for measured controlled invariance we also need the nonlinear analogue of (1.9b), although nonlinear controlled invariance and conditioned invariance are not sufficient conditions for measured controlled invariance. Some results in this direction already may be found in [2]. The approach presented here completely fits in the set-up of [3].

Some notation

Throughout this paper all our objects like manifolds, maps etc. are smooth. We recall the following canonical construction (see [3]). For a k-dimensional distribution D on a manifold M we can construct a 2(n-k)-dimensional codistribution $\dot{P}$ on TM in the following way. Define the codistribution P on M by

$$P(x) = \{\theta \in T_x^*M \mid \theta(X) = 0 \text{ for every } X \in D(x)\}, \quad x \in M.$$

Then P has a basis of n-k one-forms $\theta_1,\dots,\theta_{n-k}$. Since $\theta_i \in T^*M$ we can also consider θ_i as a real function on TM. Now we define $\dot{\theta}_i \in T^*TM$ by

$$\dot{\theta}_i(X) = X(\theta_i), \quad \text{with } X \text{ vectorfield on TM.}$$

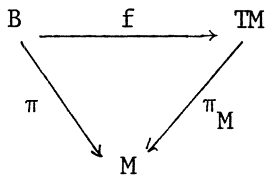
Denote the natural projection from TM onto M by π. Then also $\pi^*\theta_i \in T^*TM$. The codistribution $\dot{P}$ on TM is then defined by

$$\dot{P} = \mathrm{Span}\{\pi^*\theta_1,\dots,\pi^*\theta_{n-k}, \dot{\theta}_1,\dots,\dot{\theta}_{n-k}\}.$$

Furthermore we can also define the distribution $\dot{D}$ on TM by dualization:
$\dot{D} = \{X \text{ vectorfield on TM} \mid \theta(X) = 0, \text{ for every } \theta \in \dot{P}\}$.

2. MEASURED CONTROLLED INVARIANCE; DEFINITIONS

As in our previous paper we use the following setting for a nonlinear control system (see [3] for references). Let M be a manifold denoting the state space. Let $\pi : B \to M$ be a fiberbundle, whose fibers represent the state-dependent input spaces. Then a *control system* $\Sigma(M,B,f)$ is defined by the commutative diagram

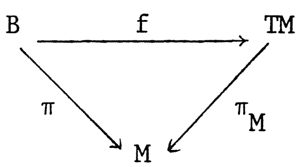


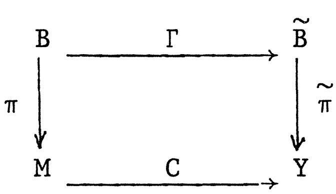
where TM denotes the tangentbundle with natural projection π_M and f is a smooth map. In local coordinates x for M, (x,u) for B this coordinate free definition simply comes down to

$$\dot{x} = f(x,u).$$

We now want to formalize the situation that the inputspace does not depend on the whole state, but only on the measurements (outputs). The following definition is very similar to the one proposed by BROCKETT [1] and related to a definition given by TAKENS [6].

DEFINITION 2.1. A *control system with measurements* $\Sigma = \Sigma(M,B,f,Y,\tilde{B},C,\Gamma)$ is given by the following. Let $\Sigma(M,B,f)$ be a control system. Let $\tilde{\pi} : \tilde{B} \to Y$ be a fiberbundle on the outputspace Y. Let $C : M \to Y$ be a surjective submersion denoting the outputfunction. Furthermore let $\Gamma : B \to \tilde{B}$ be a fiberpreserving map, such that Γ maps the fibers of B diffeomorphically onto the fibers of $\tilde{B}$. Then the control system with measurements is given by the two commutative diagrams.

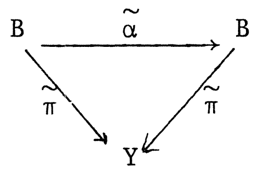




REMARK 1. The conditions on Γ are equivalent to asking that B is isomorphic to the pullback bundle of $\tilde{B}$ under C (Compare [1]).

