stichting mathematisch centrum

MC

AFDELING MATHEMATISCHE BESLISKUNDE BW 152/81 DECEMBER
(DEPARTMENT OF OPERATIONS RESEARCH)

H.C. TIJMS & H. GROENEVELT

APPROXIMATIONS FOR (s,S) INVENTORY SYSTEMS WITH STOCHASTIC LEAD TIMES AND A SERVICE LEVEL CONSTRAINT

Preprint

kruislaan 413 1098 SJ amsterdam

Printed at the Mathematical Centre, 413 Kruislaan, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

1980 Mathematics subject classification: 90B05

Approximations for (s,S) inventory systems with stochastic lead times and a service level constraint*)

by

H.C. Tijms**) & H. Groenevelt***)

ABSTRACT. This paper considers the periodic review (s,S) inventory system with stochastic lead times and backlogging of excess demand. A service level constraint is imposed on the fraction of demands that is backlogged. The purpose is to determine for given value of S-s the reorder point s such that the service constraint is satisfied. Approximations for the reorder point s are obtained by a simple and direct approach that also applies to both the periodic review lost-sales (s,S) inventory system and the continuous review (s,S) inventory system with stochastic lead times. A normal approximation using only the first two moments of the total demand in the lead time plus review time is presented. Numerical investigation shows that the normal approximations yield excellent results for the service level when the coefficient of variation of the demand in the lead time plus review time does not exceed 0.5, otherwise good approximations can be obtained when gamma densities are fitted to the demand densities by matching the first two moments.

KEY WORDS & PHRASES: *(s,S) inventory system, stochastic lead times, periodic review, service level, reorder point, approximations*

---

*) This paper will be submitted for publication elsewhere.

**) Advisor Mathematical Centre. Address: Vrije Universiteit, Amsterdam

***) Vrije Universiteit, Amsterdam

1. Introduction

A frequently used inventory control system is the periodic review (s,S) system in which the inventory position is periodically reviewed and at a review the inventory position is ordered up to the level S when it is at or below the reorder point s. Under the assumptions of deterministic lead times and backlogging of excess demand, useful approximations for (s,S) policies were obtained in Ehrhardt [4], Freeland and Porteus [7], Naddor [10,11],Porteus [13], Roberts [14], Schneider [15] and Wagner et al [18] amongst others. In most of the literature it is assumed that penalty costs for unsatisfied demand are known. However, in practice these costs are often difficult to measure and as alternative one usually requires a certain service level for e.g. the fraction of demands that is backlogged. The approximations in [15] deal with such a service constraint, cf. also [10].

For the periodic review inventory system with stochastic lead times and backlogging of excess demand, an exact but intricate analysis was given in Kaplan [9] and recently improved upon in Ehrhardt [5]. Using this exact approach, approximately average cost optimal (s,S) policies were suggested in [5] for fixed set-up costs and linear holding and backlogging costs. In this paper we shall present a much simpler and direct approach for deriving new approximations for the periodic review (s,S) inventory system in which a service level is required for the fraction of demands that is backlogged. Moreover, our simple approach can be directly extended to both the periodic review lost-sales (s,S) inventory system and the continuous review (s,S) inventory system with stochastic lead times.

In section 2 we present the derivation of the approximations. In section 3 very simple approximations suited for routine use are found by fitting a normal distribution to the distribution of the total demand in the lead time plus review time by matching the first two moments. Numerical experience with the approximations is discussed in section 4. We found that the normal approximations give excellent results for the service level when the coefficient of variation of the demand in the lead time plus review time does not exceed 0.5, otherwise good approximations can be obtained when gamma densities are fitted to the demand densities by matching the first two moments.

2. The derivation of the approximations

We first consider the single-item dynamic inventory system in which the demands in the periods $t=0,1,\ldots$ are independent random variables having common probability density $f(x)$ with mean μ_1 and standard deviation σ_1. Excess demands are backlogged. Following the terminology in Hadley and Whitin [8], we define the inventory position as the stock on hand minus backorders plus stock on order. The system is controlled by an (s,S) policy under which at the beginnings of every T periods the inventory position is reviewed and at a review the inventory position is ordered up to the level S when it is at or below the reorder point s and no ordering is done otherwise. The length T of the review interval is a fixed positive integer. The lead time of a replenishment order is a nonnegative, integer-valued random variable L with mean $\mu(L)$ and standard deviation $\sigma(L)$.

Assuming that $S-s$ is given (e.g. by the classical EOQ formula), we shall concentrate on the determination of the reorder point s such that the following service level is achieved,

$$\text{fraction of demands that is backlogged} \leq 1-\beta \tag{1}$$

with β a prescribed number between 0 and 1. In practice β will be close to 1.

Before deriving an approximation for the reorder point s, we introduce the following notation. Denote by $F^{(k)}(x)$ and $f^{(k)}(x)$ the probability distribution function and the probability density of the total demand in k periods, $k\geq 1$. Let μ_k and σ_k be the mean and the standard deviation of $f^{(k)}(x)$. Note that $\mu_k=k\mu_1$ and $\sigma_k=\sqrt{k}\sigma_1$.

To do the approximate analysis, we make the following assumptions for the (s,S) policies that are relevant for the service constraint (1).

Assumptions

(i) replenishment orders do not cross in time and moreover the marginal lead time distribution of each order is independent of the number and size of outstanding orders,

(ii) $S-s$ is sufficiently large compared with the average demand μ_T in the review interval (say $S-s>\mu_T$),

(iii) just after the delivery of a replenishment order the stock on hand is positive except for a negligible probability.

We shall now first analyse the service level of a given (s,S) rule. Define a cycle as the time interval between two consecutive arrivals of replenishment orders. Using assumption (iii), we have approximately

fraction of demands that is backlogged ≃

$$\simeq \frac{\text{average shortage at the end of a cycle}}{\text{average demand per cycle}} \ . \qquad (2)$$

To derive this ratio, we tag one of the replenishment orders. Define the random variables

γ = overshoot of s at the review at which the tagged order was placed,
ξ = total demand in the lead time of the tagged order.

Clearly, the inventory position just before the placing of the tagged order equals $s-\gamma$. Further by assumption (i), any replenishment order placed before the tagged order will have been arrived when the tagged order comes in while no order placed after the tagged order will arrive before the tagged order. Hence

stock on hand just before the tagged order arrives = $s-\gamma-\xi$.