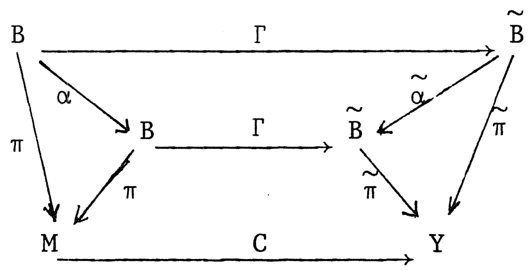
REMARK 2. This definition can be naturally interpreted as a specialization of the concept of a dynamical system with external variables given by WILLEMS (see [5] for references).

In this framework *output-feedback* is simply given by a map $\tilde{\alpha} : \tilde{B} \to \tilde{B}$ such that the diagram



commutes,

i.e. $\tilde{\alpha}$ is a bundle isomorphism. Given such an $\tilde{\alpha}$, there exists a *state-feedback* $\alpha : B \to B$ (see [3]) such that the following diagram commutes:



Then the system after output-feedback is given by $\Sigma(M,B,\hat{f})$ with $\hat{f} = f\circ\alpha$.

We now want to give a coordinatefree definition of local measured controlled invariance. This definition will be a straight-forward extension of the description of (local) controlled invariance in terms of an integrable connection on B, as given in our previous paper [3]. Afterwards we will show how this definition generates in local coordinates exactly the required properties of measured controlled invariance (see the introduction). Recall from [3] that an integrable connection on B is given by a so-called horizontal distribution on B, denoted here by H, and that it defines a lifting procedure of tangentvectors on M to tangentvectors on B. Specifically for a distribution D on M, the connection defines a distribution on B, denoted by D_ℓ.

DEFINITION 2.2 (*Local measured controlled invariance*).
Let Σ be a control system with measurements. Let D be an involutive distribution (of constant dimension) on M. We call D *locally measured controlled invariant* if there exists an integrable connection on B (i.e. a horizontal involutive distribution H) such that

(i) $f_* D_\ell \subset \dot{D}$

(ii) $\Gamma_* H$ is a horizontal involutive distribution on $\tilde{B}$.

REMARK 1. Without condition (ii) this is just the description of local controlled invariance of D as derived in [3]. Condition (ii) will ensure that we only need output-feedback.

REMARK 2. In the same way as in [3, def 3.2] we can give a definition of *global* measured controlled invariance.

Now we will show how in local coordinates this definition precisely gives the required properties. Because of our conditions on Γ we can locally find fiber respecting coordinates for B and $\tilde{B}$ such that $\Gamma = (C,\mathrm{id})$. Let $x = (x_1,\dots,x_n)$ be such coordinates for M (n-dimensional) and $(x,v) = (x_1,\dots,x_n,v_1,\dots,v_m)$ for B ((n+m)-dimensional). Then H is spanned by (see [3])

$$\frac{\partial}{\partial x_i} + h_i(x,v)\frac{\partial}{\partial v}, \qquad i = 1,\dots,n$$

where $h_i(x,v)$ are m-vectors and $\frac{\partial}{\partial v} = \left(\frac{\partial}{\partial v_1},\dots,\frac{\partial}{\partial v_m}\right)^t$.

Denote coordinates for $\tilde{B}$ ((p+m)-dimensional) as above by $(y,v) = (y_1,\dots,y_p,v_1,\dots,v_m)$. The condition that $\Gamma_* H$ is a horizontal distribution on $\tilde{B}$ is equivalent to the condition that there exist m-vectors $\tilde{h}_i(y,v)$, $i = 1,\dots,n$, defined on $\tilde{B}$ such that

$$h_i(x,v) = \tilde{h}_i(C(x),v), \qquad i = 1,\dots,n.$$

Condition (i) (D is locally controlled invariant) implies that the $h_i(x,v)$ satisfy some integrability conditions which guarantee (locally) the existence of a function $\alpha(x,v)$ such that

$$\frac{\partial d}{\partial x_i}(x,v) = h_i(x,\alpha(x,v)) \qquad i = 1,\dots,n$$