Denoting by $h(x)$ the probability density of $\gamma+\xi$, it follows that

$$\text{average shortage at the end of a cycle} \simeq \int_s^\infty (x-s)h(x)\,dx. \qquad (3)$$

Further, since the average demand per cycle is equal to the average order size,

$$\text{average demand per cycle} = S-s + E\gamma. \qquad (4)$$

By assumption ii) it follows from renewal theory that (cf. Cox [3])

$$\Pr\{\gamma\le x\} \simeq \frac{1}{\mu_T}\int_0^x \{1-F^{(T)}(y)\}dy, \quad x\ge 0,$$

independently of S-s. Further

$$E\gamma \simeq (\sigma_T^2 + \mu_T^2)/2\mu_T \ . \qquad (5)$$

To evaluate (3), note that γ and ξ are independent by the second part of assumption (i). Putting $g_e(x)=\mu_T^{-1}(1-F^{(T)}(x))$, it follows by conditioning that

$$\Pr\{\gamma+\xi\le x\} = \sum_{i\ge 0} \Pr\{L=i\}P\{\gamma+\xi\le x|L=i\} =$$

$$= \Pr\{L=0\} \int_0^x g_e(y)dy + \sum_{i\ge 1} \Pr\{L=i\} \int_0^x F^{(i)}(x-y)g_e(y)dy.$$

Hence

$$h(x) = \Pr\{L=0\}\mu_T^{-1} (1-F^{(T)}(x)) + \sum_{i\ge 1} \Pr\{L=i\} \int_0^x f^{(i)}(x-y)g_e(y)dy.$$

Using the change of variable $u=x-y$ in the latter integral, we next get

$$\frac{dh(x)}{dx} = -\Pr\{L=0\}\mu_T^{-1} f^{(T)}(x)-\mu_T^{-1} \sum_{i\ge 1} \Pr\{L=i\} \int_0^x f^{(i)}(u)f^{(T)}(x-u)du +$$

$$+ \sum_{i\ge 1} \Pr\{L=i\}f^{(i)}(x)g_e(0) =$$

$$= -\mu_T^{-1} \sum_{i\ge 0} \Pr\{L=i\}f^{(T+i)}(x)+\mu_T^{-1} \sum_{i\ge 1} \Pr\{L=i\}f^{(i)}(x),$$

where the last equality uses that $f^{(T+i)}(x)$ is the convolution of $f^{(i)}(x)$ and $f^{(T)}(x)$. Put for abbreviation

$$\eta(x) = \sum_{i\ge 0} \Pr\{L=i\}f^{(T+i)}(x) \text{ and } \xi(x) = \sum_{i\ge 1} \Pr\{L=i\}f^{(i)}(x).$$

Note that $\eta(x)$ is the probability density of the total demand η in the lead time plus review time, while $\xi(x)$ is the probability density of the total demand ξ in the lead time. Assuming that the one period's demand has a finite third moment, it follows that $E\gamma^2<\infty$ and so $\lim_{x\to\infty} x^2 h(x)=0$. We now obtain

$$\int_s^\infty (x-s)h(x)dx = \tfrac{1}{2} \int_s h(x)d(x-s)^2 = -\tfrac{1}{2} \int_s^\infty (x-s)^2 h'(x)dx =$$

$$= \frac{1}{2\mu_T} \int_s^\infty (x-s)^2\eta(x)dx - \frac{1}{2\mu_T} \int_s^\infty (x-s)^2\xi(x)dx. \qquad (6)$$

Using the relations (2)-(6) we find that for given value of S-s the service constraint (1) can be approximately satisfied by determining the reorder point s from

$$\int_s^\infty (x-s)^2\eta(x)dx - \int_s^\infty (x-s)^2\xi(x)dx = (1-\beta)2\mu_T\{S-s+(\sigma_T^2+\mu_T^2)/2\mu_T\} \qquad (7)$$

Recall that $\eta(x)$ and $\xi(x)$ are the probability densities of the demand in the lead time plus review time and the demand in the lead time. The mean and the standard deviation of the density $\eta(x)$ are given by

$$\mu=(T+EL)\mu_1 \text{ and } \sigma=\{(T+EL)\sigma_1^2+\sigma^2(L)\mu_1^2\}^{\frac{1}{2}},$$

while the mean and the standard deviation of the density $\xi(x)$ are given by the above formulae in which T=0 is put. We can simplify (7) when the service level β is close enough to 1. Therefore note that we can rewrite (6) as

$$\int_s^\infty (x-s)h(x)dx = \frac{1}{2\mu_T} E[\max(\eta-s,0)]^2 - \frac{1}{2\mu_T} E[\max(\xi-s,0)]^2.$$

The second term in this relation can be neglected for (s,S) policies having a sufficiently high service level and so for β close enough to 1 the relation (7) can be simplified to

$$\int_s^\infty (x-s)^2\eta(x)dx = (1-\beta)2\mu_T\{S-s+(\sigma_T^2+\mu_T^2)/2\mu_T^2\}. \tag{8}$$

The relation (7) for approximating the reorder point s is new. For the special case of deterministic lead times the simplified relation (8) was already obtained in [15] by a different approach using the asymptotic analysis in [14]. Our approach is not only simpler and more insightful, but it can also handle both the periodic review lost-sales (s,S) inventory system and the continuous review (s,S) inventory system with stochastic lead times.

REMARK 1. <u>The periodic review lost-sales (s,S) inventory system</u>

An examination of the above analysis shows that for the periodic review lost-sales (s,S) inventory system the average lost demand per cycle equals approximately (6) while the average demand per cycle is approximately equal to the average lost demand per cycle plus the average order size which is given by the right side of (4). Thus to achieve that the fraction of demands that is lost will not exceed $1-\beta$, it follows that for given value of S-s the reorder point can be approximately determined from (7) in which $1-\beta$ is replaced by $(1-\beta)/\beta$. Clearly for β close to 1 the backlogging and lost-sales models will not differ significantly. Other approximations for the periodic review lost-sales inventory model with a cost structure are discussed in Nahmias [12].

REMARK 2. The continuous review (s,S) inventory system

Consider a continuous review inventory model in which the demand process is described by a compound Poisson process. Customers arrive according to a Poisson process with rate λ and the demand per customer has probability density f(x) with mean μ_1 and standard deviation σ_1. Excess demands are backlogged. Under a continuous review (s,S) policy the inventory position is ordered up to the level S if at a demand epoch the inventory position falls at or below s, otherwise no ordering is done. The lead time of a replenishment order is a random variable L with mean $\mu(L)$ and standard deviation $\sigma(L)$. For the case of deterministic lead times exact methods to compute an average cost optimal (s,S) policy have been given in Archibald and Silver [1] and Federgruen et al [6]. We address ourselves to the determination of the reorder point s for given value of S-s to warrant that service constraint (1) holds. An examination of the above analysis shows that under the assumptions (i)-(iii) the reorder point s can be approximately determined from (7) provided that we put T=1 and interprete $\eta(x)$ as the probability density of the total demand of 1+N(L) customers where N(L) denotes the number of customers arriving in the lead time L. In assumption (ii) we now require that S-s is sufficiently large compared with the average demand μ_1 per customer (i.e. $S-s>\mu_1$). The mean and the standard deviation of the demand density $\eta(x)$ are now given by

$$\mu = (1+\lambda EL)\mu_1 \text{ and } \sigma = \{(1+\lambda EL)\sigma_1^2 + (\lambda^2\sigma^2(L)+\lambda EL)\mu_1^2\}^{\frac{1}{2}},$$

while the mean and the standard deviation of the density $\xi(x)$ are given by the above formulae in which $(1+\lambda EL)$ is replaced by λEL. For the continuous review lost-sales (s,S) inventory model we suggest to use (7) in which $1-\beta$ is replaced by $(1-\beta)/\beta$.

3. Approximations based on a normal demand density

In practice it may be difficult to solve (7) or (8) routinely by using the complete probability densities $\eta(x)$ and $\xi(x)$. We first give a very simple approximation that results from the simplified equation (8) when a normal density is fitted to $\eta(x)$ by matching the first two moments. It will be clear that the normal fit can be only applied when σ/μ is not too large, i.e. when $\sigma/\mu \leq 0.5$. Denote by $\Phi(x)$ and $\phi(x)$ the probability distribution function and the probability density of the standard normal distribution. Also, put for abbreviation

$$G(k) = \int_k^\infty (y-k)^2\phi(y)dy \text{ and } \rho=(1-\beta)2\mu_T\{S-s+(\sigma_T^2+\mu_T^2)/2\mu_T\}/\sigma^2. \tag{9}$$

We can write

$$s = \mu + k\sigma \tag{10}$$

for safety factor k. If we fit to $\eta(x)$ a normal density by matching the first two moments, then the simplified equation (8) becomes

$$G(k) = \rho. \tag{11}$$

Using the relation $G(k) = (1+k^2)(1-\Phi(k))-k\phi(k)$ (cf. [8]), we may tabulate the function G(k) and solve for k. However, there is an easier way to compute k. Therefore approximate the inverse function $k=G^{-1}(\rho)$ by a rational function. Using Werner et al [19], it was derived in Schneider [16] that

$$k = \frac{a_0+a_1w+a_2w^2+a_3w^3}{b_0+b_1w+b_2w^2+b_3w^3} + \varepsilon(w), \tag{12}$$

where for the case of $\rho \le 0.5$,

$w=(\ln(1/\rho^2))^{\frac{1}{2}}$, $a_0=-4.188413\cdot10^{-1}$, $a_1=-2.554696\cdot10^{-1}$, $a_2=5.189103\cdot10^{-1}$,

$a_3=0$, $b_0=1$, $b_1=2.134080\cdot10^{-1}$, $b_2=4.439934\cdot10^{-2}$, $b_3=-2.639787\cdot10^{-3}$,

and for the case $\rho > 0.5$,

$w=\rho$, $a_0=1.125946$, $a_1=-1.319002$, $a_2=-1.809643$, $a_3=-1.165009\cdot10^{-1}$,

$b_0=1$, $b_1=2.836738$, $b_2=6.559378\cdot10^{-1}$, $b_3=8.220435\cdot10^{-3}$.

Further, $\max|\varepsilon(w)|\le2.3\cdot10^{-4}$ for $-4\le k\le4$. The approximation determined by (10)-(12) uses only the first two moments of the demand in the lead time plus review time and is therefore suited for routine use in practice. This approximation will be called the normal approximation. We next discuss a very simple procedure which can be routinely used to compute the modified normal approximation that results when in (7) normal densities are fitted to $\eta(x)$ and $\xi(x)$ by matching the first two moments. Therefore denote by $\bar{\mu}$ and $\bar{\sigma}$ the mean and the standard deviation of the density $\xi(x)$. Observe that for the representation $s=\bar{\mu}+q\bar{\sigma}$ the second term in the left side of (7) can be written as $\bar{\sigma}^2G(q)$. The procedure for the modified normal approximation is as follows.

Step 0. Let $r=\rho$ with ρ defined in (9).

Step 1. Compute s from (10)-(12) with ρ replaced by r.

Step 2. Compute $q=(s-\bar{\mu})/\bar{\sigma}$ and $G(q)=(1+q^2)(1-\Phi(q))-q\phi(q)$. The function $\Phi(q)$ can be calculated from $\Phi(q)=1-\phi(q)\{a_1t+a_2t^2+a_3t^3\}+\varepsilon(q)$ where $|\varepsilon(q)|<10^{-5}$, $t=(1+\alpha q)^{-1}$, $\alpha=.33267$, $a_1=.4361836$, $a_2=-.1201676$ and $a_3=.937298$.

Step 3. Let $r=\rho+\bar{\sigma}^{-2}G(q)$ and go to step 1 until s has been sufficiently converged, i.e. when two successive values of s differ less than 0.10.

4. Numerical results

We first discuss the quality of the normal approximation for the periodic review (s,S) inventory model with stochastic lead times and backlogging of excess demands. Therefore we have tested a large number of examples in which the one period's demand has a negative binomial distribution $\{f(j),\ j\geq 0\}$. Although our primary goal is to test the service level of the approximate (s,S) policy, we assumed in the examples a fixed set-up cost K and a linear holding cost h=1. For the determination of the approximate reorder point s we have chosen S-s equal to the positive integer nearest to $\sqrt{2K\mu_1/h}$. The reorder point s determined from (10)-(11) was rounded to the integer $\lceil s\rceil$. The so-obtained approximate (s,S) policy was compared in service level (and average holding and ordering costs) with an (s^*,S^*) policy that resulted from an exact Lagrangian approach to find a policy which minimizes the average holding and ordering costs within the class of policies satisfying the service constraint (1). To compute this (s^*,S^*) policy we assumed linear shortage costs p in addition to the fixed set-up costs and the linear holding costs, where the linear holding and shortage costs are charged against the net inventory at the end of a period. For any value of p, we can compute an exact (s_p,S_p) policy which minimizes the total average costs per period. We have done this by a specialized policy-iteration algorithm as in [6], but alternatively the algorithm of Veinott and Wagner [17] could have been used. The exact service level of any (s,S) policy can be easily evaluated as will be shown below. We varied the shortage costs p until we found the smallest value of p for which the associated (s_p,S_p) policy satisfies the service constraint (1). We believe that in most cases the so-obtained (s^*,S^*) policy will minimize the average holding and ordering cost per period within the class of policies satisfying the service constraint (1). It is interesting to note that the smallest value of p is usually far away from $p(\beta)$ determined by $p(\beta)/(p(\beta)+h)=\beta$ and that the use of the $(s_{p(\beta)}, S_{p(\beta)})$ policy will in general lead to an erroneous service level.