(see [3]). In the present case, because $h_i(x,v) = \tilde{h}_i C(x),v)$ there exists a function $\alpha(y,v)$ such that

$$\frac{\partial \tilde{\alpha}}{\partial x_i}(C(x),v) = \tilde{h}_i(C(x),\tilde{\alpha}(C(x),v)), \qquad i = 1,\dots,n.$$

This function $\tilde{\alpha}(y,v)$ is the output-feedback needed; if we define the feedback $\alpha(x,v) = \tilde{\alpha}(C(x),v)$, then D is invariant with respect to the dynamics modified by this feedback. In other words $[\hat{f}(\cdot,\bar{v}),D] \subset D$ for every (constant) $\bar{v}$, with $\hat{f}(x,v) := f(x,\alpha(x,v))$.

3. NECESSARY AND SUFFICIENT CONDITIONS

In this section we will prove our main theorem about local measured controlled invariance.

THEOREM 3.1. *Let* Σ *be a control system with measurements* (definition 2.1). *Let* D *be an involutive distribution on* M. *Then* D *is locally measured controlled invariant if and only if the following three conditions are satisfied*

(i) $f_*(\pi_*^{-1}(D)) \subset \dot{D} + f_*(\Delta_0^e)$

(ii) $f^*(\overline{\dot{P}+C^*(T^*Y)}) \supset \Gamma^*(T^*\tilde{B}) \cap (f^*\dot{P}+\pi^*(T^*M))$

(iii) $f^*(\dot{P}) \cap \Gamma^*(T^*\tilde{B})$ *is an involutive codistribution*

where P *is defined by* $P = \{\theta \in T^*M \mid \theta(X) = 0$ *for every* $X \in D\}$ *i.e.* $D = \ker P$, *and* Δ_0^e *is the vertical tangentspace of* B, *i.e.* $\Delta_0^e = \{X \in TB \mid \pi_* X = 0\}$.

REMARK. We have assumed in the theorem above that D, $\dot{D} + f_*(\Delta_0^e)$ and $f^*(\dot{P}) \cap \Gamma^*(T^*\tilde{B})$ have constant dimension.

Before going on to the proof of this theorem, we will sketch, how in the linear case conditions (i) and (ii) are equivalent to conditions (1.9a) and (1.9b), while condition (iii) is automatically satisfied.
In this case

$$f(x,u) = \begin{pmatrix} x \\ Ax+Bu \end{pmatrix}, \quad \text{with } x \in X = \mathbb{R}^n,\ u \in U = \mathbb{R}^m$$

and $y = Cx$, with $y \in Y = \mathbb{R}^p$.
Instead of the distribution D we have a linear subspace $V \subset X$, and P is given by $V^\perp$.

Then because $f(x,u)$ is linear

$$f_* = \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} \quad \text{and} \quad f^* = \begin{pmatrix} I & A^T \\ 0 & B^T \end{pmatrix}.$$

Condition (i) gives

$$\begin{pmatrix} I & 0 \\ A & B \end{pmatrix} \begin{pmatrix} V \\ U \end{pmatrix} \subset \begin{pmatrix} V \\ V \end{pmatrix} + \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} \begin{pmatrix} 0 \\ U \end{pmatrix}$$

which is readily seen to be equivalent to

(3.1) $\qquad AV + \mathcal{B} \subset V + \mathcal{B}$ or $AV \subset V + \mathcal{B}$.

Condition (ii) gives

$$\begin{pmatrix} I & A^T \\ 0 & B^T \end{pmatrix} \begin{pmatrix} V^\perp + (\mathrm{Ker}C)^\perp \\ V^\perp + (\mathrm{Ker}C)^\perp \end{pmatrix} \supset \begin{pmatrix} C^T & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} Y \\ U \end{pmatrix} \cap \left(\begin{pmatrix} I & A^T \\ 0 & B^T \end{pmatrix} \begin{pmatrix} V^\perp \\ V^\perp \end{pmatrix} + \begin{pmatrix} X \\ 0 \end{pmatrix} \right)$$

or

(3.2) $$\begin{pmatrix} V^\perp + (\mathrm{Ker}C)^\perp \\ 0 \end{pmatrix} + \begin{pmatrix} A^T \\ B^T \end{pmatrix} (V^\perp + (\mathrm{Ker}C)^\perp) \supset \begin{pmatrix} (\mathrm{Ker}C)^\perp \\ B^T V^\perp \end{pmatrix}.$$