It remains to indicate how for the inventory model with stochastic lead times an average cost optimal (s,S) policy and the service level of that policy can be computed in an exact way. Therefore the assumption that orders do not cross in time is modeled as in [4] and [9], cf. also p. 911 in [12]. By doing so and assuming for ease of notation T=1, it follows from [4] that the inventory model with stochastic lead times is equivalent to the inventory model with zero lead times and one period's expected holding and shortage costs

$$L(k) = h \sum_{j=0}^{k} (k-j)\eta(j) + p \sum_{j=k}^{\infty} (j-k)\eta(j), \quad k\geq 0,$$

where k denotes the stock on hand just after ordering and $\{\eta(j),\ j\geq 0\}$ is the probability distribution of the total demand in 1+L periods. Note that for the inventory model with stochastic lead times and no cross-overs of orders L(k) represents the expected holding and shortage costs incurred in period 1+L given that at the beginning of period 0 the inventory position is k just after ordering. The inventory model with zero lead times can be exactly solved by standard methods. Similarly, assuming for ease T=1, it follows from [4] that in the inventory model with stochastic lead times and no cross-overs of orders the service level for a given policy is obtained by computing for that policy the average costs in an inventory model with zero lead times and one period's costs

$$U(k) = \sum_{j=0}^{k} \xi(j) \sum_{r=k-j}^{\infty} (r-k+j)f(j) + \mu_1 \sum_{j=k+1}^{\infty} \xi(j), \quad k\geq 0,$$

where $\{\xi(j),\ j\geq 0\}$ denotes the probability distribution of the total demand in the lead time L. Note that in the inventory model with stochastic lead times and no cross-overs of orders U(k) represents the expected amount of the demand in the period 1+L that is backlogged given that at the beginning of period 0 the inventory position is k just after ordering.

To test the quality of the approximations, we distinguish between the cases $\sigma/\mu\leq 0.5$ and $\sigma/\mu>0.5$ with σ/μ is the coefficient of variation of the total demand in the lead time plus review time. We first discuss the performance of the normal approximation for the case of $\sigma/\mu\leq 0.5$. Therefore we consider in the tables 1 and 2 a set of 90 numerical examples each having a review time T=1 and a negative binomial distribution with $\sigma_1^2/\mu_1=3$ for the one period's demand. The average demand μ_1 has five values, 8, 16, 24, 32 and 48. We consider the following three lead time distributions each with an average value of 2,

(1) $\Pr\{L=1\}=1/4$, $\Pr\{L=2\}=1/2$, $\Pr\{L=3\}=1/4$ $(\sigma^2(L)=0.5)$,

(2) $\Pr\{L=1\}=\Pr\{L=3\}=1/2$ $(\sigma^2(L)=1)$,

(3) $\Pr\{L=0\}=\Pr\{L=2\}=\Pr\{L=4\}=0.10$, $\Pr\{L=1\}=\Pr\{L=3\}=0.35$ $(\sigma^2(L)=1.5)$.

The service level is varied as $\beta=0.9$, 0.95 and 0.99. The set-up cost K has values 32 and 64 and the linear holding cost h=1. Recall that S-s was chosen as the positive integer nearest to $\sqrt{2K\mu_1/h}$. For the normal approximation (s,S) policies computed from (10)-(11) we denote by $\beta(s,S)$ the actual value of the service level. Similarly, $\beta(s^*,S^*)$ for the (s^*,S^*) policy computed by the Lagrangian approach. Although we are primarily interested in testing the service level, it may be of some secondary interest to consider the relative difference percentage in costs $\Delta C=100\{C(s,S)-C(s^*,S^*)\}/C(s^*,S^*)$ where $C(s,S)$ and $C(s^*,S^*)$ denote the average set-up and holding costs under the (s,S) policy and the (s^*,S^*) policy. For the above 90 examples we give in the tables 1 and 2 the policies (s,S) and (s^*,S^*) together with the performance measures $\beta(s,S)$, $\beta(s^*,S^*)$ and ΔC. For practical purposes it seems reasonable to allow for the approximate (s,S) policies a deviation in service of $|\beta(s,S)-\beta|=0.01$ for $\beta=0.9$, 0.95 and of $|\beta(s,S)-\beta|=0.005$ for $\beta=0.99$. By far most of the 90 examples have a deviation of the desired service that is within these limits, where the deviation in service becomes occasionally as high as 0.0175, 0.0127 and 0.0076 for $\beta=0.9$, 0.95 and 0.99. We note that the demand density $\eta(x)$ is not unimodal in the above examples with $\mu_1=8$ and $\sigma^2(L)=\frac{1}{2}$ or with $\mu_1\geq16$ and $\sigma^2(L)\geq1$. We also note that in the above examples the average time between orders $\mu_T^{-1}\sqrt{2K\mu_1/h}$ is larger than T. In case this average time is at or below T, we suggest to use an (S,S) policy with S determined by the equation $\mu_T^{-1}\int_S^\infty (x-S)\eta(x)dx=1-\beta$, cf. formula (5.2) in [8]. This equation can be very easily solved when $\eta(x)$ is approximated by a normal density, cf. also Brown [2]. The excellent performance of the normal approximation for the case of $\sigma/\mu\leq0.5$ was also found in the many other examples we have tested where the service level $\beta\geq 0.9$. The normal approximation is not recommended when $\beta< 0.9$; for the above examples with $\beta= 0.8$ we found that the service level of the normal approximation may be as much as 0.05 higher than the desired one. However, for the case of $\sigma/\mu\leq0.5$ the modified normal approximation gives also excellent results when $\beta < 0.9$.

TABLE 1. The normal (s,S) approximations and (sub)optimal (s^*,S^*) policies

	K=32			K=64			
	$\mu_1=8$	$\mu_1=16$	$\mu_1=24$	$\mu_1=8$	$\mu_1=16$	$\mu_1=24$	
(s,S)	(24,47)	(48,80)	(71,110)	(23,55)	(45,90)	(68,123)	$\sigma^2(L)=0.5$
(s^*,S^*)	(24,49)	(47,82)	(70,113)	(21,56)	(43,92)	(66,126)	$\beta=.9$
$\beta(s,S)$	.9011	.9056	.8997	.9150	.9087	.9078	$\sigma/\mu=.42,$
$\beta(s^*,S^*)$	.9075	.9045	.9014	.9005	.9011	.9037	.34, .31
$\Delta C(\%)$	-1.42	0.81	-0.16	4.32	2.53	1.61	
(s,S)	(26,49)	(52,84)	(79,118)	(24,56)	(49,94)	(75,130)	$\sigma^2(L)=1$
(s^*,S^*)	(25,52)	(51,87)	(78,121)	(23,59)	(47,98)	(73,134)	$\beta=.9$
$\beta(s,S)$	.9012	.8992	.9018	.9064	.9045	.9052	$\sigma/\mu=.49,$
$\beta(s^*,S^*)$	.9030	.9009	.9036	.9061	.9016	.9030	.42, .39
$\Delta C(\%)$	0.39	0.20	-0.14	0.92	1.58	1.13	
(s,S)	(28,51)	(57,89)	(86,125)	(26,58)	(53,98)	(81,136)	$\sigma^2(L)=1.5$
(s^*,S^*)	(26,54)	(54,92)	(82,128)	(24,61)	(50,102)	(76,141)	$\beta=.9$
$\beta(s,S)$	.9064	.9106	.9105	.9120	.9102	.9104	$\sigma/\mu=.54,$
$\beta(s^*,S^*)$	.9016	.9037	.9027	.9054	.9039	.9008	.48, .46
$\Delta C(\%)$	2.72	3.18	3.22	2.93	2.82	3.73	
(s,S)	(28,51)	(54,86)	(81,120)	(27,59)	(52,97)	(77,132)	$\sigma^2(L)=0.5$
(s^*,S^*)	(29,53)	(55,88)	(81,122)	(27,61)	(52,99)	(78,135)	$\beta=.95$
$\beta(s,S)$	.9415	.9440	.9483	.9491	.9489	.9476	$\sigma/\mu=.42,$
$\beta(s^*,S^*)$	.9508	.9501	.9504	.9517	.9507	.9524	.34, .31
$\Delta C(\%)$	-3.80	-2.69	-0.97	-0.77	-0.60	-1.92	
(s,S)	(31,54)	(60,92)	(90,129)	(29,61)	(57,102)	(86,141)	$\sigma^2(L)=1$
(s^*,S^*)	(31,57)	(60,94)	(89,128)	(29,63)	(57,104)	(85,142)	$\beta=.95$
$\beta(s,S)$	.9464	.9486	.9537	.9475	.9488	.9523	$\sigma/\mu=.49,$
$\beta(s^*,S^*)$	.9515	.9509	.9501	.9501	.9506	.9503	.42, .39
$\Delta C(\%)$	-1.91	-0.88	1.54	-0.72	-0.54	0.82	
(s,S)	(33,56)	(65,97)	(98,137)	(31,63)	(62,107)	(93,148)	$\sigma^2(L)=1.5$
(s^*,S^*)	(33,58)	(64,99)	(95,138)	(31,66)	(61,110)	(91,151)	$\beta=.95$
$\beta(s,S)$	.9471	.9513	.9559	.9489	.9527	.9536	$\sigma/\mu=.54$
$\beta(s^*,S^*)$	.9505	.9507	.9507	.9527	.9524	.9513	.48, .46
$\Delta C(\%)$	-1.14	0.58	2.64	-1.09	0.44	1.21	
(s,S)	(36,59)	(67,99)	(98,137)	(35,67)	(65,110)	(95,150)	$\sigma^2(L)=0.5$
(s^*,S^*)	(40,63)	(71,102)	(102,138)	(38,70)	(68,113)	(99,152)	$\beta=.99$
$\beta(s,S)$	.9824	.9854	.9871	.9844	.9861	.9870	$\sigma/\mu=.42,$
$\beta(s^*,S^*)$	.9911	.9907	.9905	.9904	.9901	.9907	.34, .31
$\Delta C(\%)$	-9.96	-6.63	-4.46	-6.31	-4.43	-4.31	
(s,S)	(39,62)	(75,107)	(110,149)	(38,70)	(72,117)	(107,162)	$\sigma^2(L)=1$
(s^*,S^*)	(42,66)	(76,106)	(108,144)	(41,74)	(73,117)	(105,157)	$\beta=.99$
$\beta(s,S)$	.9834	.9899	.9926	.9852	.9890	.9922	$\sigma/\mu=.49,$
$\beta(s^*,S^*)$	.9902	.9907	.9902	.9912	.9900	.9900	.42, .39
$\Delta C(\%)$	-7.54	-0.97	3.61	-6.20	-1.17	2.69	
(s,S)	(42,65)	(81,113)	(121,160)	(41,73)	(79,124)	(117,172)	$\sigma^2(L)=1.5$
(s^*,S^*)	(45,69)	(82,114)	(119,157)	(43,77)	(79,125)	(115,170)	$\beta=.99$
$\beta(s,S)$	.9842	.9891	.9921	.9862	.9898	.9915	$\sigma/\mu=.54,$
$\beta(s^*,S^*)$	.9903	.9902	.9905	.9902	.9900	.9900	.48, .46
$\Delta C(\%)$	-7.04	-1.46	2.55	-4.36	-0.21	1.97	
$\sqrt{2K/h\mu_1}$	2.83	2	1.63	4	2.83	2.31	

TABLE 2. The normal (s,S) approximations and (sub)optimal (s^*,S^*) policies

	K=32		K=64		
	μ_1=32	μ_1=48	μ_1=32	μ_1=48	
(s,S)	(96,141)	(144,199)	(91,155)	(138,216)	σ^2(L)=0.5
(s^*,S^*)	(94,143)	(143,207)	(89,158)	(135,214)	β=.9
β(s,S)	.9023	.8923	.9071	.9087	σ/μ=.29, .28
β(s^*,S^*)	.9007	.9017	.9031	.9001	
ΔC(%)	0.84	-2.58	1.52	2.76	
(s,S)	(106,151)	(161,216)	(101,165)	(154,232)	σ^2(L)=1
(s^*,S^*)	(104,154)	(160,221)	(99,169)	(151,231)	β=.9
β(s,S)	.9003	.8970	.9056	.9095	σ/μ=.38, .36
β(s^*,S^*)	.9007	.9006	.9028	.9017	
ΔC(%)	0.17	-0.71	1.08	2.01	
(s,S)	(116,161)	(176,231)	(110,174)	(168,246)	σ^2(L)=1.5
(s^*,S^*)	(110,163)	(168,235)	(104,178)	(159,246)	β=.9
β(s,S)	.9117	.9087	.9140	.9175	σ/μ=.44, .43
β(s^*,S^*)	.9014	.9012	.9018	.9007	
ΔC(%)	3.80	3.10	4.31	5.30	
(s,S)	(107,152)	(160,215)	(103,167)	(155,233)	σ^2(L)=0.5
(s^*,S^*)	(108,154)	(162,223)	(103,168)	(155,230)	β=.95
β(s,S)	.9459	.9414	.9497	.9524	σ/μ=.29, .28
β(s^*,S^*)	.9501	.9503	.9501	.9500	
ΔC(%)	-2.02	-4.21	-0.18	1.13	
(s,S)	(120,165)	(180,235)	(115,179)	(174,252)	σ^2(L)=1
(s^*,S^*)	(118,164)	(177,236)	(114,177)	(170,244)	β=.95
β(s,S)	.9558	.9554	.9553	.9627	σ/μ=.38, .36
β(s^*,S^*)	.9509	.9509	.9521	.9509	
ΔC(%)	2.08	1.94	1.22	4.64	
(s,S)	(131,176)	(198,253)	(125,189)	(190,268)	σ^2(L)=1.5
(s^*,S^*)	(127,175)	(190,254)	(122,190)	(183,261)	β=.95
β(s,S)	.9573	.9581	.9566	.9617	σ/μ=.44, .43
β(s^*,S^*)	.9506	.9503	.9516	.9503	
ΔC(%)	3.27	4.12	2.27	5.17	
(s,S)	(129,174)	(191,246)	(126,190)	(187,265)	σ^2(L)=0.5
(s^*,S^*)	(132,176)	(191,220)	(129,186)	(189,259)	β=.99
β(s,S)	.9878	.9876	.9887	.9909	σ/μ=.29, .28
β(s^*,S^*)	.9902	.9902	.9905	.9903	
ΔC(%)	-3.12	-2.88	-2.03	1.14	
(s,S)	(146,191)	(218,273)	(142,206)	(212,290)	σ^2(L)=1
(s^*,S^*)	(140,183)	(202,228)	(137,193)	(200,268)	β=.99
β(s,S)	.9946	.9964	.9943	.9970	σ/μ=.38, .36
β(s^*,S^*)	.9901	.9904	.9904	.9901	
ΔC(%)	7.27	12.44	5.63	11.98	
(s,S)	(160,205)	(240,295)	(156,220)	(234,312)	σ^2(L)=1.5
(s^*,S^*)	(155,200)	(229,289)	(151,212)	(224,295)	β=.99
β(s,S)	.9930	.9941	.9931	.9951	σ/μ=.44, .43
β(s^*,S^*)	.9900	.9902	.9901	.9902	
ΔC(%)	4.63	6.63	4.41	8.00	
$\sqrt{2K/h\mu_1}$	1.41	1.15	2	1.63	