We can define $V^\perp = W_1 \oplus W_2$ such that $B^T|W_1$ is injective, and $B^T W_2 = 0$. One can see that (3.2) is satisfied if and only if

(3.3) $\qquad A^T W_1 \subset V^\perp + (\mathrm{Ker}C)^\perp$.

From (3.1) it follows that $A^T(V^\perp \cap \mathcal{B}^\perp) \subset V^\perp$. Therefore since $W_2 \subset V^\perp \cap \mathcal{B}^\perp$, also $A^T W_2 \subset V^\perp$.
Concluding: $A^T V^\perp \subset V^\perp + (\mathrm{Ker}C)^\perp$, or by dualization

(3.4) $A(V \cap \mathrm{Ker}C) \subset V$.

PROOF (of theorem 3.1).

From [3, theorem 4.13] we know that condition (i) is necessary and sufficient for local controlled invariance of D. In other words condition (i) is equivalent to the existence of a horizontal involutive distribution H on B such that $f_* D_\ell \subset \dot{D}$ (D_ℓ defined by H). In fact when $f_*(\Delta_0^e) \cap \dot{D} = 0$, the distribution H above D, i.e. D_ℓ, is uniquely determined. Furthermore we may arbitrarily complete D_ℓ into a horizontal distribution H.
First we will prove that under condition (i) and the extra assumption $f_*(\Delta_0^e) \cap \dot{D} = 0$, conditions (ii) and (iii) are equivalent to the property that $\Gamma_*(D_\ell)$ is an involutive distribution on $\tilde{B}$, which contains no vertical vectors ($X \in T\tilde{B}$ is called vertical if $\tilde{\pi}_* X = 0$). Then we are done, because we may arbitrarily complete $\Gamma_*(D_\ell)$ into an involutive horizontal distribution $\tilde{H}$ on $\tilde{B}$, and hence we can construct a horizontal distribution H on B, such that $\Gamma_* H = \tilde{H}$ and H above D is equal to D_ℓ.
The basic observation is that $D_\ell = \ker f^* \dot{P}$. Indeed, let $(x_1,\dots,x_n)$ be local coordinates for M, such that $D = \mathrm{span}\,\{\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_k}\}$, $k \le n$, or equivalently $P = \mathrm{span}\,\{dx_{k+1},\dots,dx_n\}$.
Then D_ℓ is spanned by

$$\frac{\partial}{\partial x_i} + h_i(x,v)\,\frac{\partial f}{\partial v}, \qquad i = 1,\dots,k$$

with the h_i satisfying (see [3])

$$s^{\text{th}}\text{comp}\left(\frac{\partial f}{\partial x_i}(x,v) + \frac{\partial f}{\partial v}(x,v)h_i(x,v)\right) = 0 \qquad \begin{array}{l} i = 1,\dots,k \\ s = k+1,\dots,n. \end{array}$$

Because $f_*(\Delta_0^e) \cap \dot{D} = 0$, the h_i are uniquely determined. Now

$$\begin{aligned} f^*\dot{P} &= \mathrm{span}\,\{dx_{k+1},\dots,dx_n,\ df_{k+1},\dots,df_n\} \\ &= \mathrm{span}\{dx_{k+1},\dots,dx_n,\ \sum_{i=1}^n \frac{\partial f_{k+1}}{\partial x_i}dx_i + \frac{\partial f_{k+1}}{\partial v}dv,\dots,\sum_{i=1}^n \frac{\partial f_n}{\partial x_i}dx_i + \frac{\partial f_n}{\partial v}dv\} \end{aligned}$$

with $v = (v_1,\ldots,v_m)$.
Since $f_*(\Delta_0^e) \cap \dot{D} = 0$, the matrix

$$\begin{pmatrix} \dfrac{\partial f_{k+1}}{\partial v} \\ \vdots \\ \dfrac{\partial f_n}{\partial v} \end{pmatrix}$$

has full rank, and therefore there are no vertical vectors in ker $\dot{P}$. A close inspection shows that ker $\dot{P}$ is exactly equal to D_ℓ.