Next we discuss the approximations for the case of $\sigma/\mu > 0.5$. Since for a normal distribution the probability of negative values cannot be neglected for larger values of σ/μ, it will be clear that in general the normal approximation cannot be used for $\sigma/\mu > 0.5$. As alternative the demand densities $\eta(x)$ and $\xi(x)$ can be approximated by gamma densities by matching the first two moments. Then the relations (7) and (8) require the evaluation of incomplete gamma integrals. We found that the simplified relation (8) with a gamma density fitted to $\eta(x)$ leads in general to approximations with an erroneous service. However, for most practical situations good approximations can be obtained by using the relation (7) in which gamma densities are fitted to the demand densities $\eta(x)$ and $\xi(x)$ by matching the first two moments. These approximations can be further improved when relation (7) with the true demand densities $\eta(x)$ $\xi(x)$ can be used. Clearly, in practice it is often only possible to use a two moments method. As illustration we consider in table 3 a set of 24 examples each having a review time of T=1 and a negative binomial distribution with mean $\mu_1=8$ for the one period's demand. We take for σ_1^2/μ_1 the four values 5, 10, 15 and 25. We consider the following two lead time distributions each with an average value of 2, (i) $\Pr\{L=1\}=\Pr\{L=3\}=\frac{1}{2}$ $(\sigma(L)=1)$ and (ii) $\Pr\{L=0\}=\Pr\{L=4\}=\frac{1}{2}$ $(\sigma(L)=2)$. The service level is varied as $\beta=0.9$, 0.95 and 0.99. We choose the set-up cost K=64 and the linear holding cost h=1. We mention that the demand densities $\eta(x)$ and $\xi(x)$ are not unimodal when $\sigma_1^2/\mu_1=5$ and $\sigma(L)=2$. In table 3 we give the (s,S) policies computed by the Lagrangian approach and the approximate (s,S) policies computed from (7) with the true demand densities and with the fitted gamma densities respectively. Also, we give the normal approximation (s,S) policies. The various (s,S) policies and their associated service levels $\beta(s,S)$ are denoted by Lagr., True, Gamma and Norm., respectively.

We can conclude from our numerical investigation that for the case of $\sigma/\mu \le 0.5$ (say) the normal approximation gives excellent results when the service $\beta \ge 0.9$, while for $\beta < 0.9$ the modified normal approximation is recommended. For the case of $\sigma/\mu > 0.5$, good approximations can be obtained for most practical situations by using relation (7) in which gamma densities are fitted to the probability densities of the demand in the lead time plus review time and the demand in the lead time by matching the first two moments.

Similar conclusions can be drawn from our numerical experience with the continuous review (s,S) inventory system with backlogging. For the case of $\sigma/\mu \le 0.5$ the normal approximation and the modified normal approximation are now recommended for $\beta \ge 0.97$ (say) and $\beta < 0.97$ respectively. For the case of $\sigma/\mu > 0.5$ good approximations can be obtained by relation (7) where gamma densities are fitted to the demand densities $\eta(x)$ and $\xi(x)$ by matching the first two moments. As illustration, we consider in the tables

TABLE 3. Approximations for the case of $\sigma/\mu>0.5$

σ_1^2/μ_1	5		10		15		25		$\sigma(L)$
σ/μ	.565		.726		.858		1.074		=1
	(s,S)	β(s,S)	(s,S)	β(s,S)	(s,S)	β(s,S)	(s,S)	β(s,S)	
Lagr.	(25,62)	.9041	(31,69)	.9042	(36,76)	.9011	(48,89)	.9013	β=
True	(26,58)	.9010	(33,65)	.9046	(40,72)	.9066	(54,86)	.9090	.9
Gamma	(26,58)	.9010	(33,65)	.9046	(40,72)	.9066	(54,86)	.9090	
Norm.	(27,59)	.9095	(32,64)	.8982	(36,68)	.8854	(44,76)	.8691	
Lagr.	(33,68)	.9528	(41,79)	.9509	(50,88)	.9507	(67,107)	.9502	β=
True	(33,65)	.9491	(43,75)	.9515	(53,85)	.9530	(73,105)	.9549	.95
Gamma	(34,66)	.9541	(43,75)	.9515	(53,85)	.9530	(74,106)	.9566	
Norm.	(32,64)	.9438	(38,70)	.9315	(43,75)	.9201	(52,84)	.9021	
Lagr.	(47,82)	.9901	(63,100)	.9900	(79,117)	.9902	(110,150)	.9902	β=
True	(47,79)	.9894	(64,96)	.9897	(81,113)	.9901	(115,147)	.9908	.99
Gamma	(50,82)	.9927	(67,99)	.9918	(84,116)	.9917	(117,149)	.9915	
Norm.	(42,74)	.9808	(50,82)	.9706	(57,89)	.9621	(69,101)	.9477	
σ_1^2/μ_1	5		10		15		25		$\sigma(L)$
σ/μ	.808		.928		1.034		1.219		=2
	(s,S)	β(s,S)	(s,S)	β(s,S)	(s,S)	β(s,S)	(s,S)	β(s,S)	
Lagr.	(32,71)	.9029	(37,77)	.9030	(41,83)	.9030	(52,94)	.9006	β=
True	(34,66)	.9015	(40,72)	.9037	(46,78)	.9051	(60,92)	.9119	.9
Gamma	(30,62)	.8689	(37,69)	.8868	(44,76)	.8964	(58,90)	.9059	
Norm.	(36,68)	.9156	(40,72)	.9037	(44,76)	.8964	(50,82)	.8778	
Lagr.	(41,78)	.9506	(49,88)	.9502	(57,97)	.9502	(73,115)	.9506	β=
True	(42,74)	.9490	(52,84)	.9521	(61,93)	.9525	(80,112)	.9550	.95
Gamma	(40,72)	.9393	(49,81)	.9426	(58,90)	.9453	(78,110)	.9518	
Norm.	(42,74)	.9490	(47,79)	.9353	(51,83)	.9242	(59,91)	.9089	
Lagr.	(58,92)	.9904	(74,112)	.9902	(90,128)	.9902	(119,161)	.9901	β=
True	(58,90)	.9899	(76,108)	.9903	(93,125)	.9906	(126,158)	.9910	.99
Gamma	(59,91)	.9910	(74,106)	.9889	(90,122)	.9890	(122,154)	.9897	
Norm.	(55,87)	.9860	(62,94)	.9747	(67,99)	.9645	(77,109)	.9502	

4 and 5 a set of 48 examples each having a constant lead time of L=1 and a negative binomial distribution for the demand per customer. The mean demand per customer has two values $\mu_1=5$ and 10. The arrival rate of customers has the values $\lambda=2$ and 10. For $\lambda=10$ we consider the values $\sigma_1^2/\mu_1=2.5$, 5 and 7 so that $\sigma/\mu\le 0.5$, while for $\lambda=2$ we consider the values $\sigma_1^2/\mu_1=5$, 10, 15 and 25 so that $\sigma/\mu>0.5$. The service level is varied as $\beta=0.9$, 0.95 and 0.99. The set-up cost K=32 and the linear holding cost h=1 and we have chosen S-s equal to the integer nearest to $\sqrt{2K\lambda\mu_1/h}$. In table 4 with $\lambda=10$ we give the (s^*,S^*) policies computed by an exact Lagrangian approach and approximate (s,S) policies together with their associated service levels, where the approximate (s,S) policies correspond to the modified normal approximation when $\beta=0.9$, 0.95 and to the normal approximation when $\beta=0.99$. In table 5 with $\lambda=2$ we give

TABLE 4. Approximate (s,S) policies for the continuous review inventory system

	μ_1=5			μ_1=10			
σ_1^2/μ_1	2.5	5	7.5	2.5	5	7.5	λ=
σ/μ	.358	.417	.468	.325	.358	.388	10
(s,S)	(58,115)	(61,118)	(65,122)	(119,199)	(124,204)	(128,208)	β=
(s*,S*)	(55,122)	(59,127)	(63,132)	(115,214)	(119,219)	(123,225)	.9
β(s,S)	.9039	.8963	.8961	.8976	.8979	.8955	
β(s*,S*)	.9002	.9013	.9029	.9020	.9006	.9006	
ΔC(%)	2.62	0.65	0.14	1.27	1.74	1.11	
(s,S)	(67,124)	(71,128)	(76,133)	(135,215)	(140,220)	(146,226)	β=
(s*,S*)	(66,131)	(71,139)	(77,145)	(133,229)	(140,237)	(146,244)	.95
β(s,S)	.9484	.9417	.9401	.9460	.9423	.9417	
β(s*,S*)	.9509	.9503	.9512	.9502	.9513	.9509	
ΔC(%)	0.16	-1.87	-3.03	-0.09	-1.92	-2.11	
(s,S)	(87,144)	(95,152)	(101,158)	(170,250)	(179,259)	(187,267)	β=
(s*,S*)	(86,149)	(96,161)	(105,171)	(168,260)	(179,272)	(189,284)	.99
β(s,S)	.9901	.9884	.9856	.9900	.9889	.9876	
β(s*,S*)	.9902	.9904	.9901	.9902	.9903	.9902	
ΔC(%)	0.42	-1.93	-4.67	0.38	1.07	-2.47	

TABLE 5. Approximate (s,S) policies for the continuous review inventory system

σ_1^2/μ_1	5		10		15		25		μ_1
σ/μ	.745		.943		1.106		1.374		=5
	(s,S)	β(s,S)	(s,S)	β(s,S)	(s,S)	β(s,S)	(s,S)	β(s,S)	λ=2
True	(18,43)	.8988	(25,50)	.9073	(31,56)	.9060	(45,70)	.9111	β=
Gamma	(18,43)	.8988	(25,50)	.9073	(31,56)	.9060	(45,70)	.9111	.9
True	(25,50)	.9515	(34,59)	.9528	(43,68)	.9532	(62,87)	.9552	β=
Gamma	(25,50)	.9515	(34,59)	.9528	(43,68)	.9532	(63,88)	.9570	.95
True	(39,64)	.9904	(54,79)	.9903	(70,95)	.9907	(102,127)	.9910	β=
Gamma	(39,64)	.9904	(55,80)	.9910	(71,96)	.9912	(104,129)	.9917	.99
σ_1^2/μ_1	5		10		15		25		μ_1
σ/μ	.624		.745		.850		1.027		=10
	(s,S)	β(s,S)	(s,S)	β(s,S)	(s,S)	β(s,S)	(s,S)	β(s,S)	λ=2
True	(35,71)	.9000	(42,78)	.9036	(49,85)	.9060	(63,99)	.9089	β=
Gamma	(34,70)	.8934	(41,77)	.8986	(48,84)	.9019	(63,99)	.9089	.9
True	(46,82)	.9523	(55,91)	.9517	(65,101)	.9535	(84,120)	.9541	β=
Gamma	(44,80)	.9452	(54,90)	.9490	(64,100)	.9513	(84,120)	.9541	.95
True	(67,103)	.9903	(83,119)	.9905	(99,135)	.9905	(132,168)	.9909	β=
Gamma	(66,102)	.9895	(82,118)	.9899	(99,135)	.9905	(133,169)	.9913	.99

the approximate (s,S) policies obtained from (7) with the true demand densities $\eta(x)$ and $\xi(x)$, the approximate (s,S) policies obtained from (7) with gamma densities fitted to the demand densities and the associated service

TABLE 6. Approximate (s,S) policies for the continuous review inventory system.