From [3,5] we know that $\Gamma_*(\ker\dot{P})$ is a well defined and involutive distribution on $\tilde{B}$ if and only if ker $\dot{P}$ + ker Γ_* is an involutive distribution. By dualization this is equivalent to the involutiveness of $\dot{P} \cap \Gamma^*(T^*\tilde{B})$. Finally, under the assumption $f_*(\Delta_0^e) \cap \dot{D} = 0$, condition (ii) comes down to $f^*(\overline{\dot{P}+C^*(T^*y)}) \supset \Gamma^*(T^*\tilde{B})$ which exactly says that for an $X \in D_\ell$, such that $\pi_*X \in D \cap \ker dC$, Γ_*X has to be zero. This is equivalent to the property that Γ_*D_ℓ does not contain vertical vectors.

If we drop the assumption $f_*(\Delta_0^e) \cap \dot{D} = 0$, we know that D_ℓ is not uniquely determined. In fact we may arbitrarily (modulo involutiveness) add vectors which are elements of $\Delta_0^e \cap f_*^{-1}(\dot{D})$. In this case, ker $\dot{P}$ also contains vertical vectors, namely $\Delta_0^e \cap f_*^{-1}(\dot{D})$. However modulo $\Delta_0^e \cap f_*^{-1}(\dot{D})$ the distribution D_ℓ is uniquely determined, and if condition (iii) is satisfied $\Gamma_*(D_\ell(\mathrm{mod}\ \Delta_0^e \cap f^{-1}(\dot{D})))$ is a well defined distribution on $\tilde{B}(\mathrm{mod}\ \Gamma_*(\Delta_0^e \cap f_*^{-1}(\dot{D}))$.

Again condition (ii) is equivalent to the property that $\Gamma_*(D_\ell(\mathrm{mod}\ \Delta_0^e \cap f_*^{-1}(\dot{D})))$ does not contain vertical vectors on $\tilde{B}(\mathrm{mod}\ \Gamma_*(\Delta_0^e \cap f_*^{-1}(\dot{D}))$. Finally we can complete $\Gamma_*(D_\ell(\mathrm{mod}\ \Delta_0^e\quad f_*^{-1}(\dot{D})))$ into a horizontal involutive distribution $\tilde{H}$ on $\tilde{B}$. This generates a horizontal involutive distribution H on B such that H above D equals D_ℓ (mod $\Delta_0^e \cap f_*^{-1}(\dot{D})$). □

We will now specialize theorem 3.1 to the case of *affine* systems, thereby sharpening the results already obtained in [2], and giving necessary and sufficient conditions.

We call a control system with measurements (definition 2.1) an affine control system with measurements if B and $\tilde{B}$ are vectorbundles, $\Gamma : B \to \tilde{B}$ is a linear map from the fibers of B onto the fibers of $\tilde{B}$, and $f : B \to TM$ is an affine map from the fibers of B onto the fibers of TM. Therefore there exist (locally) vectorfields A and B_i on M, such that $f(x,u) = A(x) + \sum_{i=1}^{m} u_i B_i(x)$. Define $\Delta(x) := A(x) + \text{span}\{B_1(x),\ldots,B_m(x)\}$ and $\Delta_0(x) := \text{span}\{B_1(x),\ldots,B_m(x)\}$

THEOREM 3.2. *Let Σ be an affine control system with measurements. Let* D *be an involutive distribution on* M. *Then* D *is locally measured controlled invariant if and only if the following three conditions are satisfied*

(i) $[\Delta,D] \subset D + \Delta_0$

(ii) $[A,D\cap\ker dC] \subset D$

$[B_i,D\cap\ker dC] \subset D, \quad (i = 1,\ldots,m)$

(iii) $f^*\dot{P} \cap \Gamma^*(T^*\tilde{B})$ *is an involutive codistribution*

REMARK 1. We have assumed that D, $D + \Delta_0$ and $f^*\dot{P} \cap \Gamma^*(T^*\tilde{B})$ have constant dimension.

REMARK 2. $f^*(\dot{P}) \cap \Gamma^*(T^*\tilde{B})$ involutive implies ker dC + D involutive. However, this last condition is *not* sufficient for local measured controlled invariance (see also [2]).

PROOF. We know (see [3] for references) that condition (i) is equivalent to local controlled invariance for affine systems. Therefore we only have to prove that under conditions (i) and (iii) condition (ii) is equivalent to condition (ii) of theorem 3.1. $f^*\dot{P} \cap \Gamma^*(T^*\tilde{B})$ is involutive or equivalently $\ker f^*\dot{P} + \ker \Gamma_*$ is involutive. Therefore $\pi_*(\ker f^*\dot{P}+\ker\Gamma_*) = D + \ker dC$ is involutive. An extended version of Frobenius' theorem see([4]) gives that locally we can find coordinates $(x_1,\ldots,x_n)$ for M, such that

$$D \cap \ker dC = \text{span}\left\{\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_\ell}\right\},$$

$$D = \text{span}\left\{\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_k}\right\}, \quad k \geq \ell,$$

$$\text{and } \ker dC = \text{span}\left\{\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_\ell},\frac{\partial}{\partial x_{k+1}},\ldots,\frac{\partial}{\partial x_m}\right\}.$$

Define for simplicity of notation $x^1 = (x_1,\dots,x_\ell)$, $x^2 = (x_{\ell+1},\dots,x_k)$, $x^3 = (x_{k+1},\dots,x_m)$ and $x^4 = (x_{m+1},\dots,x_n)$.
Then $P = \text{span}\{dx^3, dx^4\}$, $dC = \text{span}\{dx^2, dx^4\}$, $P + C^*(T^*y) = \text{span}\{dx^2, dx^3, dx^4\}$ and condition (ii) of theorem 3.1. comes down to (with $A^{2,3,4}$ and $B^{2,3,4}$ denoting the 2th, 3rd and 4th components of A respectively $B = (B_1,\dots,B_m)$)

$$\text{span}\{dx^2,dx^3,dx^4,d(A^{2,3,4}+uB^{2,3,4})\} \supset$$

$$\text{span}\{dx^2,dx^4,du\} \cap (\text{span}\{dx^3,dx^4,d(A^{3,4}+uB^{3,4})\} + \text{span}\{dx^1,dx^2,dx^3,dx^4\})$$

or equivalently:
$\text{span}\{dx^2,dx^3,dx^4,d(A^{2,3,4}+uB^{2,3,4})\} \supset \text{span}\{dx^2,dx^4,B^{3,4}du\}$.
From [3] we know that the horizontal part of $\ker\{dx^2,dx^3,dx^4,d(A^{2,3,4}+uB^{2,3,4})\}$ is spanned by vectors

$$\frac{\partial}{\partial x^1} + (K_1(x)u+h_1(x))\frac{\partial}{\partial u}$$

with $K_1(x)$ $m \times m$ matrices and $h_1(x)$ m-vectors satisfying

$$\frac{\partial A^{3,4}}{\partial x^1} + B^{3,4}h_1 = 0, \text{ and } \frac{\partial B^{3,4}}{\partial x^1} + B^{3,4}K_1 = 0. \tag{3.5}$$

These vectors are contained in $\ker\{dx^2,dx^4,B^{3,4}du\}$ if and only if $B^{3,4}h_1 = 0$ and $B^{3,4}K_1 = 0$. However, by (3.5), this is equivalent to

$$\frac{\partial A^{3,4}}{\partial x^1} = 0 \quad \text{and} \quad \frac{\partial B^{3,4}}{\partial x^1} = 0.$$

These are exactly the same conditions as $[A,D\cap \ker dC] \subset D$, respectively $[B_i,D\cap \ker dC] \subset D$, $i = 1,\dots,m$. □

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