σ_1^2/μ_1	2.5		5		7.5		$\sigma(L)=$
σ/μ	.457		.481		.504		.354
	(s,S)	β(s,S)	(s,S)	β(s,S)	(s,S)	β(s,S)	
True	(139,219)	.9037	(144,224)	.9057	(148,228)	.9053	β=
Gamma	(136,216)	.8955	(141,221)	.8981	(146,226)	.9006	.9
Norm.	(136,216)	.8955	(140,220)	.8955	(144,224)	.8957	
True	(162,242)	.9514	(168,248)	.9515	(174,254)	.9518	β=
Gamma	(161,241)	.9498	(167,247)	.9501	(174,254)	.9518	.95
Norm.	(156,236)	.9413	(161,241)	.9406	(165,245)	.9385	
True	(206,286)	.9901	(216,296)	.9901	(226,306)	.9901	β=
Gamma	(211,291)	.9919	(221,301)	.9917	(232,312)	.9919	.99
Norm.	(207,287)	.9905	(214,294)	.9893	(220,300)	.9880	

σ_1^2/μ_1	2.5		5		7.5		$\sigma(L)=$
σ/μ	.644		.662		.679		.612
	(s,S)	β(s,S)	(s,S)	β(s,S)	(s,S)	β(s,S)	
True	(169,249)	.9077	(173,253)	.9081	(177,257)	.9086	β=
Gamma	(159,239)	.8844	(163,243)	.8864	(168,248)	.8906	.9
Norm.	(163,243)	.8941	(166,246)	.8933	(169,249)	.8927	
True	(195,275)	.9524	(201,281)	.9526	(208,288)	.9541	β=
Gamma	(191,271)	.9469	(196,276)	.9463	(202,282)	.9471	.95
Norm.	(189,269)	.9440	(192,272)	.9407	(196,276)	.9393	
True	(246,326)	.9903	(256,336)	.9902	(266,346)	.9902	β=
Gamma	(252,332)	.9922	(261,341)	.9917	(271,351)	.9916	.99
Norm.	(263,343)	.9948	(269,349)	.9936	(274,354)	.9923	

levels $\beta(s,S)$. Finally, we give in table 6 a set of 18 examples each having an arrival rate of $\lambda=10$ customers and a negative binomial distribution with mean $\mu_1=10$ for the demand per customer, where σ_1^2/μ_1 is varied as 2.5, 5 and 7.5. We consider the stochastic lead times (i) $\Pr\{L=\frac{1}{2}\}=\Pr\{L=1\frac{1}{2}\}=1/4$, $\Pr\{L=1\}=\frac{1}{2}$ with $EL=1$ and $\sigma(L)=.354$, and (ii) $\Pr\{L=0\}=\Pr\{L=1\}=\Pr\{L=2\}=0.1$, $\Pr\{L=\frac{1}{2}\}=\Pr\{L=1\frac{1}{2}\}=0.35$ with $EL=1$ and $\sigma(L)=.612$. For the examples with $\sigma(L)=.354$ the demand density $\xi(x)$ is bimodal, while for the examples with $\sigma(L)=.612$ both demand densities $\eta(x)$ and $\xi(x)$ are trimodal and bimodal respectively for $\sigma_1^2/\mu_1=2.5$, 5 and $\sigma_1^2/\mu_1=7.5$ respectively. We choose S-s equal to the nearest integer to $\{2K\lambda\mu_1/h\}^{\frac{1}{2}}$ where K=32 and h=1. In table 6 we give the true and the gamma approximations both from (7) for $\beta=0.9$, 0.95 and 0.99, the modified normal approximation for $\beta=0.9$, 0.95 and the normal approximation for $\beta=0.99$.

REFERENCES

1. Archibald, B. and Silver, E., "(s,S) Policies under Continuous Review and Discrete Compound Poisson Demand", *Management Sci.* 24, 899-909, 1978.
2. Brown, R.G. *Decision Rules for Inventory Management,* Holt, Rinehart & Winston, New York, 1967.
3. Cox, D.R., *Renewal Theory,* Methuen, London, 1962.
4. Ehrhardt, R., "The Power Approximation for Computing (s,S) Inventory Policies", *Management Sci.* 25, 777-786, 1979.
5. Ehrhardt, R., "(s,S) Policies for a Dynamic Inventory Model with Stochastic Lead Times", Technical Report No. 17, University of North Carolina at Chapell Hill, 1980.
6. Federgruen, A., Groenevelt, H. and Tijms, H.C., "Coordinated Replenishments in a Multi-Item Inventory System with Compound Poisson Demands and Constant Lead Times", Report No. 60, Dept. of Actuarial Sciences and Econometrics, Free University, Amsterdam, 1980.
7. Freeland, J.R. and Porteus, E.L., "Evaluating the Effectiveness of a New Method for Computing Approximately Optimal (s,S) Inventory Policies", *Operations Res.* 28, 353-363, 1980.
8. Hadley, G. and Whitin, T.M., *Analysis of Inventory Systems,* Prentice-Hall, Englewood Cliffs, N.J., 1963.
9. Kaplan, R.S., "A Dynamic Inventory Model with Stochastic Lead Times", *Management Sci.* 16, 491-507, 1970.
10. Naddor, E., "Optimal and Heuristic Decisions in Single- and Multi-Item Inventory Systems, *Management Sci.* 21, 1234-1249, 1975.
11. Naddor, E., "Sensitivity to Distributions in Inventory Systems", *Management Sci.* 24, 1769-1772, 1978.
12. Nahmias, S., "Simple Approximations for a Variety of Dynamic Leadtime Lost-Sales Inventory Models, *Operations Res.* 27, 904-924, 1979.
13. Porteus, E., "An Adjustment to the Norman-White Approach to Approximating Dynamic Programs", *Operations Res.* 27, 1203-1208, 1979.
14. Roberts, D.M., "Approximations to Optimal Policies in a Dynamic Inventory Model", in: *Studies in Applied Probability and Management Science,* K. Arrow, S. Karlin and P. Suppes (eds.), Stanford University Press, Stanford, 1962.
15. Schneider, H., "Methods for Determining the Reorder Point of an (s,S) Ordering Policy when a Service Level is Specified", *J. Opl. Res. Soc.* 12, 1181-1193, 1978.
16. Schneider, H., *Servicegrade in Lagerhaltungsmodelle,* Günter Marchal und Hans-Jochen Matzenbacher Wissenschaftsverlag, Berlin, 1979.

17. Veinott, A.F. Jr. and Wagner, H.M., "Computing Optimal (s,S) Inventory Policies", *Management Sci.* 11, 525-542, 1965.

18. Wagner, H., O'Hagan, M. and Lundh, B., "An Empirical Study of Exactly and Approximately Optimal Inventory Policies", *Management Sci.* 11, 690-723, 1965.

19. Werner, H., Stoer, J. and Bommas, W., "Rational Chebyshev Approximation", *Numerische Math.* 10, 289-306, 1967.